Plohe konstantne negativne zakrivljenosti

Davidović, Slavko

SVEUČILIŠTE U ZAGREBU

PRIRODOSLOVNO–MATEMATIČKI FAKULTET

MATEMATIČKI ODSJEK

Slavko Davidović

PLOHE KONSTANTNE NEGATIVNE ZAKRIVLJENOSTI

Diplomski rad

Voditelj rada:
Prof. dr. sc. Željka Milin-Šipuš

Zagreb, veljača 2017

Ovaj diplomski rad obranjen je dana ____________________ pred ispitnim povjerenstvom u sastavu:

1. ____________________________, predsjednik
2. ____________________________, član
3. ____________________________, član

Povjerenstvo je rad ocijenilo ocjenom ________________.

Potpisi članova povjerenstva:

1. ____________________________
2. ____________________________
3. ____________________________

Zahvaljujem mentorici, prof. dr. sc. Željki Milin-Šipuš na velikoj pomoći i vodstvu pri izradi ovoga rada; svim profesorima, asistentima, demonstratorima i drugim zaposlenicima Prirodoslovno-matematičkog fakulteta, Matematičkog odjela u Zagrebu. Također velika hvala ide mojoj obitelji i prijateljima koji su bili uvijek uz mene sve ove dane studentskog života.

Sadržaj

Uvod

Jasnije proučavanje ploha konstantne negativne zakrivljenosti vodi nas u godinu 1838. Te godine Minding je objavio veoma važan rezultat vezan za ovu vrstu ploha, a to je teorem koji kaže da su te plohe izometrične, tj. točke na dvije takve plohe mogu se postaviti u 1-1 korespondenciju tako da metrika bude sačuvana. Beltrami je kasnije takve plohe nazvao *pseudosfernim* plohama.

Bour je 1862. godine prvi postavio ono što danas nazivamo sinus-Gordonovom jednadžbom. 1879. godine Bianchi u svojoj docentskoj obrani predstavlja, u matematičkom smislu, geometrijsku konstrukciju pseudosfernih ploha. Ovaj rezultat je proširio Bäcklund 1883. godine koji uključuje ključni parametar koji dozvoljava iterativnu konstrukciju takvih pseudosfernih ploha, a 1885. Bianchi naknadno pokazuje da je Bäcklundova transformacija povezana s elegantnom invarijantom sinus-Gordonove jednadžbe. Ta invarijantna je postala poznata kao Bäcklundova transformacija sinus-Gordonove jednadžbe.

Bäcklundova transformacija ima važnu primjenu u teoriji solitona. Sve solitonske jednadžbe imaju to svojstvo invarijantnosti na Bäcklundove transformacije.

U ovom radu ćemo proučavati Bäcklundovu transformaciju, njeno geometrijsko podrijetlo i primjenu u modernoj teoriji solitona.

U prvom poglavlju pseudosferne plohe ćemo promatrati u kontekstu hiperboličkih ploha. Korisit ćemo Gauss-Weingartenove jednadžbe i svojstva hiperboličkih ploha kako bi izveli sinus-Gordonovu jednadžbu.

U drugom poglavlju pokazat ćemo da postoji nova pseudosferna ploha izometrična s početnom plohom i izvest ćemo formulu za Bäcklundovu transformaciju koja povezuje dvije pseudoferne plohe.

U trećem poglavlju 'ući ćemo' malo u primjenu Bäcklundove transformacije u teoriji solitona.

Poglavlje 1

Gauss-Weingartenove jednadžbe za hiperboličke plohe

Neka je $\mathbf{x} = \mathbf{x}(u, v)$ položajni vektor proizvoljnje točke P plohe S u prostoru $\mathbb{R}^3$ ($\mathbf{x}$ još nazivamo *parametrizacijom*, *kartom* okoline točke P). Tada su $\mathbf{x}_u$ i $\mathbf{x}_v$ tangencijalni vektori plohe S u točki P i vrijedi

$$n = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}, \tag{1.1}$$

gdje je n jedinični vektor normale plohe S. Prva i druga fundamentalna forma plohe S dane su s

$$\begin{aligned} \mathtt{I} &= d\mathbf{x} \cdot d\mathbf{x} = E\,du^2 + 2F\,du\,dv + G\,du^2, \\ \mathtt{II} &= -d\mathbf{x} \cdot dn = L\,du^2 + 2M\,du\,dv + N\,dv^2, \end{aligned} \tag{1.2}$$

gdje su

$$\begin{gathered} E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v, \\ L = -\mathbf{x}_u \cdot n_u = \mathbf{x}_{uu} \cdot n, \quad N = -\mathbf{x}_v \cdot n_v = \mathbf{x}_{vv} \cdot n, \\ M = -\mathbf{x}_u \cdot n_v = -\mathbf{x}_v \cdot n_u = \mathbf{x}_{uv} \cdot n. \end{gathered} \tag{1.3}$$

Šestorka $\{E, F, G; L, M, N\}$ određuje položaj plohe u prostoru. Gaussove jednadžbe povezane s plohom S su

$$\begin{aligned} \mathbf{x}_{uu} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + L\,n, \\ \mathbf{x}_{uv} &= \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + M\,n, \\ \mathbf{x}_{vv} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + N\,n, \end{aligned} \tag{1.4}$$

a Weingartenove

$$
\begin{aligned}
n_u &= \frac{MF - LG}{W^2}\mathbf{x}_u + \frac{LF - ME}{W^2}\mathbf{x}_v,\\
n_v &= \frac{NF - MG}{W^2}\mathbf{x}_u + \frac{MF - NE}{W^2}\mathbf{x}_v,
\end{aligned}
\tag{1.5}
$$

gdje je $W^2 = \|\mathbf{x}_u \times \mathbf{x}_v\|^2 = EG - F^2$.
Izraze Γ^i_{jk}, za koje vrijedi

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{GE_u - 2FF_u + FE_v}{2W^2},\\
\Gamma^2_{11} &= \frac{2EF_u - EE_v - FE_u}{2W^2},\\
\Gamma^1_{12} &= \frac{GE_v - FG_u}{2W^2},\\
\Gamma^2_{12} &= \frac{EG_u - FE_v}{2W^2},\\
\Gamma^1_{22} &= \frac{2GF_v - GG_u - FG_v}{2W^2},\\
\Gamma^2_{22} &= \frac{EG_v - 2FF_v + FG_u}{2W^2},
\end{aligned}
\tag{1.6}
$$

nazivamo Christofellovim simbolima 2. vrste.
Uvjeti usklađenosti $(\mathbf{x}_{uu})_v = (\mathbf{x}_{uv})_u$ i $(\mathbf{x}_{uv})_v = (\mathbf{x}_{vv})_u$, prošireni Gaussovim *Veličanstvenim teoremom* (*Theorema egregium*) i primjenjeni na *linearni* Gaussov sustav, daju *nelinearni* Mainardi-Codazzijev sustav

$$
\begin{aligned}
\left(\frac{L}{W}\right)_v - \left(\frac{M}{W}\right)_u + \frac{L}{W}\Gamma^2_{22} - 2\frac{M}{W}\Gamma^2_{12} + \frac{N}{W}\Gamma^2_{11} &= 0,\\
\left(\frac{N}{W}\right)_u - \left(\frac{M}{W}\right)_v + \frac{L}{W}\Gamma^1_{22} - 2\frac{M}{W}\Gamma^1_{12} + \frac{N}{W}\Gamma^1_{11} &= 0,
\end{aligned}
\tag{1.7}
$$

ili, ekvivalentno tome,

$$
\begin{aligned}
L_v - M_u &= L\Gamma^1_{12} + M\left(\Gamma^2_{12} - \Gamma^1_{11}\right) - N\Gamma^2_{11},\\
M_v - N_u &= L\Gamma^1_{22} + M\left(\Gamma^2_{22} - \Gamma^1_{12}\right) - N\Gamma^2_{12},
\end{aligned}
\tag{1.8}
$$

Iz tog teorema proizlazi i izraz za *Gaussovu (totalnu) zakrivljenost*

$$
K = \frac{LN - M^2}{EG - F^2},
\tag{1.9}
$$

ili, u Louville-ovu prikazu,

$$K = \frac{1}{W}\left[\left(\frac{W}{E}\Gamma^2_{11}\right)_v - \left(\frac{W}{E}\Gamma^2_{12}\right)_u\right]. \tag{1.10}$$

U fizičkom smislu, *Veličanstveni teorem* podrazumijeva da je totalna zakrivljenost plohe S invarijantna pod savijanjem bez rastezanja.

U nastavku ćemo navesti propozicije važne za daljnji rad.

Propozicija 1.0.1. *Za točku $P \in S$ kažemo da je hiperbolička ako je totalna zakrivljenost u toj točki negativna.*

Odavde slijedi da za plohu čija je totalna zakrivljenost negativna za sve točke $P \in S$ kažemo da je hiperbolička ploha.

Propozicija 1.0.2. *U hiperboličkoj točki plohe postoje točno dva asimptotska smjera.*

Ta dva smjera čine *asimptotsku mrežu* plohe.

Propozicija 1.0.3. *Parametarska mreža plohe se podudara s asimptotskom mrežom ako i samo ako vrijedi $L = N = 0$.*

Ako je totalna zakrivljenost plohe S negativna, što znači da je S hiperbolička ploha, onda se asimptotska mreža plohe podudara s parametarskom mrežom i vrijedi $L = N = 0$. Tada se Mainardi-Codazzijeve jednadžbe (1.7) skrate u

$$\left(\frac{M}{W}\right)_u + 2\Gamma^2_{12}\frac{M}{W} = 0, \quad \left(\frac{M}{W}\right)_v + 2\Gamma^1_{12}\frac{M}{W} = 0, \tag{1.11}$$

odnosno jednadžbe (1.8) u

$$\begin{aligned} -M_u &= M\left(\Gamma^2_{12} - \Gamma^1_{11}\right), \\ M_v &= M\left(\Gamma^2_{22} - \Gamma^1_{12}\right). \end{aligned} \tag{1.12}$$

Totalna zakrivljenost se reducira na

$$K = -\frac{M^2}{W^2} =: -\frac{1}{\rho^2}. \tag{1.13}$$

Pokažimo da je (1.11) $\Leftrightarrow$ (1.12).

Dokaz. Krenimo od

$$\left(\frac{M}{W}\right)_u + 2\Gamma^2_{12}\frac{M}{W} = 0 \quad \Leftrightarrow \quad -M_u = M\left(\Gamma^2_{12} - \Gamma^1_{11}\right). \tag{1.14}$$

Imamo

$$\left(\frac{M}{W}\right)_u = \frac{M_u W - M W_u}{W^2}.$$

Pribrojimo li lijevoj i desnoj strani jednadžbe $2\Gamma^2_{12}\,\frac{M}{W}$ i potom ih pomnožimo s W, dobivamo

$$M_u - M\frac{W_u}{W} + 2M\Gamma^2_{12} = 0$$

$$\Leftrightarrow -M_u = M\left(2\Gamma^2_{12} - \frac{W_u}{W}\right).$$

Iz dobivene jednadžbe i (1.12) vidimo da treba vrijediti $\frac{W_u}{W} = \Gamma^2_{12} + \Gamma^1_{11}$. Provjerimo.

$$\frac{W_u}{W} = \frac{1}{2W^2}\cdot 2W\,W_u = \frac{1}{2W^2}\left(W^2\right)_u$$

Nadalje, imamo

$$\left(W^2\right)_u = \left(EG - F^2\right)_u = E_uG + EG_u - 2FF_u,$$

$$\begin{aligned}\Gamma^2_{12} + \Gamma^1_{11} &= \frac{1}{2W^2}\left(EG_u - FE_v + GE_u - 2FF_u + FE_v\right)\\ &= \frac{1}{2W^2}\left(W^2\right)_u,\end{aligned}$$

čime smo pokazali da vrijedi $\frac{W_u}{W} = \Gamma^2_{12} + \Gamma^1_{11}$, odnosno dokazali tvrdnju (1.14).
Analogno pokažemo

$$\left(\frac{M}{W}\right)_v + 2\Gamma^1_{12}\frac{M}{W} = 0 \quad \Leftrightarrow \quad M_v = M\left(\Gamma^2_{22} - \Gamma^1_{12}\right). \tag{1.15}$$

□

Za kut ω između parametarskih krivulja vrijedi

$$\begin{aligned}\cos\omega &= \frac{\mathbf{x}_u\mathbf{x}_v}{\|\mathbf{x}_u\|\,\|\mathbf{x}_v\|} = \frac{F}{\sqrt{\mathbf{x}_u^2}\,\sqrt{\mathbf{x}_v^2}} = \frac{F}{\sqrt{EG}},\\ \sin^2\omega &= 1 - \cos^2\omega = 1 - \frac{F^2}{EG} = \frac{EG - F^2}{EG} = \frac{W^2}{EG}\\ &\Rightarrow \sin\omega = \frac{W}{\sqrt{EG}}.\end{aligned} \tag{1.16}$$

Budući da su $E, G > 0$ ($E = \mathbf{x}_u^2, G = \mathbf{x}_v^2$), bez smanjenja općenitosti možemo uzeti

$$E = \rho^2 a^2, \quad G = \rho^2 b^2. \tag{1.17}$$

S obzirom na (1.16) i (1.17), za F, W i M dobivamo

$$\begin{aligned} F &= \sqrt{EG}\cos\omega = \sqrt{\rho^2\, a^2 \cdot \rho^2\, b^2}\cos\omega = \rho^2\, ab\cos\omega, \\ W &= \sqrt{EG}\sin\omega = \sqrt{\rho^2\, a^2 \cdot \rho^2\, b^2}\sin\omega = \rho^2\, ab\sin\omega. \\ M^2 &= \frac{W^2}{\rho^2} = \frac{\rho^4\, a^2 b^2 \sin^2\omega}{\rho^2} = \rho^2\, a^2 b^2 \sin^2\omega \quad \Rightarrow \quad M = \rho\, ab\sin\omega. \end{aligned} \tag{1.18}$$

Tada se prva i druga fundamentalnu forma reduciraju u

$$\begin{aligned} \mathtt{I} &= \rho^2 \left(a^2\, du^2 + 2ab\cos\omega\, du\, dv + b^2\, dv^2\right), \\ \mathtt{II} &= 2\rho\, ab\sin\omega\, du\, dv. \end{aligned} \tag{1.19}$$

Pogledajmo kako sada izgledaju Mainardi-Codazzijeve jednadžbe. Krenimo od prve jednadžbe u (1.11):

$$\left(\frac{M}{W}\right)_u + 2\Gamma_{12}^2 \left(\frac{M}{W}\right) = 0.$$

Prisjetimo se da smo definirali $\frac{M^2}{W^2} = \frac{1}{\rho^2}$, pa bez smanjenja općenitosti možemo napisati $\frac{M}{W} = \frac{1}{\rho}$, i to uvrstimo, pa imamo:

$$\begin{aligned} & \left(\frac{1}{\rho}\right)_u + 2\frac{1}{\rho}\Gamma_{12}^2 = 0 \\ \Leftrightarrow \quad & -\frac{\rho_u}{\rho^2} + 2\frac{1}{\rho}\Gamma_{12}^2 = 0 \quad / \cdot \rho^2 \\ \Leftrightarrow \quad & -\rho_u + 2\rho\Gamma_{12}^2 = 0. \end{aligned} \tag{1.20}$$

Γ_{12}^2 izračunamo:

$$\begin{aligned} \Gamma_{12}^2 &= \frac{EG_u - FE_v}{2W^2} \\ &= \frac{1}{2W^2}\left[\rho^2\, a^2 \left(2\rho\rho_u\, b^2 + 2\rho^2 b\, b_u\right)\right] - \rho^2\, a\, b\, \cos\omega \left(2\rho\rho_v\, a^2 + 2\rho^2\, a\, a_v\right) \\ &= \frac{1}{2\rho^4\, a^2\, b^2\, \sin^2\omega} \cdot 2\rho^3\, a^2\, b\left[\rho_u\, b + \rho\, b_u - \cos\omega\left(\rho_v\, a + \rho\, a_v\right)\right] \\ &= \frac{1}{\rho\, b\sin^2\omega}\left[\rho_u\, b + \rho\, b_u - \cos\omega\left(\rho_v\, a + \rho\, a_v\right)\right]. \end{aligned}$$

Analogno, za drugu Mainardi-Codazzijevu jednadžbu iz (1.11) vrijedi:

$$
\begin{aligned}
&\left(\frac{M}{W}\right)_v + 2\Gamma^1_{12}\left(\frac{M}{W}\right) = 0 \\
\Leftrightarrow\quad &\left(\frac{1}{\rho}\right)_v + 2\frac{1}{\rho}\Gamma^1_{12} = 0 \\
\Leftrightarrow\quad &-\frac{\rho_v}{\rho^2} + 2\frac{1}{\rho}\Gamma^1_{12} = 0 \quad / \cdot \rho^2 \\
\Leftrightarrow\quad &-\rho_v + 2\rho\Gamma^1_{12} = 0,
\end{aligned}
\tag{1.21}
$$

gdje je

$$
\begin{aligned}
\Gamma^1_{12} &= \frac{GE_v - FG_u}{2W^2} \\
&= \frac{1}{2W^2}\left[\rho^2 b^2 \left(2\rho\rho_v a^2 + 2\rho^2 a\, a_v\right)\right] - \rho^2 a\, b\, \cos\omega \left(2\rho\rho_u b^2 + 2\rho^2 b\, b_u\right) \\
&= \frac{1}{2\rho^4 a^2 b^2 \sin^2\omega} \cdot 2\rho^3 a\, b^2 \left[\rho_v a + \rho\, a_v - \cos\omega\,(\rho_u b + \rho\, b_u)\right] \\
&= \frac{1}{\rho\, a \sin^2\omega}\left[\rho_v a + \rho\, a_v - \cos\omega\,(\rho_u b + \rho\, b_u)\right].
\end{aligned}
$$

Napokon, dobivamo:

$$
-\rho_u + \frac{2}{b\sin^2\omega}\left[\rho_u b + \rho\, b_u - \cos\omega\,(\rho_v a + \rho\, a_v)\right] = 0, \tag{1.22a}
$$

$$
-\rho_v + \frac{2}{a\sin^2\omega}\left[\rho_v a + \rho\, a_v - \cos\omega\,(\rho_u b + \rho\, b_u)\right] = 0. \tag{1.22b}
$$

Iz prve jednakosti imamo

$$
\begin{aligned}
&\frac{2\cos\omega}{b\sin^2\omega}\rho\, a_v = -\frac{2\cos\omega}{b\sin^2\omega}\rho_v a + \frac{2}{b\sin^2\omega}\left[\rho_u b + \rho\, b_u\right] - \rho_u \quad / \cdot \frac{b\sin^2\omega}{2\rho\cos\omega} \\
\Rightarrow\quad & a_v = -\frac{b\sin^2\omega}{2\rho\cos\omega}\,\frac{2\cos\omega}{b\sin^2\omega}\rho_v a + \frac{b\sin^2\omega}{2\rho\cos\omega}\,\frac{2}{b\sin^2\omega}\left[\rho_u b + \rho\, b_u\right] - \rho_u\frac{b\sin^2\omega}{2\rho\cos\omega} \\
\Leftrightarrow\quad & a_v = \frac{\rho_v}{\rho}a + \frac{1}{\rho\cos\omega}\left[\rho_u b + \rho\, b_u\right] - \rho_u\frac{b\sin^2\omega}{2\rho\cos\omega}.
\end{aligned}
$$

To uvrstimo u (1.22b) te dobijemo

$$
\begin{aligned}
-\rho_v + \frac{2}{a\sin^2\omega}\rho_v a - \frac{2\cos\omega}{a\sin^2\omega}(\rho_u b + \rho b_u) \\
+ \frac{2\rho}{a\sin^2\omega}\left[\frac{-\rho_v}{\rho}a + \frac{1}{\rho\cos\omega}(\rho_u b + \rho b_u) - \rho_u \frac{b\sin^2\omega}{2\rho\cos\omega}\right] = 0
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow \quad & -\rho_v + \frac{2}{\sin^2\omega}\rho_v - \frac{2\cos\omega}{a\sin^2\omega}(\rho_u b + \rho b_u) - \frac{2\rho_v}{\sin^2\omega} \\
& \qquad + \frac{2}{a\cos\omega\sin^2\omega}(\rho_u b + \rho b_u) - \rho_u \frac{b}{a\cos\omega} = 0 \\
\Leftrightarrow \quad & -\rho_v + (\rho_u b + \rho b_u)\left(\frac{2}{a\cos\omega\sin^2\omega} - \frac{2\cos\omega}{a\sin^2\omega}\right) - \rho_u\frac{b}{a\cos\omega} = 0 \\
\Leftrightarrow \quad & -\rho_v + (\rho_u b + \rho b_u)\frac{2 - 2\cos^2\omega}{a\cos\omega\sin^2\omega} - \rho_u\frac{b}{a\cos\omega} = 0 \\
\Leftrightarrow \quad & -\rho_v + (\rho_u b + \rho b_u)\frac{2\sin^2\omega}{a\cos\omega\sin^2\omega} - \rho_u\frac{b}{a\cos\omega} = 0 \\
\Leftrightarrow \quad & -\rho_v + \rho_u b\frac{2}{a\cos\omega} - \rho_u b\frac{1}{a\cos\omega} + \rho b_u \frac{2}{a\cos\omega} = 0 \\
\Leftrightarrow \quad & -\rho_v + \rho_u b\frac{1}{a\cos\omega} + \rho b_u\frac{2}{a\cos\omega} = 0 \quad / \cdot \frac{a\cos\omega}{2\rho} \\
\Leftrightarrow \quad & -\rho_v\frac{a\cos\omega}{2\rho} + \rho_u\frac{b}{2\rho} + b_u = 0
\end{aligned}
$$

$$
\boxed{\Leftrightarrow \quad b_u + \frac{1}{2}\frac{\rho_u}{\rho}b - \frac{1}{2}\frac{\rho_v}{\rho}a\cos\omega = 0.}
$$

Analogno, iz jednakosti (1.22b) imamo:

$$
\frac{2\cos\omega}{a\sin^2\omega}\rho b_u = -\frac{2\cos\omega}{a\sin^2\omega}\rho_u b + \frac{2}{a\sin^2\omega}[\rho_v a + \rho a_v] - \rho_v \quad / \cdot \frac{a\sin^2\omega}{2\rho\cos\omega}
$$

$$
\Rightarrow \quad b_u = -\frac{a\sin^2\omega}{2\rho\cos\omega}\frac{2\cos\omega}{a\sin^2\omega}\rho_u b + \frac{a\sin^2\omega}{2\rho\cos\omega}\frac{2}{a\sin^2\omega}[\rho_v a + \rho a_v] - \rho_v\frac{a\sin^2\omega}{2\rho\cos\omega}
$$

$$
\Rightarrow \quad b_u = \frac{\rho_u}{\rho}b + \frac{1}{\rho\cos\omega}[\rho_v a + \rho a_v] - \rho_v\frac{a\sin^2\omega}{2\rho\cos\omega},
$$

što uvrštavamo u (1.22a):

$$
\begin{aligned}
-\rho_u + \frac{2}{b\sin^2\omega}\rho_u b - \frac{2\cos\omega}{b\sin^2\omega}(\rho_v a + \rho a_v)& \\
+ \frac{2\rho}{b\sin^2\omega}\left[\frac{-\rho_u}{\rho} b + \frac{1}{\rho\cos\omega}(\rho_v a + \rho a_v) - \rho_v \frac{a\sin^2\omega}{2\rho\cos\omega}\right] &= 0
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow \quad -\rho_u + \frac{2}{\sin^2\omega}\rho_u - \frac{2\cos\omega}{b\sin^2\omega}(\rho_v a + \rho a_v) - \frac{2\rho_u}{\sin^2\omega}& \\
+ \frac{2}{b\cos\omega\sin^2\omega}(\rho_v a + \rho a_v) - \rho_v \frac{a}{b\cos\omega} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow \quad -\rho_u + (\rho_v a + \rho a_v)\left(\frac{2}{b\cos\omega\sin^2\omega} - \frac{2\cos\omega}{b\sin^2\omega}\right) - \rho_v \frac{a}{b\cos\omega} &= 0 \\
\Leftrightarrow \quad -\rho_u + (\rho_v a + \rho a_v)\frac{2 - 2\cos^2\omega}{b\cos\omega\sin^2\omega} - \rho_v \frac{a}{b\cos\omega} &= 0 \\
\Leftrightarrow \quad -\rho_u + (\rho_v a + \rho a_v)\frac{2\sin^2\omega}{b\cos\omega\sin^2\omega} - \rho_v \frac{a}{b\cos\omega} &= 0 \\
\Leftrightarrow \quad -\rho_u + \rho_v a \frac{2}{b\cos\omega} - \rho_v a \frac{1}{b\cos\omega} + \rho a_v \frac{2}{b\cos\omega} &= 0 \\
\Leftrightarrow \quad -\rho_u + \rho_v a \frac{1}{b\cos\omega} + \rho a_v \frac{2}{b\cos\omega} &= 0 \quad / \cdot \frac{b\cos\omega}{2\rho} \\
\Leftrightarrow \quad -\rho_u \frac{b\cos\omega}{2\rho} + \rho_v \frac{a}{2\rho} + a_v &= 0
\end{aligned}
$$

$$
\boxed{\Leftrightarrow \quad a_v + \frac{1}{2}\frac{\rho_v}{\rho} a - \frac{1}{2}\frac{\rho_u}{\rho} b\cos\omega = 0.}
$$

Dakle, ovo su Mainardi-Codazzijeve jednadžbe u novom zapisu:

$$
\begin{aligned}
a_v + \frac{1}{2}\frac{\rho_v}{\rho} a - \frac{1}{2}\frac{\rho_u}{\rho} b\cos\omega &= 0, \\
b_u + \frac{1}{2}\frac{\rho_u}{\rho} b - \frac{1}{2}\frac{\rho_v}{\rho} a\cos\omega &= 0.
\end{aligned}
\tag{1.23}
$$

Za totalnu zakrivljenost (1.10) dobivamo

$$
\omega_{uv} + \frac{1}{2}\left(\frac{\rho_u}{\rho}\frac{b}{a}\sin\omega\right)_u + \frac{1}{2}\left(\frac{\rho_v}{\rho}\frac{a}{b}\sin\omega\right)_v - ab\sin\omega = 0 \tag{1.24}
$$

Postavimo sada uvjet da je totalna zakrivljenost $K = -1/\rho^2 < 0$ konstantna; tada S nazivamo *pseudosfernom* plohom. Iz Mainardi-Codazzijevih jednadžbi (1.22a) i (1.22b) tada proizlazi $a = a(u)$, $b = b(v)$. Ako je S parametrizirana duljinom luka na asimptotskoj mreži (tako da odgovara transformaciji $du \to d\overline{u} = \sqrt{E(u)}\,du$, $dv \to d\overline{v} = \sqrt{G(v)}\,dv$), tada

$$E = G = 1, \quad F = \cos\omega, \quad W = \sin\omega, \quad M = \frac{1}{\rho}\sin\omega, \tag{1.25}$$

a prva i druga fundamentalna forma postaju

$$\begin{aligned} \mathtt{I} &= du^2 + 2\cos\omega\,du\,dv + b^2\,dv^2, \\ \mathtt{II} &= \frac{2}{\rho}\,\sin\omega\,du\,dv. \end{aligned} \tag{1.26}$$

Totalna zakrivljenost iz (1.24) reducira se u poznatu *sinus-Gordonovu* jednadžbu

$$\boxed{\omega_{uv} = \frac{1}{\rho^2}\sin\omega.} \tag{1.27}$$

Pogledajmo što se događa s Gauss-Weingartenovim jednadžbama (1.4) i (1.5) nakon transformacije.

$$\begin{aligned} \mathbf{x}_{uu} &= \frac{GE_u - 2FF_u + FE_v}{2w^2}\mathbf{x}_u + \frac{2EF_u - EE_v - FE_u}{2W^2}\mathbf{x}_v \\ &= \frac{\rho^2 b^2\left(\rho^2 a^2\right)_u - 2\rho^2\,ab\cos\omega\left(\rho^2\,ab\cos\omega\right)_u + 2\rho^2\,ab\cos\omega\left(\rho^2\,a^2\right)_v}{2\rho^4\,a^2b^2\sin^2\omega}\mathbf{x}_u \\ &+ \frac{2\rho^2\,a^2\left(\rho^2\,ab\cos\omega\right)_u - \rho^2\,a^2\left(\rho^2\,a^2\right)_v - \rho^2\,ab\cos\omega\left(\rho^2\,a^2\right)_u}{2\rho^4\,a^2b^2\sin^2\omega}\mathbf{x}_v = \text{(transformiramo)} \\ &= \frac{1\,(1)_u - 2\cos\omega\,(\cos\omega)_u + 2\cos\omega\,(1)_v}{2\sin^2\omega}\mathbf{x}_u \\ &+ \frac{2\cdot 1\,(\cos\omega)_u - 1\,(1)_v\cos\omega\,(1)_u}{2\sin^2\omega}\mathbf{x}_v \\ &= \frac{\omega_u\sin\omega\cos\omega}{\sin^2\omega}\mathbf{x}_u - \frac{\omega_u\sin\omega}{\sin^2\omega}\mathbf{x}_v \\ &= \omega_u\operatorname{ctg}\omega\,\mathbf{x}_u + \omega_u\frac{1}{\sin\omega}\,\mathbf{x}_v \end{aligned}$$

$$
\begin{aligned}
\mathbf{x}_{uv} &= \frac{GE_v - FG_u}{2W^2}\mathbf{x}_u + \frac{EG_u - FE_v}{2W^2}\mathbf{x}_v + M\,n \\
&= \frac{\rho^2\, b^2 \left(\rho^2\, a^2\right)_v - \rho^2\, ab\cos\omega \left(\rho^2\, b^2\right)_u}{2\rho^4\, a^2 b^2 \sin^2\omega}\mathbf{x}_u \\
&+ \frac{\rho^2\, a^2 \left(\rho^2\, b^2\right)_u - \rho^2\, ab\cos\omega \left(\rho^2\, a^2\right)_v}{2\rho^4\, a^2 b^2 \sin^2\omega}\mathbf{x}_v + \rho\, ab\sin\omega \cdot \mathbf{n} \\
&= (\text{transformiramo}) = \frac{1\,(1)_v - \cos\omega\,(1)_u}{2\sin^2\omega}\mathbf{x}_u + \frac{1\,(1)_u - \cos\omega\,(1)_v}{2\sin^2\omega}\mathbf{x}_v + \frac{1}{\rho}\sin\omega\,\mathbf{n} \\
&= \frac{1}{\rho}\sin\omega\,\mathbf{n}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_{vv} &= \frac{2GF_v - GG_u - FG_v}{2W^2}\mathbf{x}_u + \frac{EG_v - 2FF_v + FG_u}{2W^2}\mathbf{x}_v \\
&= \frac{2\rho^2\, b^2 \left(\rho^2\, ab\cos\omega\right)_v - \rho^2\, b^2 \left(\rho^2\, b^2\right)_u - \rho^2\, ab\cos\omega \left(\rho^2\, b^2\right)_v}{2\rho^4\, a^2 b^2 \sin^2\omega}\mathbf{x}_u \\
&+ \frac{\rho^2\, a^2 \left(\rho^2\, b^2\right)_v - 2\rho^2\, ab\cos\omega \left(\rho^2\, ab\cos\omega\right)_v + \rho^2\, ab\cos\omega \left(\rho^2\, b^2\right)_u}{2\rho^4\, a^2 b^2 \sin^2\omega}\mathbf{x}_v = (\text{transformiramo}) \\
&= \frac{2\,(\cos\omega)_v - 1\,(1)_u - \cos\omega\,(1)_v}{2\sin^2\omega}\mathbf{x}_u + \frac{1\,(1)_v - 2\cos\omega\,(\cos\omega)_v + \cos\omega\,(1)_u}{2\sin^2\omega}\mathbf{x}_v \\
&= \frac{-2\omega_u\sin\omega}{2\sin^2\omega}\mathbf{x}_u + \frac{2\omega_u\cos\omega\sin\omega}{2\sin^2\omega}\mathbf{x}_v \\
&= -\omega_u\frac{1}{\sin\omega}\,\mathbf{x}_u + \omega_u\,\mathrm{ctg}\,\omega\,\mathbf{x}_v
\end{aligned}
$$

$$
\begin{aligned}
n_u &= \frac{\rho\, ab\sin\omega \cdot \rho^2\, ab\cos\omega}{\left(\rho^2\, ab\sin\omega\right)^2}\mathbf{x}_u - \frac{\rho\, ab\sin\omega \cdot \rho^2\, a^2}{\left(\rho^2\, ab\sin\omega\right)^2}\mathbf{x}_v = (\text{transformiramo}) \\
&= \frac{\frac{1}{\rho}\sin\omega\cos\omega}{\sin^2\omega}\mathbf{x}_u - \frac{\frac{1}{\rho}\sin\omega}{\sin^2\omega}\mathbf{x}_v \\
&= \frac{1}{\rho}\,\mathrm{ctg}\,\omega\,\mathbf{x}_u - \frac{1}{\rho}\frac{1}{\sin\omega}\,\mathbf{x}_v
\end{aligned}
$$

$$
\begin{aligned}
n_v &= \frac{-\rho\, ab \sin\omega \cdot \rho^2\, b^2}{(\rho^2\, ab \sin\omega)^2}\mathbf{x}_u + \frac{\rho\, ab \sin\omega \cdot \rho^2\, ab \cos\omega}{(\rho^2\, ab \sin\omega)^2}\mathbf{x}_v = (\text{transformiramo}) \\
&= \frac{-\frac{1}{\rho}\sin\omega}{\sin^2\omega}\mathbf{x}_u + \frac{\frac{1}{\rho}\sin\omega\cos\omega}{\sin^2\omega}\mathbf{x}_v \\
&= -\frac{1}{\rho}\frac{1}{\sin\omega}\,\mathbf{x}_u + \frac{1}{\rho}\operatorname{ctg}\omega\,\mathbf{x}_v
\end{aligned}
$$

Dakle, Gaussove jednadžbe reduciraju se u

$$
\begin{aligned}
\mathbf{x}_{uu} &= \omega_u \operatorname{ctg}\omega\,\mathbf{x}_u - \omega_u\frac{1}{\sin\omega}\,\mathbf{x}_v, \\
\mathbf{x}_{uv} &= \frac{1}{\rho}N\sin\omega, \\
\mathbf{x}_{vv} &= -\omega_v\frac{1}{\sin\omega}\,\mathbf{x}_u + \omega_v\operatorname{ctg}\omega\,\mathbf{x}_v,
\end{aligned}
\tag{1.28}
$$

a Weingartenove

$$
\begin{aligned}
n_u &= \frac{1}{\rho}\operatorname{ctg}\omega\,\mathbf{x}_u - \frac{1}{\rho}\operatorname{cosec}\omega\,\mathbf{x}_v, \\
n_v &= -\frac{1}{\rho}\operatorname{cosec}\omega\,\mathbf{x}_u + \frac{1}{\rho}\operatorname{ctg}\omega\,\mathbf{x}_v.
\end{aligned}
\tag{1.29}
$$

U 20. stoljeću se pokazalo da sinus-Gordonova jednadžba nalazi svoje primjene u fizici. Tako, na primjer, Bäcklundova transformacija za ovu jednadžbu ima važnu primjenu u teoriji širenja kristalnih dislokacija, teoriji napredovanja ultrakratkih optičkih impulsa, kretanju Blochovih ploha u magnetskim kristalima, unitarnoj teoriji elementarnih čestica i tako dalje. To je dobar motiv za početak proučavanja klasične Bäcklundove transformacije za sinus-Gordonovu jednadžbu. Pokazat ćemo da Bäcklundova transformacija odgovara spoju invarijantnih transformacija prema Bianchiju i Lie-u. Lie-eva simetrija nameće ključni Bäcklundov parametar u Bianchijevu transformaciju koji omogućuje stvaranje višesolitonskih rješenja. O solitonima ćemo nešto više reći kasnije.

Poglavlje 2

Klasična Bäcklundova transformacija za sinus-Gordonovu jednadžbu

Temelj originalne Bäcklundove transformacije za sinus-Gordonovu jednadžbu jest jednostavna geometrijska konstrukcija pseudosferne plohe. Ako uzmemo točku P na početnoj plohi S i dužinu $\overline{P\overline{P}}$ konstantne duljine, i tangenta u točki P je konstruirana po zahtjevima Bäcklundove transformacije, kako ćemo vidjeti kasnije, onda graf točaka $\overline{P}$ kad P ocrtava plohu S jest druga pseudosferna ploha $\overline{S}$ s jednakom totalnom zakrivljenošću kao S. Proces možemo ponavljati kako bi generirali niz pseudosfernih ploha s jednakom totalnom zakrivljenošću početne plohe S.

Neka je S preudosferna ploha s totalnom zakrivljenošću $K = -1/\rho^2$ i neka je $\mathbf{x} = \mathbf{x}(u, v)$ proizvoljna karta, gdje u i v odgovaraju parametrizaciji duljinom luka duž asimptotskih krivulja. U ovoj parametrizaciji $\mathbf{x}_u$, $\mathbf{x}_v$ i n su jedinični vektori, no $\mathbf{x}_u$ i $\mathbf{x}_v$ nisu ortogonalni. Prema tome, zgodno je uvesti ortonormiranu trijadu $\{A, B, C\}$, za koju vrijedi

$$\begin{aligned} A = \mathbf{x}_u, \quad C = n = \frac{\mathbf{x}_u \times \mathbf{x}_v}{W} &= \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sin\omega}, \\ B = C \times A = n \times \mathbf{x}_u = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sin\omega} \times \mathbf{x}_u &= \frac{1}{\sin\omega}\mathbf{x}_v - \operatorname{ctg}\omega\, \mathbf{x}_u. \end{aligned} \tag{2.1}$$

Koristeći Gaussove (1.28) i Weingertenove jednadžbe (1.29) izračunajmo A_u, B_u, C_u, A_v, B_v i C_v. Počnimo s A_u:

$$A_u = \mathbf{x}_{uu} = \omega_u \operatorname{ctg}\omega\, \mathbf{x}_u + \omega_u \frac{1}{\sin\omega}\, \mathbf{x}_v.$$

Znamo da je $\mathbf{x}_u = A$, ali još ne znamo izraziti $\mathbf{x}_v$ preko A, B ili C. Zapišimo $\mathbf{x}_v$ kao linearnu kombinaciju vektora A i B:

$$\mathbf{x}_v = \alpha A + \beta B \tag{2.2}$$

Pomnožimo li izraz (2.2) skalarno s A, dobivamo:

$$A \cdot \mathbf{x}_v = \alpha,$$

odnosno

$$\mathbf{x}_u \cdot \mathbf{x}_v = \alpha.$$

Izraz na lijevoj strani jednak je $F = \cos\omega$, odakle slijedi

$$\alpha = \cos\omega.$$

Pomnožimo li (2.2) skalarno s B, dobivamo:

$$B \cdot \mathbf{x}_v = \beta$$

Lijeva strana je jednaka:

$$(n \times \mathbf{x}_u) \cdot \mathbf{x}_v = n \cdot (\mathbf{x}_u \times \mathbf{x}_v) = n \cdot W n = W n^2 = W,$$

odakle slijedi $\beta = W = \sin\omega$.
Dobili smo

$$\mathbf{x}_v = \cos\omega\, A + \sin\omega\, B.$$

Napokon,

$$\begin{aligned} A_u &= \omega_u \operatorname{ctg}\omega\, A - \omega_u \frac{1}{\sin\omega} (\cos\omega\, A + \sin\omega\, B) \\ &= \omega_u \operatorname{ctg}\omega\, A - \omega_u \operatorname{ctg}\omega\, A - \omega_u B \\ &= -\omega_u B. \end{aligned}$$

Izračunajmo B_u i C_u.

$$\begin{aligned} B_u &= \left(\frac{1}{\sin\omega}\, \mathbf{x}_v - \operatorname{ctg}\omega\, \mathbf{x}_u \right)_u \\ &= -\omega_u \frac{1}{\sin^2\omega} \cos\omega\, \mathbf{x}_v - \frac{1}{\sin\omega}\, \mathbf{x}_{vu} - \left(-\omega_u \frac{1}{\sin^2\omega}\, \mathbf{x}_u + \operatorname{ctg}\omega\, \mathbf{x}_{uu} \right) \\ &= -\omega_u \operatorname{ctg}\omega \frac{1}{\sin\omega} (\cos\omega\, A + \sin\omega\, B) - \frac{1}{\sin\omega} \cdot \frac{1}{\rho} \sin\omega\, n + \omega_u \frac{1}{\sin^2\omega} A + \omega_u \operatorname{ctg}\omega\, B \\ &= \left(-\operatorname{ctg}^2\omega + \frac{1}{\sin^2\omega} \right) \omega_u A - \frac{1}{\rho} C \\ &= \omega_u A - \frac{1}{\rho} C \end{aligned}$$

$$
\begin{aligned}
C_u = n_u &= \frac{1}{\rho}\operatorname{ctg}\omega\,\mathbf{x}_u - \frac{1}{\rho}\frac{1}{\sin\omega}\,\mathbf{x}_v = \frac{1}{\rho}\operatorname{ctg}\omega\,A - \frac{1}{\sin\omega}\,(\cos\omega\,A + \sin\omega\,B)\\
&= -\frac{1}{\rho}\,B
\end{aligned}
$$

Analogno, izračunajmo čemu su jednaki i A_v, B_v i C_v.

$$A_v = \mathbf{x}_{uv} = \frac{1}{\rho}\sin\omega\,n = \frac{1}{\rho}\sin\omega\,C$$

$$
\begin{aligned}
B_v &= \left(\frac{1}{\sin\omega}\,\mathbf{x}_v - \operatorname{ctg}\omega\,\mathbf{x}_u\right)_v\\
&= -\omega_v\frac{1}{\sin^2\omega}\cos\omega\,\mathbf{x}_v - \frac{1}{\sin\omega}\,\mathbf{x}_{vv} - \left(-\omega_v\frac{1}{\sin^2\omega}\,\mathbf{x}_u + \operatorname{ctg}\omega\,\mathbf{x}_{uv}\right)\\
&= -\omega_v\frac{1}{\sin^2\omega}\cos\omega\,(\cos\omega\,A + \sin\omega\,B) + \frac{1}{\sin\omega}\left(-\omega_v\frac{1}{\sin\omega}\,\mathbf{x}_u + \omega_v\operatorname{ctg}\omega\,\mathbf{x}\right)\\
&+ \omega_v\frac{1}{\sin\omega}\,A + \omega_v\frac{\operatorname{ctg}\omega}{\sin\omega}\,(\cos\omega\,A + \sin\omega\,B) + \omega_v\frac{1}{\sin^2\omega}\,A - \frac{1}{\rho}\cos\omega\,C\\
&= \ldots = -\frac{1}{\rho}\cos\omega\,C
\end{aligned}
$$

$$
\begin{aligned}
C_v = n_v &= -\frac{1}{\rho}\csc\omega\,\mathbf{x}_u + \frac{1}{\rho}\operatorname{ctg}\omega\,\mathbf{x}_v = -\frac{1}{\rho}\frac{1}{\sin\omega}\,A + \frac{1}{\rho}\operatorname{ctg}\omega\,(\cos\omega\,A + \sin\omega\,B)\\
&= \ldots = -\frac{1}{\rho}\sin\omega\,A + \frac{1}{\rho}\cos\omega\,B
\end{aligned}
$$

Dobivene rezultate možemo zapisati matrično:

$$
\begin{aligned}
\begin{pmatrix} A\\ B\\ C\end{pmatrix}_u &= \begin{pmatrix} 0 & -\omega_u & 0\\ \omega_u & 0 & 1/\rho\\ 0 & -1/\rho & 0\end{pmatrix}\begin{pmatrix} A\\ B\\ C\end{pmatrix},\\
\begin{pmatrix} A\\ B\\ C\end{pmatrix}_v &= \begin{pmatrix} 0 & 0 & (1/\rho)\sin\omega\\ 0 & 0 & -(1/\rho)\cos\omega\\ -(1/\rho)\sin\omega & (1/\rho)\cos\omega & 0\end{pmatrix}\begin{pmatrix} A\\ B\\ C\end{pmatrix}.
\end{aligned}
\tag{2.3}
$$

Ovaj linearni sustav je kompatibilan ako i samo ako ω zadovoljava sinus-Gordonovu jednadžbu.

Novu pseudosfernu plohu $\overline{S}$ s kartom $\overline{\mathbf{x}}$ zapisujemo u obliku

$$\overline{\mathbf{x}} = \mathbf{x} + L\cos\phi\, A + L\sin\phi\, B, \tag{2.4}$$

gdje je $L = \left\|\overline{\mathbf{x}} - \mathbf{x}\right\|$ konstanta. Važno je napomenuti da ne pomiješamo ovu konstantu L i L iz druge fundamentalne forme početne plohe.
Pokažimo da zaista vrijedi $L = \left\|\overline{\mathbf{x}} - \mathbf{x}\right\|$:

$$\begin{aligned}
\overline{\mathbf{x}} - \mathbf{x} &= L(\cos\phi\, A + \sin\phi\, B) \\
\left\|\overline{\mathbf{x}} - \mathbf{x}\right\| &= L\sqrt{(\cos\phi\, A + \sin\phi\, B)^2} = L\sqrt{A^2\cos^2\phi + 2AB\cos\phi\sin\phi + B^2\sin^2\phi} \\
&= L\sqrt{\cos^2\phi + \sin^2\phi} = L\sqrt{1} = L.
\end{aligned}$$

Kut $\phi(u, v)$ je ograničen zahtjevom da na plohi $\overline{S}$, kao i na S, koordinate u i v odgovaraju parametrizaciji duž asimptotske mreže, a nužan uvjet za to je da $\overline{S}$ ima prvu fundamentalnu formu oblika (1.24). Zbog toga mora vrijediti

$$\overline{\mathbf{x}}_u \cdot \overline{\mathbf{x}}_u = 1, \quad \overline{\mathbf{x}}_v \cdot \overline{\mathbf{x}}_v = 1. \tag{2.5}$$

Prije nego smo u mogućnosti primijeniti uvjete, vidimo da moramo najprije odrediti $\overline{\mathbf{x}}_u$ i $\overline{\mathbf{x}}_v$.

$$\begin{aligned}
\overline{\mathbf{x}}_u &= (\mathbf{x} + L\cos\phi\, A + L\sin\phi\, B)_u \\
&= \mathbf{x}_u + L(-\phi_u\sin\phi\, A + \cos\phi\, A_u) + L(\phi_u\cos\phi\, B + \sin\phi\, B_u) \\
&= A - L\phi_u\sin\phi\, A - L\omega_u\cos\phi\, B + L\phi_u\cos\phi\, B + L\sin\phi\left(\omega_u A + \frac{1}{\rho}C\right) \\
&= \left[1 - L(\phi_u - \omega_u)\sin\phi\right] A + L(\phi_u - \omega_u)\cos\phi\, B + \frac{L}{\rho}\sin\phi\, C
\end{aligned}$$

$$\begin{aligned}
\overline{\mathbf{x}}_v &= (\mathbf{x} + L\cos\phi\, A + L\sin\phi\, B)_v \\
&= \mathbf{x}_v + L(-\phi_v\sin\phi\, A + \cos\phi\, A_v) + L(\phi_v\cos\phi\, B + \sin\phi\, B_v) \\
&= \cos\omega\, A + \sin\omega\, B - L\phi_v\sin\phi\, A + \frac{L}{\rho}\cos\phi\sin\omega\, C + L\phi_v\cos\phi\, B - \frac{L}{\rho}\sin\phi\cos\omega\, C \\
&= (\cos\omega - L\phi_v\sin\phi)A + (\sin\omega + L\phi_v\cos\phi)B + \frac{L}{\rho}\sin(\omega - \phi)C
\end{aligned}$$

Dakle, vrijedi

$$\begin{aligned}\overline{\mathbf{x}}_u &= \left[1 - L\,(\phi_u - \omega_u)\sin\phi\right]A + L\,(\phi_u - \omega_u)\cos\phi\, B + \frac{L}{\rho}\sin\phi\, C\\ \overline{\mathbf{x}}_v &= (\cos\omega - L\phi_v\sin\phi)\,A + (\sin\omega + L\phi_v\cos\phi)\,B + \frac{L}{\rho}\sin(\omega-\phi)C.\end{aligned} \tag{2.6}$$

Izračunajmo što dobivamo iz uvjeta $\overline{\mathbf{x}}_u \cdot \overline{\mathbf{x}}_u = 1$.

$$\begin{aligned}1 &= (1 - L\,(\phi_u - \omega_u)\sin\phi)^2 + L^2\,(\phi_u - \omega_u)^2\cos^2\phi + \frac{L^2}{\rho^2}\sin^2\phi\\ &= 1 - 2L\,(\phi_u - \omega_u)\sin\phi + L^2\,(\phi_u - \omega_u)^2\sin^2\phi + L^2\,(\phi_u - \omega_u)^2\cos^2\phi + \frac{L^2}{\rho^2}\sin^2\phi\end{aligned}$$

$$\begin{aligned}&\Rightarrow \quad L^2\,(\phi_u - \omega_u)^2 - 2L\,(\phi_u - \omega_u)\sin\phi + \frac{L^2}{\rho^2}\sin^2\phi = 0\\ &\Rightarrow \quad (\phi_u - \omega_u)^2 - \frac{2}{L}\,(\phi_u - \omega_u)\sin\phi + \frac{1}{\rho^2}\sin^2\phi = 0\end{aligned}$$

Uvest ćemo supstituciju $\phi_u - \omega_u = t$ te pronaći rješenja t_1 i t_2 kvadratne jednadžbe

$$t^2 - \frac{2}{L}t^2\sin\phi + \frac{1}{\rho^2}\sin^2\phi = 0.$$

$$\begin{aligned}t_{1,2} &= \frac{1}{2}\left(\frac{2}{L}\sin\phi \pm \sqrt{\frac{4}{L^2}\sin^2\phi - 4\cdot\frac{\sin^2\phi}{\rho^2}}\right) = \frac{1}{2}\left(\frac{2}{L}\sin\phi \pm 2\ \sin\phi\sqrt{\frac{1}{L^2} - \frac{1}{\rho^2}}\right)\\ &= \frac{1}{L}\sin\phi \pm \frac{1}{L}\sin\phi\sqrt{1 - \frac{L^2}{\rho^2}} = \frac{1}{L}\sin\phi\left(1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}\right)\end{aligned}$$

Vratimo $\phi_u - \omega_u = t$ i dobivamo

$$\phi_u = \omega_u + \frac{1}{L}\left(1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}\right)\sin\phi. \tag{2.7}$$

Iz uvjeta $\overline{\mathbf{x}}_v \cdot \overline{\mathbf{x}}_v = 1$ imamo

$$\begin{aligned}
1 &= (\cos\omega - L\,\phi_v \sin\phi)^2 + (\sin\omega + L\,\phi_v \cos\phi)^2 + \frac{L^2}{\rho^2}\sin^2(\omega-\phi) \\
&= \cos^2\omega - 2L\,\phi_v \cos\omega\sin\phi + L^2\,\phi_v^2\sin^2\phi \\
&+ \sin^2\omega + 2L\,\phi_v\sin\omega\cos\phi + L^2\,\phi_v^2\cos^2\phi + \frac{L^2}{\rho^2}\,\sin^2(\omega-\phi) \\
&= \cos^2\omega + \sin^2\omega + 2L\,(\sin\omega\cos\phi - \cos\omega\sin\phi) + L^2\,\phi_v^2\left(\sin^2\phi + \cos^2\phi\right) \\
&+ \frac{L^2}{\rho^2}\,\sin^2(\omega-\phi)\,.x
\end{aligned}$$

$$\Rightarrow \quad L^2\,\phi_v^2 + 2L\,\phi_v\sin(\omega-\phi) + \frac{L^2}{\rho^2}\,\sin(\omega-\phi) = 0$$

Uvodimo supstituciju $t = \phi_v$ i tražimo rješenja kvadratne jednadžbe

$$L^2\,t^2 + 2L\,t\sin(\omega-\phi) + \frac{L^2}{\rho^2}\,\sin(\omega-\phi) = 0.$$

$$\begin{aligned}
t_{1,2} &= \frac{1}{2L^2}\left[-2L\,\sin(\omega-\phi) \pm \sqrt{4L^2\,\sin^2(\omega-\phi) - 4\frac{L^2}{\rho^2}\,\sin^2(\omega-\phi)}\right] \\
&= \frac{1}{2L^2}\left[-2L\,\sin(\omega-\phi) \pm \sqrt{4L^2\,\sin^2(\omega-\phi)\left(1-\frac{L^2}{\rho^2}\right)}\right] \\
&= \frac{1}{2L^2}\left[-2L\,\sin(\omega-\phi) \pm 2L\,\sin(\omega-\phi)\,\sqrt{1-\frac{L^2}{\rho^2}}\right) \\
&= -\frac{1}{L}\,\sin(\omega-\phi)\left(1 \mp \sqrt{1-\frac{L^2}{\rho^2}}\right) \\
&= \frac{1}{L}\,\sin(\phi-\omega)\left(1 \mp \sqrt{1-\frac{L^2}{\rho^2}}\right)
\end{aligned}$$

Vratimo $\phi_v = t$ i dobivamo

$$\phi_v = \frac{1}{L}\left(1 \mp \sqrt{1-\frac{L^2}{\rho^2}}\right)\sin(\phi-\omega). \tag{2.8}$$

Ako uvedemo oznaku β takvu da vrijedi

$$\beta = \frac{\rho}{L}\left(1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}\right) = \frac{L}{\rho}\left(1 \mp \sqrt{1 - \frac{L^2}{\rho^2}}\right)^{-1}, \tag{2.9}$$

onda (2.7) i (2.8) zapisujemo kao

$$\begin{aligned} \phi_u &= \omega_u + \frac{\beta}{\rho}\sin\phi, \\ \phi_v &= \frac{1}{\beta\rho}\sin(\phi - \omega). \end{aligned} \tag{2.10}$$

Koristeći (2.10), izrazi iz (2.6) postaju

$$\begin{aligned} \overline{\mathbf{x}}_u &= \left(1 - \frac{L}{\rho}\beta\,\sin^2\phi\right)A + \frac{L}{\rho}\beta\,\sin\phi\cos\phi B + \frac{L}{\rho}\sin\phi C, \\ \overline{\mathbf{x}}_v &= \left[\cos\omega - \frac{L}{\rho\beta}\sin\phi\sin(\phi-\omega)\right]A \\ &+ \left[\sin\omega + \frac{L}{\rho\beta}\cos\phi\sin(\phi-\omega)\right]B - \frac{L}{\rho}\sin(\phi-\omega)C. \end{aligned} \tag{2.11}$$

Kako bismo izračunali prvu fundamentalnu formu plohe $\overline{S}$, trebamo odrediti $\overline{E}$, $\overline{F}$ i $\overline{G}$. Znamo: $\overline{E} = \overline{G} = 1$ ($\overline{E} = \overline{\mathbf{x}}_u \cdot \overline{\mathbf{x}}_u$, $\overline{G} = \overline{\mathbf{x}}_v \cdot \overline{\mathbf{x}}_v$), još moramo izračunati $\overline{F}$.

$$\begin{aligned} \overline{F} &= \overline{\mathbf{x}}_u \cdot \overline{\mathbf{x}}_v \\ &= \left(1 - \frac{L}{\rho}\beta\,\sin^2\phi\right)\cdot\left[\cos\omega - \frac{L}{\rho\beta}\sin\phi\sin(\phi-\omega)\right] \\ &+ \frac{L}{\rho}\beta\,\sin\phi\cos\phi\cdot\left[\sin\omega + \frac{L}{\rho\beta}\cos\phi\sin(\phi-\omega)\right] - \frac{L}{\rho}\sin\phi\cdot\frac{L}{\rho}\sin(\phi-\omega) \\ &= \cos\omega - \frac{L}{\rho\beta}\sin\phi\sin(\phi-\omega) - \frac{L\beta}{\rho}\sin^2\phi\cos\omega + \frac{L^2}{\rho^2}\sin^3\phi\sin(\phi-\omega) \\ &+ \frac{L\beta}{\rho}\sin\phi\cos\phi\sin\omega + \frac{L^2}{\rho^2}\sin\phi\cos^2\phi\sin(\phi-\omega) - \frac{L^2}{\rho^2}\sin\phi\sin(\phi-\omega) \end{aligned}$$

Kako je

$$\frac{L^2}{\rho^2}\sin\phi\cos^2\phi\sin(\phi-\omega) = \frac{L^2}{\rho^2}\sin\phi\left(1-\sin^2\phi\right)\sin(\phi-\omega)$$

$$= \frac{L^2}{\rho^2} \sin\phi \sin(\phi - \omega) - \frac{L^2}{\rho^2} \sin^3\phi \sin(\phi - \omega),$$

to dobivamo

$$\overline{F} = \cos\omega - \frac{L}{\rho\beta} \sin\phi \sin(\phi - \omega) - \frac{L\beta}{\rho} \sin^2\phi \cos\omega + \frac{L\beta}{\rho} \sin\phi \cos\phi \sin\omega.$$

Vrijedi

$$-\frac{L\beta}{\rho} \sin^2\phi \cos\omega + \frac{L\beta}{\rho} \sin\phi \cos\phi \sin\omega = -\frac{L\beta}{\rho} \sin\phi \left(\sin\phi \cos\omega - \cos\phi \sin\omega\right)$$
$$= -\frac{L\beta}{\rho} \sin\phi \sin(\phi - \omega),$$

pa slijedi

$$\begin{aligned}\overline{F} &= \cos\omega - \frac{L}{\rho\beta} \sin\phi \sin(\phi - \omega) - \frac{L\beta}{\rho} \sin\phi \sin(\phi - \omega) \\ &= \cos\omega - \frac{L}{\rho} \sin\phi \sin(\phi - \omega) \left(\frac{1}{\beta} + \beta\right).\end{aligned}$$

Izračunajmo kolika je vrijednost izraza $1/\beta + \beta$.

$$\begin{aligned}\frac{1}{\beta} + \beta &= \frac{1}{\beta}\left[1 + \beta^2\right] \\ &= \frac{1}{\beta}\left[1 + \frac{\rho^2}{L^2}\left(1 \pm 2\sqrt{1 - \frac{L^2}{\rho^2}} + 1 - \frac{L^2}{\rho^2}\right)\right] \\ &= \frac{1}{\beta}\left[1 + \frac{\rho^2}{L^2} \pm 2\frac{\rho^2}{L^2}\sqrt{1 - \frac{L^2}{\rho^2}} + \frac{\rho^2}{L^2} - 1\right] \\ &= \frac{1}{\beta} \cdot 2\frac{\rho^2}{L^2}\left[1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}\right] \\ &= \frac{L}{\rho} \frac{1}{1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}} \cdot 2\frac{\rho^2}{L^2}\left(1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}\right) \\ &= \frac{2\rho}{L}\end{aligned}$$

Napokon, za $\overline{F}$ dobivamo

$$\overline{F} = \cos\omega - 2\sin\phi \sin(\phi - \omega).$$

Zapišimo $\cos\omega$ kao $\cos(\phi - \phi + \omega) = \cos(\phi - (\phi - \omega))$.

$$\begin{aligned}\overline{F} &= \cos(\phi - (\phi - \omega)) - 2\sin\phi\sin(\phi - \omega)\\ &= \cos\phi\cos(\phi - \omega) + \sin\phi\sin(\phi - \omega) - 2\sin\phi\sin(\phi - \omega)\\ &= \cos\phi\cos(\phi - \omega) - \sin\phi\sin(\phi - \omega)\\ &= \cos(\phi + (\phi - \omega))\\ &= \cos(2\phi - \omega)\end{aligned}$$

Sada kada imamo sve podatke, prvu fundamentalnu formu plohe $\overline{S}$ zapisujemo

$$\overline{\mathrm{I}} = du^2 + 2\cos(2\phi - \omega)\,du\,dv + dv^2. \tag{2.12}$$

Normala $\overline{n}$ plohe $\overline{S}$ dana je s

$$\overline{n} = \frac{\overline{\mathbf{x}}_u \times \overline{\mathbf{x}}_v}{\left\|\overline{\mathbf{x}}_u \times \overline{\mathbf{x}}_v\right\|} = \frac{\overline{\mathbf{x}}_u \times \overline{\mathbf{x}}_v}{\overline{W}} = -\frac{L}{\rho}\sin\phi\, A + \frac{L}{\rho}\cos\phi B + \left(1 - \frac{L\beta}{\rho}\right)C, \tag{2.13}$$

a njene derivacije po u i v

$$\begin{aligned}\overline{n}_u &= -\frac{L\beta}{\rho^2}\sin\phi\cos\phi A + \left(\frac{L\beta}{\rho^2}\cos^2\phi - \frac{1}{\rho}\right)B + \frac{L}{\rho^2}\cos\phi C,\\ \overline{n}_v &= \left[\frac{L}{2\rho^2\beta}\sin(\omega - 2\phi) + \frac{1}{\rho}\left(1 - \frac{L}{2\rho\beta}\right)\sin\omega\right]A\\ &\quad + \left[\frac{L}{2\rho^2\beta}\cos(\omega - 2\phi) + \frac{1}{\rho}\left(1 - \frac{L}{2\rho\beta}\right)\cos\omega\right]B - \frac{L}{\rho^2}\cos(\omega - \phi)C.\end{aligned} \tag{2.14}$$

Pokažimo kako smo došli do gornjih izraza. Podijelit ćemo račun na dva dokaza, jedan za n i jedan za n_u i n_v.

Dokaz. Izračunajmo $\overline{\mathbf{x}}_u \times \overline{\mathbf{x}}_v$. Kako su A, B i C međusobno ortonormirani, to vrijedi

$$A \times B = C, \quad B \times C = A, \quad C \times A = B.$$

$$
\begin{aligned}
\overline{\mathbf{x}}_u \times \overline{\mathbf{x}}_v &= \left(1 - \frac{L\beta}{\rho}\sin^2\phi\right)\left(\sin\omega + \frac{L}{\rho\beta}\cos\phi\sin(\phi-\omega)\right) C \\
&+ \left(1 - \frac{L\beta}{\rho}\sin^2\phi\right)\cdot\frac{L}{\rho}\sin(\phi-\omega)\, B \\
&- \frac{L\beta}{\rho}\sin\phi\cos\phi\cdot\left(\cos\omega - \frac{L}{\rho\beta}\sin\phi\sin(\phi-\omega)\right) C - \frac{L\beta}{\rho}\sin\phi\cos\phi\cdot\frac{L}{\rho}\sin(\phi-\omega)\, A \\
&+ \frac{L}{\rho}\sin\phi\cdot\left(\cos\omega - \frac{L}{\rho\beta}\sin\phi\sin(\phi-\omega)\right) B \\
&- \frac{L}{\rho}\sin\phi\cdot\left(\sin\omega + \frac{L}{\rho\beta}\cos\phi\sin(\phi-\omega)\right) A \\
&= \left[-\frac{L^2\beta}{\rho^2}\sin\phi\cos\phi\sin(\phi-\omega) - \frac{L}{\rho}\sin\phi\sin\omega - \frac{L^2}{\rho^2\beta}\sin\phi\cos\phi\sin(\phi-\omega)\right] A \\
&+ \left[\frac{L}{\rho}\sin(\phi-\omega) - \frac{L^2\beta}{\rho^2}\sin^2\phi\sin(\phi-\omega) + \frac{L}{\rho}\sin\phi\cos\omega - \frac{L^2}{\rho^2\beta}\sin^2\phi\sin(\phi-\omega)\right] B \\
&+ \left[\sin\omega + \frac{L}{\rho\beta}\cos\phi\sin(\phi-\omega) - \frac{L\beta}{\rho}\sin^2\phi\sin\omega - \frac{L\beta}{\rho}\sin\phi\cos\phi\cos\omega\right] C
\end{aligned}
$$

Izračunajmo izraz u zagradi koja stoji uz A.

$$
\begin{aligned}
&-\frac{L^2}{\rho^2}\sin\phi\cos\phi\sin(\phi-\omega)\cdot\left(\beta + \frac{1}{\beta}\right) - \frac{L}{\rho}\sin\phi\sin\omega \\
&\quad = \left\{\text{znamo: } \beta + \frac{1}{\beta} = \frac{2\rho}{L}\right\} = -\frac{L}{\rho}\sin\phi\,[2\cos\phi\sin(\phi-\omega) + \sin\omega] \\
&\quad = \{\text{zamjena: } \sin\omega = \sin\phi\cos(\phi-\omega) - \cos\phi\sin(\phi-\omega)\} \\
&\quad = -\frac{L}{\rho}\sin\phi\,[\cos\phi\sin(\phi-\omega) + \sin\phi\cos(\phi-\omega)] \\
&\quad = -\frac{L}{\rho}\sin\phi\sin(2\phi-\omega)
\end{aligned}
$$

Analogno, izračunajmo izraz uz B.

$$
\begin{aligned}
&\frac{L}{\rho}\sin(\phi-\omega)-\frac{L^2}{\rho^2}\sin^2\phi\sin(\phi-\omega)\cdot\left(\beta+\frac{1}{\beta}\right)+\frac{L}{\rho}\sin\phi\cos\omega\\
&\quad=\left\{\text{znamo: }\beta+\frac{1}{\beta}=\frac{2\rho}{L}\right\}\{\text{zamjena: }\cos\omega=\cos\phi\cos\,(\phi-\omega)+\sin\phi\sin(\phi-\omega)\}\\
&\quad=\frac{L}{\rho}\left[\sin(\phi-\omega)-\sin^2\phi\sin(\phi-\omega)+\sin\phi\cos\phi\cos(\phi-\omega)\right]\\
&\quad=\ldots=\frac{L}{\rho}\cos\phi\,[\cos\phi\sin\phi-\omega+\sin\phi\cos(\phi-\omega)]\\
&\quad=\frac{L}{\rho}\cos\phi\sin(2\phi-\omega)
\end{aligned}
$$

Izračunajmo izraz uz C.

$$
\begin{aligned}
\sin\omega+\frac{L}{\rho\beta}&\cos\phi\sin(\phi-\omega)-\frac{L\beta}{\rho}\sin^2\phi\sin\omega-\frac{L\beta}{\rho}\sin\phi\cos\phi\cos\omega\\
&=\sin\omega+\frac{L}{\rho\beta}\cos\phi\sin(\phi-\omega)-\frac{L\beta}{\rho}\sin\phi\,(\sin\phi\sin\omega+\cos\phi\cos\omega)\\
&=\sin\omega+\frac{L}{\rho\beta}\cos\phi\sin(\phi-\omega)-\frac{L\beta}{\rho}\sin\phi\cos(\phi-\omega)\\
&=\{\text{zamjena: }\sin\omega=\sin\phi\cos\,(\phi-\omega)-\cos\phi\sin(\phi-\omega)\}\\
&=\sin\phi\cos(\phi-\omega)\cdot\left(1-\frac{L\beta}{\rho}\right)+\cos\phi\sin(\phi-\omega)\cdot\left(\frac{L}{\rho\beta}-1\right)
\end{aligned}
$$

Odredimo čemu su jednaki izrazi $1-L\beta/\rho$ i $L/\rho\beta-1$:

$$
1-\frac{L\beta}{\rho}=1-\left(1\pm\sqrt{1-\frac{L^2}{\rho^2}}\right)=\mp\sqrt{1-\frac{L^2}{\rho^2}}
$$

$$
\frac{L}{\rho\beta}-1=1\mp\sqrt{1-\frac{L^2}{\rho^2}}-1=\mp\sqrt{1-\frac{L^2}{\rho^2}}
$$

Vidimo da oba izraza daju istu vrijednost, pa možemo odabrati bilo koji od njih i nastaviti gdje smo bili stali.

$$
\Rightarrow\quad\left(1-\frac{L\beta}{\rho}\right)(\sin\phi\cos(\phi-\omega)+\cos\phi\sin(\phi-\omega))=\left(1-\frac{L\beta}{\rho}\right)\sin(2\phi-\omega)
$$

Dakle, dobili smo

$$
\overline{\mathbf{x}}_u\times\overline{\mathbf{x}}_v=\sin(2\phi-\omega)\left[-\frac{L}{\rho}\sin\phi\cdot A+\frac{L}{\rho}\cos\phi\cdot B+\left(1-\frac{L\beta}{\rho}\cdot C\right)\right].
$$

Kako bismo odredili $\overline{n}$ moramo još izračunati vrijednost $\overline{W}$.

$$\overline{W} = \sqrt{\overline{EG} - \overline{F}^2} = \sqrt{1 - \cos^2(2\phi - \omega)} = \sin(2\phi - \omega)$$

Sada je očigledno da (2.13) vrijedi. □

U redu, sada pokažimo da vrijedi (2.14).

Dokaz.

$$\begin{aligned}
\overline{n}_u &= -\frac{L}{\rho}\left(\phi_u \cos\phi \cdot A + \sin\phi \cdot A_u\right) + \frac{L}{\rho}\left(-\phi_u \sin\phi \cdot B + \cos\phi \cdot B_u\right) + \left(1 - \frac{L\beta}{\rho}\right) C_u \\
&= \{\text{koristimo } (2.3)\} = -\frac{L}{\rho}\phi_u \cos\phi \cdot A + \frac{L}{\rho}\omega_u \sin\phi \cdot B - \frac{L}{\rho}\phi_u \sin\phi \cdot B \\
&+ \frac{L}{\rho}\cos\phi\left(\omega_u \cdot A + \frac{1}{\rho}\cdot C\right) - \frac{1}{\rho}\left(1 - \frac{L\beta}{\rho}\right)\cdot B \\
&= -\frac{L}{\rho}\cos\phi \cdot A\left[\phi_u - \omega_u\right] + B\left[-\frac{L}{\rho}\sin\phi\left(\phi_u - \omega_u\right) - \frac{1}{\rho} + \frac{L\beta}{\rho^2}\right] \\
&+ \frac{L}{\rho^2}\cos\phi \cdot C \\
&= \{\text{koristimo } (2.10)\} = -\frac{L}{\rho}\cos\phi \cdot A \cdot \frac{\beta}{\rho}\sin\phi + B\left[-\frac{L\beta}{\rho^2}\sin^2\phi - \frac{1}{\rho} + \frac{L\beta}{\rho^2}\right] + \frac{L}{\rho^2}\cos\phi\, C \\
&= -\frac{L\beta}{\rho^2}\cos\phi\sin\phi A + B\left[\frac{L\beta}{\rho^2}\left(-\sin^2\phi + 1\right) - \frac{1}{\rho}\right] + \frac{L}{\rho^2}\cos\phi\, C \\
&= -\frac{L\beta}{\rho^2}\cos\phi\sin\phi A + \left[\frac{L\beta}{\rho^2}\cos^2\phi - \frac{1}{\rho}\right] B + \frac{L}{\rho^2}\cos\phi\, C
\end{aligned}$$

$$\begin{aligned}
\overline{n}_v &= -\frac{L}{\rho}\left(\phi_v \cos\phi \cdot A + \sin\phi \cdot A_v\right) + \frac{L}{\rho}\left(-\phi_v \sin\phi \cdot B + \cos\phi \cdot B_v\right) + \left(1 - \frac{L\beta}{\rho}\right) C_v \\
&= \{\text{koristimo } (2.3)\} = -\frac{L}{\rho}\phi_v \cos\phi A - \frac{L}{\rho}\sin\phi \cdot \frac{1}{\rho}\sin\omega\, C - \frac{L}{\rho}\phi_v \sin\phi B \\
&+ \frac{L}{\rho}\cos\phi \cdot \left(-\frac{1}{\rho}\cos\omega\, C\right) + \left(1 - \frac{L\beta}{\rho}\right)\left(-\frac{1}{\rho}\sin\omega A + \frac{1}{\rho}\cos\omega B\right) \\
&= \{\text{koristimo } (2.10)\} = A\left[-\frac{L}{\rho^2\beta}\sin(\phi - \omega)\cos\phi - \frac{1}{\rho}\sin\omega\left(1 - \frac{L\beta}{\rho}\right)\right] \\
&+ B\left[\frac{L}{\rho^2\beta}\sin(\phi - \omega)\sin\phi + \frac{1}{\rho}\cos\omega\left(1 - \frac{L\beta}{\rho}\right)\right] \\
&- C\left[\frac{L}{\rho^2}\sin\omega\sin\phi + \frac{L}{\rho^2}\cos\omega\cos\phi\right]
\end{aligned}$$

Sredimo malo $\overline{n}_v$. Izračunajmo izraz koji je pomnožen s A. Pogledajmo prvi sumand u zagradi.

$$\begin{aligned}
-\frac{L}{\rho^2\beta}\sin(\phi-\omega)\cos\phi &= \frac{L}{\rho^2\beta}\sin(\omega-\phi)\cos\phi \\
&= \frac{1}{2}\frac{L}{\rho^2\beta}\left[\sin((\omega-\phi)-\phi)+\sin((\omega-\phi)+\phi)\right] \\
&= \frac{L}{2\rho^2\beta}\left[\sin(\omega-2\phi)+\sin\omega\right]
\end{aligned}$$

Sada vratimo dobiveno u zagradu i imamo

$$\frac{L}{2\rho^2\beta}\sin(\omega-2\phi)+\frac{L}{2\rho^2\beta}\sin\omega-\frac{1}{\rho}\left(1-\frac{L\beta}{\rho}\right)\sin\omega,$$

odnosno

$$\frac{L}{2\rho^2\beta}\sin(\omega-2\phi)+\sin\omega\left(\frac{L}{2\rho^2\beta}-\frac{1}{\rho}\left(1-\frac{L\beta}{\rho}\right)\right).$$

Za izraz uz $\sin\omega$ vrijedi

$$\frac{L}{2\rho^2\beta}-\frac{1}{\rho}\left(1-\frac{L\beta}{\rho}\right)=\frac{L}{2\rho^2\beta}+\frac{L\beta}{\rho^2}-\frac{1}{\rho}=\frac{L}{\rho^2}\left(\frac{1}{2\beta}+\beta\right)-\frac{1}{\rho}.$$

Odredimo koliko je $\frac{1}{2\beta}+\beta$.

$$\begin{aligned}
\frac{1}{2\beta}+\beta &= \frac{1}{2\beta}\left(1+2\beta^2\right)=\frac{1}{2\beta}\left[1+2\frac{\rho^2}{L^2}\left(1\pm 2\sqrt{1-\frac{L^2}{\rho^2}}+1-\frac{L^2}{\rho^2}\right)\right] \\
&= \frac{1}{2\beta}\left[1+2\frac{\rho^2}{L^2}\pm 4\frac{\rho^2}{L^2}\sqrt{1-\frac{L^2}{\rho^2}}+2\frac{\rho^2}{L^2}-2\right] \\
&= \frac{1}{2\beta}\left[4\frac{\rho^2}{L^2}\pm 4\frac{\rho^2}{L^2}\sqrt{1-\frac{L^2}{\rho^2}}-1\right]=\frac{1}{2\beta}\cdot 4\frac{\rho^2}{L^2}\left[1\pm\sqrt{1-\frac{L^2}{\rho^2}}\right]-\frac{1}{2\beta} \\
&= \left\{\text{koristimo } 1\pm\sqrt{1-\frac{L^2}{\rho^2}}=\frac{L\beta}{\rho}\right\}=\frac{\rho^2}{L^2\beta}\cdot\frac{L\beta}{\rho}-\frac{1}{2\beta} \\
&= \frac{2\rho}{L}-\frac{1}{2\beta}
\end{aligned}$$

Vratimo rezultat gore, pa imamo:

$$\frac{L}{\rho^2}\left(\frac{2\rho}{L}-\frac{1}{2\beta}\right)-\frac{1}{\rho}=\frac{2}{\rho}-\frac{L}{2\rho^2\beta}-\frac{1}{\rho}=\frac{1}{\rho}-\frac{L}{2\rho^2\beta}=\frac{1}{\rho}\left(1-\frac{L}{2\rho\beta}\right).$$

Konačno, cijeli izraz uz A zapisujemo

$$A\left[\frac{L}{2\rho^2\beta}\sin(\omega-2\phi)+\frac{1}{\rho}\left(1-\frac{L}{2\rho\beta}\right)\sin\omega\right].$$

Izračunajmo sada izraz koji se nalazi uz B. Pogledajmo prvi sumand u zagradi.

$$\begin{aligned}-\frac{L}{\rho^2\beta}\sin(\phi-\omega)\,\sin\phi &= \frac{L}{\rho^2\beta}\sin(\omega-\phi)\,\sin\phi\\ &= \frac{1}{2}\frac{L}{\rho^2\beta}\left[\cos((\omega-\phi)-\phi)+\cos((\omega-\phi)+\phi)\right]\\ &= \frac{L}{2\rho^2\beta}\left[\cos(\omega-2\phi)+\cos\omega\right]\end{aligned}$$

Vratimo dobiveno u zagradu i imamo

$$\frac{L}{2\rho^2\beta}\sin\{(\omega-2\phi)\}+\frac{L}{2\rho^2\beta}\sin\omega-\frac{1}{\rho}\left(1-\frac{L\beta}{\rho}\right)\sin\omega,$$

odnosno

$$\frac{L}{2\rho^2\beta}\sin\{(\omega-2\phi)\}+\sin\omega\left(\frac{L}{2\rho^2\beta}-\frac{1}{\rho}\left(1-\frac{L\beta}{\rho}\right)\right).$$

Uz $\cos\omega$ imamo isti izraz kao u prethodnom slučaju uz $\sin\omega$, pa naposlijetku uz B imamo

$$B\left[\frac{L}{2\rho^2\beta}\cos(\omega-2\phi)+\frac{1}{\rho}\left(1-\frac{L}{2\rho\beta}\right)\cos\omega\right].$$

Izračunajmo napokon i izraz koji množi C.

$$\frac{L^2}{\rho^2}(\sin\omega\sin\phi+\cos\omega\cos\phi)=\frac{L^2}{\rho^2}\cos(\omega-\phi)$$

Konačno, sredili smo zapis za $\overline{n}_v$:

$$\begin{aligned}\overline{n}_v=&\left[\frac{L}{2\rho^2\beta}\sin(\omega-2\phi)+\frac{1}{\rho}\left(1-\frac{L}{2\rho\beta}\right)\sin\omega\right]A\\ &+\left[\frac{L}{2\rho^2\beta}\cos(\omega-2\phi)+\frac{1}{\rho}\left(1-\frac{L}{2\rho\beta}\right)\cos\omega\right]B\\ &-\frac{L^2}{\rho^2}\cos(\omega-\phi)\,C.\end{aligned}$$

□

Da bismo izračunali drugu fundamentalnu formu plohe $\overline{S}$, trebamo izračunati $\overline{L}$, $\overline{M}$ i $\overline{N}$:

$$\overline{L} = \overline{\mathbf{x}}_u \cdot \overline{n}_u = 0, \quad \overline{N} = \overline{\mathbf{x}}_v \cdot \overline{n}_v = 0,$$
$$\overline{M} = \overline{\mathbf{x}}_u \cdot \overline{n}_v = \overline{\mathbf{x}}_v \cdot \overline{n}_u = \frac{1}{\rho} \sin(2\phi - \omega)$$

Primjetimo, kao što je za početnu plohu S vrijedilo $L = N = 0$, tako i za plohu $\overline{S}$ pripadajući $\overline{L}$ i $\overline{N}$ moraju biti jednaki 0. Dakle, drugu fundamentalnu formu plohe $\overline{S}$ zapisujemo

$$\overline{\mathtt{II}} = \frac{2}{\rho} \sin(2\phi - \omega)\, du\, dv, \tag{2.15}$$

a zajedno s prvom fundamentalnom formom (2.12) pokazuje da je $\overline{S}$ zaista pseudosferna ploha parametrizirana duljinom luka duž asimptotske mreže, između kojih je kut određen s

$$\overline{\omega} = 2\phi - \omega, \tag{2.16}$$

gdje $\overline{\omega}$ ima istu ulogu na plohi $\overline{S}$ kao kut ω na S. Posebno, $\overline{\omega}$ mora zadovoljavati sinus-Gordonovu jednadžbu

$$\overline{\omega}_{uv} = \frac{1}{\rho^2} \sin \overline{\omega}. \tag{2.17}$$

Primjenjujući jednakost (2.16) na izraze iz (2.10) dobivamo

$$\boxed{\begin{aligned} \left(\frac{\overline{\omega} - \omega}{2}\right)_u &= \frac{\beta}{\rho}\left(\frac{\overline{\omega} + \omega}{2}\right), \\ \left(\frac{\overline{\omega} + \omega}{2}\right)_v &= \frac{1}{\beta\rho}\left(\frac{\overline{\omega} - \omega}{2}\right). \end{aligned}} \tag{2.18}$$

Ovo je klasični oblik Bäcklundove transformacije koja povezuje sinus-Gordonove jednadžbe (1.27) i (2.17). Standardna oznaka je $\mathbb{B}_\beta$.
Poznato je da uslijed $\mathbb{B}_\beta$ vrijedi

$$\overline{n} \cdot n = 1 - \frac{L\beta}{\rho} = const, \tag{2.19}$$

što znači da se tangencijalne ravnine u odgovarajućim točkama ploha S i $\overline{S}$ sijeku pod konstantnim kutem ζ, gdje je $\beta = \operatorname{tg} \zeta/2$.
Dokažimo to.

Dokaz. Tvrdimo da vrijedi $\beta = \operatorname{tg} \zeta/2$. Mjera kuta između tangencijalnih ravnina u odgovarajućim točkama ploha S i $\overline{S}$ jednaka je mjeri kuta između normala u odgovarajućim točkama, pa vrijedi

$$\overline{n} \cdot n = \cos\zeta = 1 - \frac{L\beta}{\rho}. \tag{2.20}$$

Znamo da je

$$\beta = \frac{\rho}{L}\left(1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}\right).$$

Za $\cos\zeta$ vrijedi

$$\begin{aligned}
\cos\zeta &= \cos^2\frac{\zeta}{2} - \sin^2\frac{\zeta}{2} \\
&= \cos^2\frac{\zeta}{2}\left(1 - \operatorname{tg}^2\frac{\zeta}{2}\right) \\
&= \frac{1}{1 + \operatorname{tg}^2\frac{\zeta}{2}} \cdot \left(1 - \operatorname{tg}^2\frac{\zeta}{2}\right)
\end{aligned}$$

Kako imamo da je $\cos\zeta = 1 - \frac{L\beta}{\rho}$, to uz početnu tvrdnju vrijedi

$$\begin{aligned}
& 1 - \frac{L\beta}{\rho} = \frac{1-\beta^2}{1+\beta^2} \\
\Leftrightarrow \quad & \left(1 - \frac{L\beta}{\rho}\right)\left(1+\beta^2\right) = 1 - \beta^2 \\
\Leftrightarrow \quad & 1 + \beta^2 - \frac{L}{\rho}\beta - \frac{L}{\rho}\beta^3 = 1 - \beta^2 \quad / : \beta \neq 0 \\
\Leftrightarrow \quad & -\frac{L}{\rho}\beta^2 + 2\beta - \frac{L}{\rho} = 0 \quad / \cdot \left(-\frac{\rho}{L}\right) \\
\Leftrightarrow \quad & \beta^2 - 2\frac{\rho}{L}\beta + 1 = 0
\end{aligned}$$

Pronađimo rješenja gornje kvadratne jednadžbe.

$$\begin{aligned}\beta_{1,2} &= \frac{1}{2}\left(2\frac{\rho}{L} \pm \sqrt{4\frac{\rho^2}{L^2} - 4}\right) \\ &= \frac{1}{2}\left(2\frac{\rho}{L} \pm 2\sqrt{\frac{\rho^2}{L} - 1}\right) \\ &= \frac{1}{2}\cdot 2\left(\frac{\rho}{L} \pm \sqrt{\frac{\rho^2}{L^2}}\right) \\ &= \frac{\rho}{L}\left(1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}\right),\end{aligned}$$

čime smo dokazali tvrdnju. □

Bianchi je u svojoj originalnoj konstrukciji uzeo $L = \rho$, $\beta = 1$, tako da su maloprije spomenute tangencijalne ravnine ortogonalne, tj. $\overline{n} \cdot n = 0$. Dakle, možemo reći da je Bäcklundov rezultat proširenje Bianchijevog. Bäcklundovo ispuštanje zahtjeva ortogonalnosti dozvoljava parametru β da bude ubačen u Bianchijevu transformaciju. Ustvari, Bäcklundovu tranformaciju $\mathbb{B}_\beta$ možemo promatrati kao kompoziciju Bianchijeve transformacije i invarijantnosti jednostavne Lie-ve grupe. Prema tome, sinus-Gordonova jednadžba (1.27) je invarijantna s obzirom na

$$u^* = \beta u, \quad v^* = \frac{v}{\beta}, \qquad \beta \neq 0 \tag{2.21}$$

tako da svako rješenje $\omega = \omega(u, v)$ generira jednoparametarsku klasu rješenja $\omega^*(u^*, v^*) = \omega(\beta u, v/\beta)$. Lie je zapazio da spajanje invarijantnosti (2.21) s originalnom Bianchijevom transformacijom

$$\begin{aligned}\left(\frac{\overline{\omega} - \omega}{2}\right)_{u^*} &= \frac{1}{\rho}\sin\left(\frac{\overline{\omega} + \omega}{2}\right), \\ \left(\frac{\overline{\omega} + \omega}{2}\right)_{v^*} &= \frac{1}{\rho}\sin\left(\frac{\overline{\omega} - \omega}{2}\right),\end{aligned}$$

daje Bäcklundovu transformaciju (2.18).

U smislu konstrukcije pseudosfernih ploha, Bäcklundova transformacija odgovara sljedećem rezultatu: neka je $\mathbf{x}$ karta pseudosferne plohe S koja odgovara rješenju ω sinus-Gordonove jednadžbe (1.27); neka je $\overline{\omega}$ dobiven Bäcklundovom transformacijom kuta ω. Tada kartu $\overline{\mathbf{x}}$

pseudosferne plohe $\overline{S}$ dobijemo ovako: iz (2.4) imamo

$$\begin{aligned}
\overline{\mathbf{x}} &= \mathbf{x} + L\cos\phi\, A + L\sin\phi\, B = \{\text{primjenjujemo (2.1)}\} = \\
&= \mathbf{x} + L\cos(\omega+\overline{\omega})\,\mathbf{x}_u + L\sin(\omega+\overline{\omega})\left(\frac{1}{\sin\omega}\,\mathbf{x}_v - \frac{\cos\omega}{\sin\omega}\mathbf{x}_u\right) \\
&= \mathbf{x} + \frac{L}{\sin\omega}\left[\cos(\omega+\overline{\omega})\cdot\sin\omega - \sin(\omega+\overline{\omega})\cdot\cos\omega\right]\mathbf{x}_u + \frac{L}{\sin\omega}\cdot\sin(\omega+\overline{\omega}) \\
&= \mathbf{x} + \frac{L}{\sin\omega}\left[\sin\left(\omega - \frac{\omega+\overline{\omega}}{2}\right)\right]\mathbf{x}_u + \frac{L}{\sin\omega}\cdot\sin(\omega+\overline{\omega})\,\mathbf{x}_v \\
&= \mathbf{x} + \frac{L}{\sin\omega}\left[\sin\left(\frac{\omega-\overline{\omega}}{2}\right)\right]\mathbf{x}_u + \frac{L}{\sin\omega}\cdot\sin(\omega+\overline{\omega})\,\mathbf{x}_v \\
&= \mathbf{x} + \frac{L}{\sin\omega}\left[\sin\left(\frac{\omega-\overline{\omega}}{2}\right)\mathbf{x}_u + \sin(\omega+\overline{\omega})\,\mathbf{x}_v\right],
\end{aligned}$$

gdje je $L = \rho\sin\zeta$.

Poglavlje 3

Solitoni i višesolitonska rješenja

3.1 O solitonima

Izraz *soliton* izveden je od engleskog *solitary wave* (u prijevodu: *usamljeni val*). U širem smislu, pod njim se podrazumijevaju valovi koji su ograničeni u prostoru (lokalizirani) i kreću se ne mijenjajući svoj oblik. U užem smislu, solitonima se označavaju rješenja određenih nelinearnih diferencijalnih jednadžbi (solitonskih jednadžbi) uz odgovarajuće rubne uvjete (koji osiguravaju lokaliziranost). Pri sudarima, dva solitona prolaze jedan kroz drugog uz nelinearnu interakciju—asimptotski zadržavaju oblik i brzinu, no pri raspršenju dolazi do pomaka u fazi.

Solitonski val prvi je uočio škotski inženjer John Scott Russel 1834. godine dok je promatrao par konja kako vuče brod duž uskog kanala. Brod se naglo zaustavio, a voda koja je gurana brodom se nastavila kretati velikom brzinom uzimajući oblik velikog usamljenog uzdignuća zaobljene, glatke i dobro definirane gomile vode, koja je nastavila put duž kanala bez vidljive promjene oblika i smanjenja brzine. Russelovu oduševljenost, međutim, nisu podijelili fizičari njegovog vremena, tako da je fenomen dosta dugo ostao neistražen, a Scott Russel zapamćen po drugim dostignućima.

Rješenja solitonskih jednadžbi mogu biti jednosolitonska, dvosolitonska, trosolitonska—višesolitonska, ovisno o tome koliko solitona u međusobnoj interakciji promatramo.

3.2 Generiranje višesolitonskih rješenja

Pogledajmo primjenu auto-Bäcklundove transformacije pri određivanju višesolitonskih rješenja sinus-Gordonove jednadžbe.
Krenimo s početnim rješenjem $\omega = 0$ sinus-Gordonove jednadžbe (1.27) koje uvrstimo u

(2.18) i dobivamo:

$$\begin{aligned} \overline{\omega}_u &= \frac{2\beta}{\rho} \sin\left(\frac{\overline{\omega}}{2}\right), \\ \overline{\omega}_v &= \frac{2}{\beta\rho} \sin\left(\frac{\overline{\omega}}{2}\right). \end{aligned} \tag{3.1}$$

Drugo, netrivijalno, rješenje $\overline{\omega}$ sinus-Gordonove jednadžbe (2.17) možemo dobiti integracijom gornjeg para jednadžbi. Počnimo s

$$\overline{\omega}_u = \frac{2\beta}{\rho} \sin \frac{\overline{\omega}}{2}.$$

Podijelimo li jednadžbu sa $\sin(\overline{\omega}/2)$, imamo

$$\frac{\overline{\omega}_u}{\sin\left(\frac{\overline{\omega}}{2}\right)} = \frac{2\beta}{\rho}.$$

Integracijom po u dobivamo

$$\int \frac{\overline{\omega}_u}{\sin\left(\frac{\overline{\omega}}{2}\right)}\, du = \int \frac{2\beta}{\rho}\, du.$$

Uvedimo supstituciju

$$\begin{aligned} \frac{\overline{\omega}}{2} &= t \\ \Rightarrow \quad \overline{\omega}_u\, du &= 2\, dt. \end{aligned}$$

Sada imamo:

$$\begin{aligned} 2\int \frac{1}{\sin t}\, dt &= 2\frac{\beta u}{\rho} + c_1 \\ \Leftrightarrow \quad \int \frac{1}{\sin t}\, dt &= \frac{\beta u}{\rho} + c_1 \\ \Leftrightarrow \quad -\ln\left(\operatorname{ctg} t + \frac{1}{\sin t}\right) &= \frac{\beta u}{\rho} + c_1 \\ \Leftrightarrow \quad e^{-\ln\left(\operatorname{ctg} t + \frac{1}{\sin t}\right)} &= e^{\frac{\beta u}{\rho} + c_1} \\ \Leftrightarrow \quad \frac{1}{e^{\ln\left(\operatorname{ctg} t + \frac{1}{\sin t}\right)}} &= e^{\frac{\beta u}{\rho} + c_1} \\ \frac{1}{\left(\operatorname{ctg} t + \frac{1}{\sin t}\right)} &= e^{\frac{\beta u}{\rho} + c_1}, \end{aligned}$$

gdje je c_1 proizvoljna konstanta integracije.
Lijeva strana jednakosti iznosi $\frac{\sin t}{\cos t+1} = \operatorname{tg}(t/2)$, pa slijedi

$$\operatorname{tg}\left(\frac{t}{2}\right) = e^{\frac{\beta u}{\rho}+c_1},$$

odnosno, kada vratimo $t = \overline{\omega}/2$, imamo

$$\operatorname{tg}\left(\frac{\overline{\omega}}{4}\right) = e^{\frac{\beta u}{\rho}+c_1}.$$

Primjenjujući funkciju arctg i zatim množeći jednadžbu s 4 dobivamo

$$\overline{\omega} = 4\ \operatorname{arctg}\left(e^{\frac{\beta u}{\rho}+c_1}\right). \tag{3.2}$$

Analognim postupkom, za

$$\overline{\omega}_v = \frac{2}{\beta\rho}\sin\left(\frac{\overline{\omega}}{2}\right)$$

iz (3.1) dobivamo rješenje

$$\overline{\omega} = 4\ \operatorname{arctg}\left(e^{\frac{v}{\beta\rho}+c_2}\right), \tag{3.3}$$

gdje je c_2 proizvoljna konstanta integracije. Iz rješenjâ (3.2) i (3.3) vidimo da imamo jedno rješenje

$$\overline{\omega} = 4\ \operatorname{arctg}\left(e^{\frac{\beta u}{\rho}+\frac{v}{\beta\rho}+c}\right), \tag{3.4}$$

gdje je c proizvoljna konstanta integracije.

Zanimljivo je napomenuti da su veličine

$$\begin{aligned}\overline{\omega}_u &= \frac{2\beta}{\rho}\operatorname{sech}\left(\frac{\beta u}{\rho}+\frac{v}{\beta\rho}+c\right),\\ \overline{\omega}_v &= \frac{2}{\beta\rho}\operatorname{sech}\left(\frac{\beta u}{\rho}+\frac{v}{\beta\rho}+c\right),\end{aligned} \tag{3.5}$$

one koje imaju karakteristični oblik grbe povezan sa solitonom.
Do prethodnih jednadžbi (3.5) smo došli deriviranjem (3.4) po u i v. Ovdje ćemo pokazati postupak samo za računanje $\overline{\omega}_u$, dok je za $\overline{\omega}_v$ postupak analogan.

$$\begin{aligned}\overline{\omega}_u &= \left[4\ \operatorname{arctg}\left(e^{\frac{\beta u}{\rho}+\frac{v}{\beta\rho}+c}\right)\right]_u = \frac{4\beta}{\rho}\,\frac{e^{\frac{\beta u}{\rho}+\frac{v}{\beta\rho}+c}}{1+e^{2\left(\frac{\beta u}{\rho}+\frac{v}{\beta\rho}+c\right)}} = \left\{\text{supst. } t = \frac{\beta u}{\rho}+\frac{v}{\beta\rho}+c\right\}\\ &= \frac{4\beta}{\rho}\,\frac{e^t}{1+e^{2t}} = \frac{4\beta}{\rho}\,\frac{e^t}{e^t\left(e^{-t}+e^t\right)} = \frac{4\beta}{\rho}\,\frac{1}{e^{-t}+e^t} = \left\{\operatorname{ch} t = \frac{e^t+e^{-t}}{2}\right\}\\ &= \frac{4\beta}{\rho\cdot 2\operatorname{ch} t} = \frac{2\beta}{\rho\operatorname{ch} t} = \frac{2\beta}{\rho}\operatorname{sech} t\\ &= \frac{2\beta}{\rho}\operatorname{sech}\left(\frac{\beta u}{\rho}+\frac{v}{\beta\rho}+c\right)\end{aligned}$$

Značajno je to da se analitički izrazi za višesolitonska rješenja, koji obuhvaćaju njihovu nelinearnu interakciju, mogu dobiti potpuno algebarskim postupkom. Ovo je posljedica elegantnog *nelinearnog načela superpozicije* proizašlog iz auto-Bäcklundove transformacije $\mathbb{B}_\beta$ koji je postavio Bianchi 1892. godine i poznat je po nazivu *Bianchijev teorem permutabilnosti*: ukratko,

$$\boxed{\Omega = \omega + 4 \text{ arctg}\left[\frac{\beta_2+\beta_1}{\beta_2-\beta_1}\,\text{tg}\left(\frac{\omega_2-\omega_1}{4}\right)\right],} \tag{3.6}$$

gdje su ω_1 i ω_2 Bäcklundove transformacije od ω preko $\mathbb{B}_\beta$, i to tako da je $\omega_1 = \mathbb{B}_{\beta_1}(\omega)$, $\omega_2 = \mathbb{B}_{\beta_2}(\omega)$. Taj teorem će nam biti od važnosti kod spominjanja dvosolitonskih rješenja, u našem slučaju *breather*-a.

3.3 Pseudosferne solitonske plohe i *breather*-i

U sljedećem primjeru Bäcklundovu transformaciju ćemo koristiti u njenoj linernoj verziji

$$\overline{\mathbf{x}} = \mathbf{x} + \frac{L}{\sin\omega}\left[\sin\left(\frac{\omega-\overline{\omega}}{2}\right)\mathbf{x}_u + \sin\left(\omega+\overline{\omega}\right)\mathbf{x}_v\right],$$

gdje je $L = \rho\sin\zeta$, za konstrukciju pseudosferne plohe koja odgovara solitonskim rješenjima sinus-Gordonove jednadžbe. Stoga je primjerenije parametrizirati pseudosferne plohe koordinatama zakrivljenosti

$$x = u + v, \quad y = u - v.$$

Ako stavimo $\omega = 2\theta$, prva i druga fundamentalna forma plohe postaju

$$\begin{aligned} \mathtt{I} &= \cos^2\theta\,dx^2 + \sin^2\theta dy^2\,, \\ \mathtt{II} &= \frac{1}{\rho}\sin\theta\cos\theta\left(dx^2 - dy^2\right). \end{aligned} \tag{3.7}$$

Ortonormiranu trijadu možemo zapisati

$$A^* = \frac{\mathbf{x}_x}{\cos\theta}, \quad B^* = \frac{\mathbf{x}_y}{\sin\theta}, \quad C^* = n, \tag{3.8}$$

dok Gauss-Weingartenove jednadžbe (1.28) i (1.29) daju

$$\begin{aligned} \begin{pmatrix} A^* \\ B^* \\ C^* \end{pmatrix}_x &= \begin{pmatrix} 0 & \theta_y & \frac{1}{\rho}\sin\theta \\ -\theta_y & 0 & 0 \\ -\frac{1}{\rho}\sin\theta & 0 & 0 \end{pmatrix}\begin{pmatrix} A^* \\ B^* \\ C^* \end{pmatrix}, \\ \begin{pmatrix} A^* \\ B^* \\ C^* \end{pmatrix}_y &= \begin{pmatrix} 0 & \theta_x & 0 \\ -\theta_x & 0 & -\frac{1}{\rho}\cos\theta \\ 0 & \frac{1}{\rho}\cos\theta & 0 \end{pmatrix}\begin{pmatrix} A^* \\ B^* \\ C^* \end{pmatrix}. \end{aligned} \tag{3.9}$$

Ovaj linearni sustav $\{A^*,B^*,C^*\}$ je kompatibilan ako i samo ako vrijedi

$$\theta_{xx} - \theta_{yy} = \frac{1}{\rho^2} \sin\theta \cos\theta. \tag{3.10}$$

Ova verzija sinus-Gordonove jednadžbe je najprisutnija u fizikalnim primjenama. Tako, x označava prostornu varijablu, a y vrijeme, pa se ova jednadžba obično naziva 1 + 1-dimenzionalna sinus-Gordonova jednadžba (jer sadrži jednu prostornu i jednu vremensku nezavisnu varijablu).

Pseudosfera

U još jednom primjeru vidimo da stacionarno jednosolitonsko rješenje jednadžbe (3.10),

$$\theta = 2 \operatorname{arctg}\left(e^{\frac{x}{\rho}+c}\right), \tag{3.11}$$

dobiveno namještanjem $u = v = x/2$, $\beta = 1$ u (3.4), odgovara pseudosfernoj rotacijskoj plohi, poznatoj kao *Beltramijeva pseudosfera*.

Kako bi se uspostavila veza između (3.11) i pseudosfere, podsjećamo da je karta $\mathbf{x}$ rotacijske plohe, koja je nastala rotacijom ravninske krivulje $z = \phi(r)$ oko z-osi, dana s

$$\mathbf{x} = \begin{pmatrix} r \cos\eta \\ r \sin\eta \\ \phi(r) \end{pmatrix}. \tag{3.12}$$

Kružnice r = konst su paralele, a krivulje η = konst su meridijani.

Kako bismo odredili rješenje sinus-Gordonove jednadžbe (3.10) koje odgovara pseudosferi, moramo je parametrizirati prema (3.7). U pogledu parametara ψ i η, karta pseudosfere je dana s

$$\mathbf{x} = \begin{pmatrix} \rho \sin\psi \cos\eta \\ \rho \sin\psi \sin\eta \\ \rho\left(\cos\psi + \ln\left|\operatorname{tg}\left(\frac{\psi}{2}\right)\right|\right) \end{pmatrix}, \tag{3.13}$$

dok prva i druga fundamentalna forma glase

$$\begin{aligned} \mathtt{I} &= \rho^2 \operatorname{ctg}^2\psi \, d\psi^2 + \rho^2 \sin^2\psi \, d\eta^2, \\ \mathtt{II} &= \rho \operatorname{ctg}\psi \, d\psi^2 - \rho \sin\psi \cos\psi \, d\eta^2. \end{aligned} \tag{3.14}$$

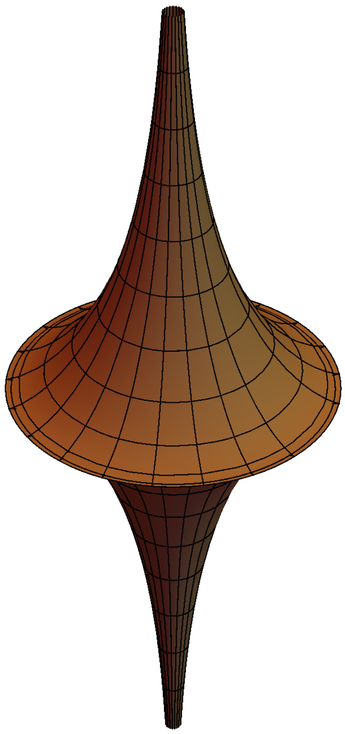
Ako sada uvedemo koordinate x i y u skladu s

$$dx = \rho \frac{d\psi}{\sin\psi}, \quad y = \rho\eta, \tag{3.15}$$

onda $\mathtt{I}$ i $\mathtt{II}$ poprimaju oblik (3.7), s time da je $\theta = \psi$. Integriranjem (3.15) dobivamo jednosolitonsko rješenje (3.11). U pogledu parametara x i y karta pseudosfere je dana s

$$\mathbf{x}(x, y) = \begin{pmatrix} \rho \operatorname{sech}\left(\frac{x}{\rho} + c\right)\cos\left(\frac{y}{\rho}\right) \\ \rho \operatorname{sech}\left(\frac{x}{\rho} + c\right)\sin\left(\frac{y}{\rho}\right) \\ \rho\left[\frac{x}{\rho} + c - \operatorname{th}\left(\frac{x}{\rho} + c\right)\right] \end{pmatrix}. \tag{3.16}$$

U ovoj parametrizaciji $x =$ konst i $y =$ konst su redom paralele i meridijani.



Slika 3.1: Pseudosfera

Slika 3.1 predstavlja pseudosferu s parametrima $\rho = 1$ i $c = 1$.
Navedimo još neke plohe.

Dinijeva ploha

Dinijeva ploha je helikoid generiran traktrisom i u paramterima x i y njegova karta je dana s

$$\mathbf{x}(x,y) = \begin{pmatrix} \rho\ \sin\zeta\,\mathrm{sech}\,\chi\cos\left(\frac{y}{\rho}\right) \\ \rho\ \sin\zeta\,\mathrm{sech}\,\chi\sin\left(\frac{y}{\rho}\right) \\ x-\rho\ \sin\zeta\,\mathrm{th}\,\chi \end{pmatrix}, \tag{3.17}$$

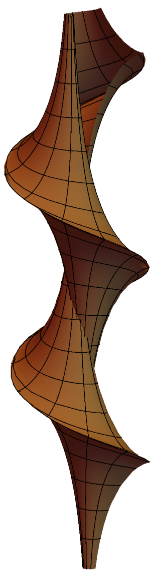
gdje je

$$\chi = \frac{x - y\ \cos\zeta}{\rho\ \sin\zeta},$$

a ζ je konstanta. Pseudosferu dobijemo kada je $\zeta = \pi/2$. Pripadajuće rješenje $1+1$-dimenzionalne sinus-Gordonove jednadžbe (3.10) je jednosolitonsko rješenje koje se kreće, dano s (3.4) zapisano u pogledu koordinata x i y i s $\beta = \mathrm{tg}\,(\zeta/2)$, odnosno

$$\theta = \frac{\omega}{2} = 2\,\mathrm{arctg}\,e^{\frac{1}{2\rho}\left(\beta+\frac{1}{\beta}\right)x+\frac{1}{2\rho}\left(\beta-\frac{1}{\beta}\right)y} = 2\ \mathrm{arctg}\,e^{\chi}. \tag{3.18}$$

Slika 3.2 predstavlja Dinijevu plohu s parametrima $\rho = 1$ i $\zeta = 1$.



Slika 3.2: Dinijeva ploha

Dvosolitonske plohe i *breather*-i

Dvosolitonsko rješenje dano je formulom

$$\Theta^{\pm} = \pm 2 \text{ arctg}\left[\frac{\sin\left(\frac{\zeta_2+\zeta_1}{2}\right)\text{sh}\left(\frac{\chi_1-\chi_2}{2}\right)}{\sin\left(\frac{\sin\zeta_2-\zeta_1}{2}\right)\text{ch}\left(\frac{\chi_1+\chi_2}{2}\right)}\right], \tag{3.19}$$

gdje je

$$\chi_i = \frac{1}{\rho\,\sin\zeta_i}\,(x - y\,\cos\zeta_i)\,, i = 1, 2, \zeta_1 \neq \zeta_2.$$

Breather-i su podklasa periodičnih dvosolitonskih rješenja.

U pogledu Bäcklundovih parametara $\beta_i = \text{tg}\,(\zeta_i/2)$ dvosolitonsko rješenje Θ^+ iz (3.19) postaje

$$\Theta^{+} = 2 \text{ arctg}\left[\frac{\beta_2+\beta_1}{\beta_2-\beta_1}\frac{\text{sh}\left(\frac{\chi_1-\chi_2}{2}\right)}{\text{ch}\left(\frac{\chi_1+\chi_2}{2}\right)}\right], \tag{3.20}$$

gdje je

$$\chi_i = \frac{1}{2\rho\beta_i}\left[\left(1+\beta_i^2\right)x - \left(1-\beta_i^2\right)y\right].$$

Kako bi dobili periodično rješenje, uvodimo kompleksno-konjugirane Bäckludnove parametre $\beta_1 = c + di$ i $\beta_2 = c - di$, pa (3.19) daje

$$\Theta^{+} = 2 \text{ arctg}\left[\frac{c\,\sin\left(\frac{d}{2\rho\left(c^2+d^2\right)}\xi\right)}{d\,\text{ch}\left(\frac{c}{2\rho\left(c^2+d^2\right)}\eta\right)}\right], \tag{3.21}$$

gdje je

$$\begin{aligned}\xi &= \left[1-\left(c^2+d^2\right)\right]x - \left[1+\left(c^2+d^2\right)\right]y,\\ \eta &= \left[1+\left(c^2+d^2\right)\right]x - \left[1-\left(c^2-d^2\right)\right]y.\end{aligned}$$

Dakle, Θ^+ je realno i periodično rješenje u varijabli ξ.

Ako zahtijevamo da $|\beta_1| = 1$ tako da $c^2 + d^2 = 1$, tada dobivamo rješenje

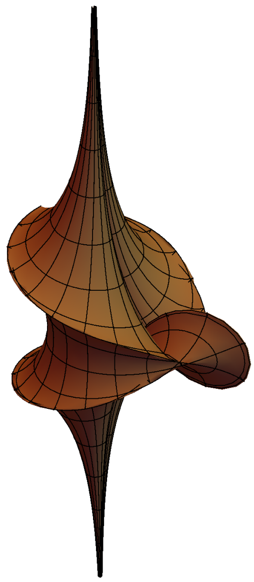
$$\Theta^{+} = -2 \text{ arctg}\left[\frac{c\,\sin\left(\frac{d}{\rho}\,y\right)}{d\,\text{ch}\left(\frac{c}{\rho}\,x\right)}\right], \tag{3.22}$$

koje je periodično u y. Nazivamo ga *stacionarnim breather*-om budući da se ne translatira kako se y razvija. Pripadajuća karta dana je s

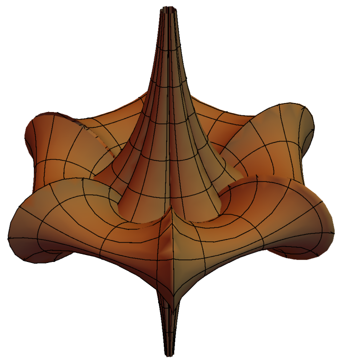
$$\begin{aligned}\mathbf{x}_{breather} = \begin{pmatrix}0\\0\\x\end{pmatrix} &+ \frac{2d}{c}\frac{\sin(dy)\operatorname{ch}(cx)}{d^2\operatorname{ch}^2(cx)+c^2\sin^2(dy)}\begin{pmatrix}\sin y\\-\cos y\\0\end{pmatrix}\\ &+\frac{2d^2}{c}\frac{\operatorname{ch}(cx)}{d^2\operatorname{ch}^2(cx)+c^2\sin^2(dy)}\begin{pmatrix}\cos y\,\cos(dy)\\ \sin y\,\cos(dy)\\ -\operatorname{sh}(cx)\end{pmatrix}\end{aligned} \tag{3.23}$$

gdje je $c = \sqrt{1-d^2}$.

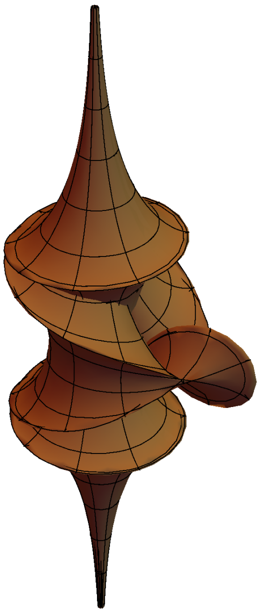
Svakom racionalnom broju $0 < d < 1$ odgovara stacionarni breather koji je periodičan u y. Ako zapišemo $d = p/q$, gdje s p i q relativno prosti i vrijedi $p < q$, tada je period breather-a jednak $2\pi\, q/p$. Na slikama ispod prikazane su pseudosferne plohe stacionarnih breather-a s parametrima $d = \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{1}{2}$.



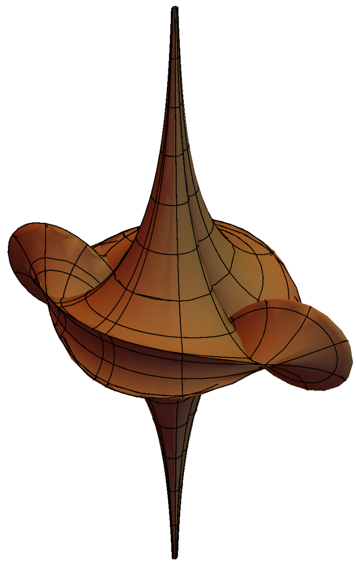
Slika 3.3: Breather 1, $d = \frac{1}{4}$



Slika 3.4: Breather 2, $d = \frac{3}{4}$



Slika 3.5: Breather 3, $d = \frac{1}{5}$



Slika 3.6: Breather 4, $d = \frac{1}{2}$

Bibliografija

[1] C. Rogers, W. K. Schief: *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory*, Cambridge University Press, UK, 2002.

[2] Željka Milin-Šipuš, Stipe Vidak: *Uvod u diferencijalnu geometriju*, skripta, verzija 2.3 web: https://web.math.pmf.unizg.hr/nastava/udg/skripta-2016.pdf

[3] Bojan Đuričković: *Solitoni, osnovni pojmovi*, web: http://bojand.org/seminarski/solitoni/1-solitoni.html

Sažetak

U ovom radu promatrali smo hiperboličke plohe koje pod posebnim uvjetima možemo nazvati pseudosfernim plohama. Veliku ulogu tu je imala Gaussova zakrivljenost za koju smo odredili da bude negativna i konstantna, odakle potječe naslov ovoga rada. Zatim smo izveli sinus-Gordonovu jednadžbu plohe i njenu Bäcklundovu transformaciju koja daje vezu između nove i početne pseudosferne plohe. Na kraju smo vidjeli neke primjere primjena Bäcklundove transformacije pri generiranju multisolitonskih rješenja.

Summary

In this thesis we observe hyperbolic surfaces which, under special conditions, are also called pseudospherical surfaces. The crucial part in this case has the Gaussian curvature which we set to be negative and constant. This also gives the title of this thesis. We show that the Bäcklund transformation for the sine-Gordon equation gives connection between new and initial pseudospherical surfaces. Finally, we present some examples of applications of the Bäcklund transformation in construction of multi-soliton solutions.

Životopis

Zovem se Slavko Davidović. Svoj životni put započeo sam 3. kolovoza 1986. godine u malom selu pokraj Požege, Vetovu. Od malih nogu sam volio pamtiti telefonske brojeve i registracijske oznake auta te sam uvijek smišljao način kako da što brže izračunam je li neki od brojeva djeljiv s 3. Nakon osnovne škole, koju sam završio u Kutjevu, upisujem Prirodoslovno-matematičku gimnaziju u Požegi u kojoj sam maturirao 2005. godine. Iste godine upisujem preddiplomski sveučilišni studij matematike na Prirodoslovno-matematičkom fakultetu u Zagrebu, da bih 2008. godine upisao nastavnički smjer matematike na istom fakultetu, koji završavam 2012. godine. Isti smjer odabirem i za diplomski studij koji završavam 2017. godine.

Uz to što želim biti dobar profesor, sviram nekoliko instrumenata, od kojih posebno volim harmoniku.