Generating function for angular momentum multiplicities

Sunko, Denis K.; Svrtan, D.

Generating function for angular momentum multiplicities

D. K. Sunko

Department of Theoretical Physics, University of Zagreb, 41000 Zagreb, Yugoslavia

D. Svrtan

Department of Mathematics, University of Zagreb, 41000 Zagreb, Yugoslavia
(Received 6 June 1984)

It is shown that the multiplicities of the total angular momentum projections for n-particle systems are given by the coefficients of certain Gaussian polynomials, for both bosons and fermions. These polynomials are parametrized in the same way as would be the corresponding binomial coefficients which solve the Bose-Einstein or Fermi-Dirac counting problems for the total number of allowed states. A comment on spin and statistics is included.

I. INTRODUCTION

When considering a system of n identical particles, one is faced with the problem of deducing the allowed total angular momenta I to which they may couple. Some values of I are forbidden by (anti)symmetrization; some occur more than once. A method for solving the problem has been known for a long time.[1] One can write all possible states in the m scheme which are consistent with the statistics. Then the state with maximal total projection $M = M_{max}$ is simultaneously the state with $I = M_{max}$, $M = M_{max}$. The states of a given $M \geq 0$ are counted, and if there are more of them than those with $M' = M + 1$, the difference gives the number of states with $I = M$. Otherwise there is no state of that I.

A more practical solution was given by Racah[1] in terms of the seniority scheme. Unfortunately, it does not resolve the multiplicities completely, and additional quantum numbers have to be introduced *ad hoc*, usually by computer choice.[2]

In this paper an efficient method is described by which one may deduce the J multiplicities of any system of identical particles. Since it is based on the m scheme, it gives primarily the number of states with a given total projection M. They turn out to be equal to the coefficients of certain Gaussian polynomials. A recursion relation is also available to calculate these coefficients.

Not all Gaussian polynomials correspond to solutions of multiplicity problems. Some are excluded because they solve problems with the wrong spin-statistics connection. An interesting analogy exists between the classification scheme for Gaussian polynomials with correct spin statistics and the classification of classical simple Lie groups according to the metric they preserve.

The principal result of this paper is obtained by demonstrating a connection between the multiplicity problem and some problems in combinatorics whose solutions may be traced in the literature. For example, Eqs. (11) and (12) are also found on p. 19 of Macdonald;[4] the derivation of (13) from (5) and (6) is given, e.g., in pp. 33—35 of Andrews,[5] in a slightly different notation. Indeed, much of Secs. III and IV could have been replaced by references to

the literature, once (5) and (6) [or (17) and (18)] have been established. We have chosen instead to carry out the derivations in full, believing that a cryptic style would have made our results appear unnecessarily obscure.

II. NOTATION AND DEFINITIONS

We first give a compact notation for m scheme states. It is best explained by an example. Take $n = 3$ fermions of $j = \frac{5}{2}$. Then the possible states of total projection $M = \frac{3}{2}$ are

$$| 100011 \rangle , \quad | 010101 \rangle , \quad | 001110 \rangle , \qquad (1)$$

where each column is weighted with a single particle projection, from $m = -\frac{5}{2}$ to $\frac{5}{2}$. The total number of columns is $2j + 1 = 6$. The total projection is lowered (raised) by moving a single unit to the left (right). This is somewhat easier to work with than the usual "odometer" listing,[3] since the number of columns is fixed by j and not n. Also, when constructing overlaps between the m and jj schemes,[1] each column can be assigned a permanent factor $[(j + m)(j - m + 1)]^{1/2}$.

For bosons, several can occupy one single-particle state; e.g., the $M = 7$ states of $n = 5$ bosons of $l = 2$ are

$$| 01004 \rangle , \quad | 00113 \rangle , \quad | 00032 \rangle . \qquad (2)$$

When constructing overlaps, in addition to the factor already mentioned, one has to include the square roots of boson occupation numbers. These are explicitly exhibited by scheme (2). Finally, let us introduce the Gaussian polynomial[4,5]

$$\begin{bmatrix} p \\ r \end{bmatrix}_q = \frac{(1 - q^p)(1 - q^{p-1}) \cdots (1 - q^{p-r+1})}{(1 - q)(1 - q^2) \cdots (1 - q^r)} . \qquad (3)$$

It is a generalization (q analog) of the ordinary binomial coefficient, to which it specializes for $q = 1$,

$$\begin{bmatrix} p \\ r \end{bmatrix}_{q=1} = \begin{bmatrix} p \\ r \end{bmatrix} .$$

The degree of polynomial (3) is $r(p - r)$.

III. FERMIONS

Generally, if we have n fermions of spin j, the number of possible states with total projection M is equal to the number of all arrangements of length $2j+1$ consisting of 0's and n 1's [see (1)] for which

$$m_1 + m_2 + \cdots + m_n = M , \tag{4}$$

where m_i ($m_i < m_{i+1}$) are the single-particle projections ranging from $-j$ to j. By letting

$$b_i = m_i + j + 1 ,$$

(4) becomes equivalent to

$$b_1 + b_2 + \cdots + b_n = M + n(j+1) \equiv \overline{M} , \tag{5}$$

where

$$1 \leq b_1 < b_2 < \cdots < b_n \leq 2j+1 . \tag{6}$$

The total number of arrangements satisfying (5) is found by using certain properties of polynomials.

Any polynomial in x may be written in terms of its roots x_i. By collecting powers of x, one gets

$$\prod_{i=1}^{p} (x - x_i) = \sum_{k=0}^{p} (-1)^k e_k(x_1, x_2, \ldots, x_p) x^{p-k} , \tag{7}$$

where e_k is the kth elementary symmetric function, e.g.,

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_1 x_3$$

and, generally,

$$e_k(x_1, \ldots, x_p) = \sum x_{i_1} x_{i_2} \cdots x_{i_k} , \tag{8}$$

where the summation extends over all $(i_1, i_2, \ldots, i_k)$ for which

$$1 \leq i_1 < i_2 \cdots < i_k \leq p . \tag{9}$$

It is important to notice that condition (9) is the same as (6). So (7) will be a natural device for counting our arrangements, if we are careful of two things. First, the total number of b_i's in (5) is n, the number of fermions. By putting

$$x_i \sim t$$

in (7), one only has to consider coefficients of t^n. (In our case, $p = 2j + 1$.) Similarly, to accommodate (5), we demand

$$x_{b_i} \sim q^{b_i}$$

on the right-hand side (rhs) of (8). Thus, by putting

$$x_i = q^i t \tag{10}$$

on the left-hand side (lhs) of (8), we get the result that the number of states satisfying (5) is equal to the coefficient of

$$q^{\overline{M}} t^n$$

in the polynomial

$$\prod_{i=1}^{2j+1} (1 + q^i t) = \sum_{k=0}^{2j+1} t^k e_k(q, q^2, q^3, \ldots, q^{2j+1}) . \tag{11}$$

It is easy to prove by induction on p that

$$\prod_{i=1}^{p} (1 + q^i t) = \sum_{r=0}^{p} q^{r(r+1)/2} \begin{bmatrix} p \\ r \end{bmatrix}_q t^r . \tag{12}$$

The multiplicity we are seeking is then the coefficient of q^M in (12) with $r = n$ and $p = 2j + 1$. By moving back to M via (5), we complete the proof of *Theorem 1:* The number of states with total projection M for n fermions of spin j is given by the coefficient of q^M in

$$q^{-(n/2)(2j+1-n)} \begin{bmatrix} 2j+1 \\ n \end{bmatrix}_q . \tag{13}$$

In our example, $j = \frac{5}{2}$, $n = 3$, we get

$$q^{-9/2} \begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = q^{-9/2} + q^{-7/2} + 2q^{-5/2} + 3q^{-3/2}$$
$$+ 3q^{-1/2} + 3q^{1/2} + 3q^{3/2}$$
$$+ 2q^{5/2} + q^{7/2} + q^{9/2} , \tag{14}$$

which shows that there is one state for each of $I = \frac{9}{2}$, $\frac{5}{2}$, and $\frac{3}{2}$ and no states of $I = \frac{7}{2}$ or $\frac{1}{2}$. As mentioned before, this result is obtained by subtracting successive multiplicities appearing in (14).

Notice that the result (13) has a very suggestive form. The binomial coefficient

$$\begin{bmatrix} 2j+1 \\ n \end{bmatrix}$$

is just the solution of the Fermi-Dirac counting problem, giving the total number of allowed configurations for n fermions distributed over $2j + 1$ single particle states. The factor in (13) is equal to

$$q^{-I_{\max}} ,$$

where

$$I_{\max} = \frac{1}{2} n(2j + 1 - n) ,$$

the maximal allowed I [put all "ones" to the right in (1)].

IV. BOSONS

Following the discussion at the end of Sec. III, we can immediately state *Theorem 2:* The number of states with total projection M for n bosons of spin l is given by the coefficient of q^M in

$$q^{-nl} \begin{bmatrix} 2l+n \\ n \end{bmatrix}_q . \tag{15}$$

Note that for bosons $L_{\max} = nl$, and the solution of the Bose-Einstein counting problem for n bosons in $r = 2l + 1$, single-particle states are given by

$$\begin{bmatrix} r + n - 1 \\ n \end{bmatrix} .$$

We shall only sketch the proof, since it is dual of the one given. Conditions (4)–(6) are changed, since more than one boson can occupy a projection state. Instead of (4) we have

$$a_1 m_1 + a_2 m_2 + \cdots + a_r m_r = M , \tag{16}$$

$$a_1 + a_2 + \cdots + a_r = n \ , \quad 0 \leq a_i \leq n \ . \qquad (17)$$

A shift by $l+1$ produces

$$1 \leq c_1 < c_2 \cdots < c_r \leq 2l+1 \ ,$$

$$a_1 c_1 + a_2 c_2 + \cdots + a_r c_r = \overline{M} = M + n(l+1). \qquad (18)$$

The problem is now reduced to counting all possible monomials

$$(q^{c_1} t)^{a_1} (q^{c_2} t)^{a_2} \cdots (q^{c_r} t)^{a_r} = q^{\overline{M}} t^n \qquad (19)$$

with $\overline{M}$ and n fixed. In order to do this, instead of (7), we now use the following expansion:[4,5]

$$\prod_{i=1}^{p} \frac{1}{(1-x_i)} = \sum_{k=0}^{\infty} h_k(x_1, \ldots, x_p) \ , \qquad (20)$$

obtained by computing the lhs as a product of the geometric series, and where the rhs denotes the kth complete homogeneous symmetric function, e.g.,

$$h_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_1 x_3$$

and in general

$$h_k(x_1, \ldots, x_p) = \sum x_1^{a_1} x_2^{a_2} \cdots x_p^{a_p} \ , \qquad (21)$$

where the sum extends over all $(a_i \geq 0)$ such that

$$a_1 + a_2 + \cdots + a_p = k \ . \qquad (22)$$

Again, (22) is the same as (17), and we need the coefficient of $q^{\overline{M}} t^n$ in

$$\prod_{i=1}^{2l+1} \frac{1}{(1-q^i t)} = \sum_{n=0}^{\infty} q^n \begin{bmatrix} 2l+n \\ n \end{bmatrix}_q t^n \ , \qquad (23)$$

where (23) [like (12)] can be proved by induction. Replacing $\overline{M}$ by M from (18), Eq. (15) is justified.

V. CALCULATING THE COEFFICIENTS

The Gaussian polynomials in (13) and (15) contain powers of q from zero to $2I_{\max}$. The factors $q^{-I_{\max}}$ will be omitted in the following. We shall give a procedure for calculating the coefficients of Gaussian polynomials without using definition (3). Any formal series $P(q)$ may be expressed in two ways (if $c_0 = 1$):

$$P(q) = \sum_{M \geq 0} c_m q^m \equiv \exp \left[\sum_{k \geq 1} \frac{1}{k} p_k q^k \right] \ . \qquad (24)$$

Then the c's and p's are connected by the identity[6]

$$c_m = \frac{1}{m!} \det \begin{bmatrix} p_1 & -1 & 0 & - & - & - & 0 \\ p_2 & p_1 & -2 & & & & \\ & & & & & & 0 \\ & & & & & & \\ & & & & & p_1 & (-n+1) \\ p_m & - & - & - & - & p_2 & p_1 \end{bmatrix} \ , \qquad (25)$$

or, equivalently,

$$mc_m = p_1 c_{m-1} + p_2 c_{m-2} + \cdots + p_{m-1} c_1 + p_m \ . \qquad (26)$$

The point is that it is easy to write a closed expression for the p's of a Gaussian polynomial because the logarithm of (3) is a simple expression. Using the expansion of $\ln(1-x)$, one gets

$$p_m = \sum_{\substack{s=1 \\ s \mid m}}^{\min(r,p-r)} s - \sum_{\substack{s=\max(r,p-r)+1 \\ s \mid m}}^{p} s \ , \qquad (27)$$

where r and p on the rhs are the parameters of (3), and $s \mid m$ means that s divides m. This additional restriction severely reduces the number of terms which contribute to (27), and the p's are usually small integers. As an example, consider the case of $n=5$ bosons of spin $l=2$. The relevant polynomial has $p=9$, $r=5$ [see (15)]. It has terms up to q^{20}, but we need to calculate only up to q^{10}, since it is symmetric (i.e., positive and negative projections are equivalent). Using (27), we obtain

$$p_1 = 1 \ , \quad p_4 = 7 \ , \quad p_7 = -6 \ , \quad p_{10} = 3 \ .$$

$$p_2 = 3 \ , \quad p_5 = 1 \ , \quad p_8 = -1 \ , \qquad (28)$$

$$p_3 = 4 \ , \quad p_6 = 0 \ , \quad p_9 = -5 \ ,$$

The recursion (26) then gives

$$c_0 = 1 \ ,$$

$$c_1 = 1 \ , \quad c_4 = 5 \ , \quad c_7 = 9 \ , \quad c_{10} = 12 \ .$$

$$c_2 = 2 \ , \quad c_5 = 6 \ , \quad c_8 = 11 \ , \qquad (29)$$

$$c_3 = 3 \ , \quad c_6 = 8 \ , \quad c_9 = 11 \ ,$$

Thus we have one state of $L=10$; $c_1 - c_0 = 0$ states of $L=9$; the $c_2 - c_1 = 1$ state of $L=8$; one state of $L=7$; $c_4 - c_3 = 2$ states of $L=6$; the $c_5 - c_4 = 1$ state of $L=5$; two states of $L=4$; one of $L=3$; two of $L=2$; none of $L=1$; and $c_{10} - c_9 = 1$ of $L=0$.

VI. COMMENT ON SPIN AND STATISTICS

We have seen in (13) and (15) how Gaussian polynomials generate the angular momentum multiplicities for fermions and bosons. Let us now pose the inverse question: Given a polynomial (3), of what problem is it a solution? For instance, if p is odd, (3) cannot solve any fermion problem (13), and it can solve a boson problem only if r is also odd since $2l$ in (15) is an even number. Table I is constructed by similar considerations. We note that $r(p-r)$ is the degree of (3), so it is not unnatural to consider $(p-r)$ instead of p. We have implicitly assumed the correct spin-statistics relation. The box labeled "none" actually solves both problems with the relation reversed.

TABLE I. A classification of Gaussian polynomials (3) according to which multiplicity problem they solve. The correct spin-statistics relation is assumed.

	r odd	r even
$(p-r)$ even	Bosons	Both
$(p-r)$ odd	Fermions	None

TABLE II. A classification of classical simple groups according to the metric they preserve (from Ref. 9).

Symmetry property of metric	Functional form of metric Bilinear	Sesquilinear
Symmetric	$g_{ij} = g_{ji}$ Orthogonal	$g_{ij} = g_{ji}^{*}$ Unitary
Antisymmetric	$g_{ij} = -g_{ji}$ Symplectic	None

Similarly, the column (r odd) also gives the solutions for "fermions" for ($p - r$) even and "bosons" for $p - r$ odd, but with the wrong statistics. A moment's reflection shows that our derivation of (13) and (15) did not need the spin-statistics relation; it is an external condition.

As an example, we may recall the Jordan-Schwinger realization of the SU(2) group in terms of what are sometimes pedantically called "two kinds of bosons."[7] They may be considered as one boson of spin $\frac{1}{2}$ and two projections.[8] Using (15) with $l = \frac{1}{2}$ (!), we get

$$\begin{bmatrix} n+1 \\ n \end{bmatrix}_q = \frac{1-q^{n+1}}{1-q} = \sum_{i=0}^{n} q^i , \tag{30}$$

the known result that n Jordan-Schwinger bosons realize a state of total $I = n/2$ and only that I, all the coefficients in (30) being unity. Thus our formalism is indeed not prejudiced as to spin and statistics.

There is, however, a very interesting analogy between the classification of Table I, which assumes the correct spin-statistics relation, and a classification of classical simple Lie groups. Table II classifies them according to the metric preserved by group operations.[9] The similarity between Tables I and II seems to be more than superficial. Generators of a unitary group U($2l+1$) or U($2j+1$) can be constructed from bilinear combinations of both boson and fermion operators. If these bilinear combinations are coupled to good angular momentum, those coupled to odd angular momentum close the algebra of the seniority subgroup.[10] It is the orthogonal group O($2l+1$) for bosons and the symplectic group Sp($2j+1$) for fermions. The fact that some Gaussian polynomials solve no multiplicity problem with the correct statistics seems to be reflected in the conspicuous absence of a class of simple groups from Table II.

At present, this is nothing more than an analogy. If one were able to construct an explicit connection between Tables I and II, this would independently corroborate the spin-statistics relation. We shall briefly return to that point in the following section.

VII. DISCUSSION

The operational way to use the results of this work is given by Eqs. (26) and (27). They are suitable for computer applications, and a compact program can be written

which generates the allowed states of any configuration of identical particles. This is useful when basis sets are constructed for diagonalizations, either in the shell model or any phonon model.

From a more abstract point of view, it is interesting that both boson and fermion problems are reduced to one and the same task of calculating coefficients of Gaussian polynomials. The polynomials even give multiplicities for problems with the wrong spin-statistics connections. This suggests that it is not possible to deduce the physical spin-statistics relation from mathematical arguments alone. Indeed, the very existence and usefulness of the Jordan-Schwinger construction shows that an essential ingredient of this relation is ascribing a physical reality to the particles in question. This is also implied by the traditional derivation of the relation,[11] which begins by writing a Hamiltonian. A way to give a physical distinction to those Gaussian polynomials which solve the multiplicity problems with the correct spin statistics would be to provide a connection between them and compact simple groups. The latter could be physically interpreted as describing the conservation of probability. The analogy between Tables I and II seems to imply that such a connection is possible. Its actual construction remains an open problem.

The main result of this work has been to remove a principal disadvantage of the m scheme in determining multiplicities, that one had to write out all possible states of type (1) or (2) and then count them. This procedure has been replaced by a recursive algorithm. Another disadvantage has remained, since no definition of additional quantum numbers is given.

Traditionally, quantum numbers have been assigned in connection with subgroup chains. The Gel'fand-Tseitlin chain which completely resolves the multiplicity problem does not exhibit the physical angular momentum group.[12] The Racah chain is physical, but the seniority subgroup does not resolve the multiplicities completely.[1,3] The seniority quantum number may be read off from the m-scheme content of a jj state, if the appropriate overlaps are constructed.[3] It is conceivable that a definition of additional quantum numbers could be constructed based on the m scheme. One would not expect them all to be connected with subgroups of the maximal compact unitary group. The situation would then be analogous to the separation of variables in differential equations, where there exist separable coordinate systems which do not lie on subgroup orbits of the symmetry group of the equation.[13] We hope that our results for the multiplicity problem will stimulate efforts to solve the quantum numbers problem by combinatorial arguments in the m scheme.

[1] A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic, New York, 1963).

[2] B. F. Bayman and A. Lande, Nucl. Phys. **77**, 1 (1966).

[3] R. D. Lawson, *Theory of the Nuclear Shell Model* (Clarendon, Oxford, 1980).

[4] I. G. Macdonald, *Symmetric Functions and Hall Polynomials* (Oxford University, New York, 1979).

[5] G. Andrews, *Theory of Partitions* (Addison-Wesley, Reading, Mass., 1976).

[6] F. Faà di Bruno, *Einleitung in die Theorie der Binären Formen* (B. G. Teubner, Leipzig, 1881), p. 126.

[7] L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Physics,* Encyclopedia of Mathematics, Vol. 8, edited by Gian-Carlo Rota (Addison-Wesley, Reading, Mass., 1981).

[8] J. Schwinger, in *Quantum Theory of Angular Momentum,* edited by L. C. Biedenharn and H. van Dam (Academic, New York, 1965), p. 229.

[9] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974).

[10] J. C. Parikh, *Group Symmetries in Nuclear Structures* (Plenum, New York, 1978).

[11] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[12] L. C. Biedenharn, Group Theoretical Approaches to Nuclear Spectroscopy, presented at the Theoretical Physics Institute, University of Colorado, 1962, p. 258.

[13] W. Miller, *Symmetry and Separation of Variables,* Encyclopedia of Mathematics and its Applications, Vol. 4, edited by Gian-Carlo Rota (Addison-Wesley, Reading, Mass., 1977).