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LETTER TO THE EDITOR

CLEBSCH–GORDAN COEFFICIENTS FOR THE QUANTUM ALGEBRA
 $SU(2)_{p,q}$

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The Clebsch–Gordan coefficients for the $SU(2)_{p,q}$ algebra are calculated using the covariant – tensor method for quantum groups. It is shown that the C.-G. coefficients depend on a single parameter $Q = \sqrt{pq}$.

During the past few years, much attention has been paid to the quantum deformations of Lie algebras (quantum groups)¹⁾, both from the mathematical and physical point of view. The main idea of physical application of the quantum groups is a generalization of the concept of symmetry. For example, the rules for the addition of angular momenta in q -deformed $SU(2)_q$ algebra are generalized in accordance with q -deformed algebra and co-algebra²⁾.

Multiparameter deformations of Lie algebras (with more than one deforming parameter) were also studied³⁾.

In this Letter we calculate for the Clebsch–Gordan coefficients for two-parameter (p, q) deformed $SU(2)_{p,q}$ algebra. We show that the C.-G. coefficients depend effec-

tively on only one parameter $Q = \sqrt{pq}$. Our result are in agreement with Drinfeld and Reshetikhin's theorem⁴⁾.

We recall the $SU(2)_{p,q}$ algebra defined in references [3] and [5] (p and q are real parameters):

$$\begin{aligned} [J_0, J_{\pm}] &= \pm J_{\pm} \\ [J_+, J_-]_{p,q} &= J_+ J_- - qp^{-1} J_- J_+ = [2J_0]_{p,q} \\ [2J_0]_{p,q} &= \frac{p^{2J_0} - q^{-2J_0}}{p - q^{-1}} \\ (J_0)^\dagger &= J_0 \\ (J_{\pm})^\dagger &= J_{\mp} . \end{aligned} \tag{1}$$

The coproduct Δ is:

$$\begin{aligned} \Delta(J_{\pm}) &= J_{\pm} \otimes p^{J_0} + q^{-J_0} \otimes J_{\pm} \\ \Delta(J_0) &= J_0 \otimes 1 + 1 \otimes J_0 . \end{aligned} \tag{2}$$

The finite dimensional unitary irreducible representation (IRREP) D^j of spin j contains the highest weight vector $|jj\rangle$, satisfying

$$\begin{aligned} J_0|jj\rangle &= j|jj\rangle \\ J_+|jj\rangle &= 0 \\ \langle jj|jj\rangle &= 1 . \end{aligned} \tag{3}$$

The other orthonormalized states of IRREP D^j , $|jm\rangle$, with $-j \leq m \leq j$, satisfy

$$\begin{aligned} J_+|jm\rangle &= \left(\frac{q}{p}\right)^{\frac{1}{2}(j-m-1)} \sqrt{[j-m]_{p,q}[j+m+1]_{p,q}} |j\ m+1\rangle \\ J_-|jm\rangle &= \left(\frac{q}{p}\right)^{\frac{1}{2}(j-m)} \sqrt{[j-m]_{p,q}[j-m+1]_{p,q}} |j\ m-1\rangle \\ J_0|jm\rangle &= m|jm\rangle . \end{aligned} \tag{4}$$

We calculate the C.-G. coefficients for the $SU(2)_{p,q}$ quantum algebra using the covariant – tensor method recently proposed by us⁶⁾. The main results are written in tensor notation. The basis vectors in the tensor space $(V_2)^{\otimes k}$ are $|e_{a_1} \dots e_{a_k}\rangle$, with $a_1, \dots, a_k = 1, 2$.

Then

$$|jm\rangle = |e_{a_1, \dots, a_k}\rangle = \frac{1}{\sqrt{f}} q^{-\frac{M}{2}} \sum_{\text{perm}(a_1 \dots a_k)} (pq)^{\frac{1}{2}\chi(a_1 \dots a_k)} |e_{a_1} \dots e_{a_k}\rangle \quad (5)$$

where the curly bracket $\{a_1 \dots a_k\}$ denotes the q -symmetrization. The summation runs over all the allowed permutations of the fixed set of indices (n_1 1's and n_2 2's). $\chi(a_1 \dots a_k)$ is the number of inversions with respect to the normal order 11...122...2, and

$$\begin{aligned} M &= n_1 n_2 = (j+m)(j-m) \\ j &= \frac{1}{2}(n_1 + n_2) \quad m = \frac{1}{2}(n_1 - n_2) \\ f &= \binom{2j}{j+m}_{p,q} = \frac{[2j]_{p,q}!}{[j+m]_{p,q}! [j-m]_{p,q}!}. \end{aligned} \quad (6)$$

The important relations are:

$$\begin{aligned} f &= q^{-M} \sum_{\text{perm}(a_1 \dots a_k)} (pq)^{\chi(a_1 \dots a_k)} \\ [n]_{p,q} &= \frac{p^n - q^{-n}}{p - q^{-1}} = \left(\frac{p}{q}\right)^{\frac{1}{2}(n-1)} [n]_Q \end{aligned} \quad (7)$$

with $Q = \sqrt{pq}$.

The dual states are

$$\begin{aligned} \langle ce_{a_1} \dots e_{a_k} | &= (|e_{a_1} \dots e_{a_k}\rangle)^\dagger \\ \langle jm| &= (|jm\rangle)^\dagger = \frac{1}{\sqrt{f}} q^{-\frac{M}{2}} \sum_{\text{perm}(a_1 \dots a_k)} Q^{\chi(a_1 \dots a_k)} \langle e_{a_k} \dots e_{a_1} |. \end{aligned} \quad (8)$$

From the orthonormal condition

$$\langle e_{a_k} \dots e_{a_1} | e_{b_1} \dots e_{b_k} \rangle = \delta_{a_1 b_1} \dots \delta_{a_k b_k} \quad (9)$$

and equation (7) it follows that $\langle jm_1 | jm_2 \rangle = \delta_{m_1 m_2}$.

Applying $\Delta(J_\pm)$ and $\Delta(J_0)$ from equations (2), we obtain equations (4). It is important to note that $|jm\rangle_{p,q} = |jm\rangle_Q$.

Furthermore, the quadratic form, invariant under the action of the coproduct Δ (equations (2)), is

$$I = |e_{a_k} \dots e_{a_1}\rangle |e_{b_1} \dots e_{b_k}\rangle \varepsilon_{a_1 b_1} \dots \varepsilon_{a_k b_k} \quad (10)$$

where

$$\begin{aligned}
 \varepsilon &= \begin{pmatrix} 0 & p^{\frac{1}{2}} \\ -q^{-\frac{1}{2}} & 0 \end{pmatrix} \\
 \left(\frac{\varepsilon}{\sqrt{[2]}} \right)_{p,q} &= \left(\frac{\varepsilon}{\sqrt{[2]}} \right)_Q \\
 \varepsilon_{ab}\varepsilon_{bc} &= -\sqrt{\frac{p}{q}}\delta_{ac} & \varepsilon_{ab}\varepsilon_{cb} &= \sqrt{\frac{p}{q}}(Q^{2J_0})_{ac} \\
 \varepsilon_{ab}\varepsilon_{ab} &= [2]_{p,q} & (\varepsilon_{ba})_{p,q} &= -(\varepsilon_{ab})_{q^{-1},p^{-1}}.
 \end{aligned} \tag{11}$$

The general form of the C.-G. coefficients for the $SU(2)_{p,q}$ algebra is⁶⁾

$$\begin{aligned}
 \langle j_1 m_1 j_2 m_2 | JM \rangle_{p,q} &= N_{p,q} \cdot F(p, q) = \\
 &= N_{p,q} \sum_{m=-j}^{+j} \langle j_1 m_1 | (j_1 - j)(m_1 - m)jm \rangle_{p,q} \times \\
 &\times \langle j_2 m_2 | j - m(j_2 - j)(m_2 + m) \rangle_{p,q} \langle j m j - m | 00 \rangle_{p,q} \times \\
 &\times \langle (j_1 - j)(m_1 - m)(j_2 - j)(m_2 + m) | JM \rangle_{p,q}
 \end{aligned} \tag{12}$$

where $2j = j_1 + j_2 - J$ and $N_{p,q}$ is the norm depending on j_1, j_2 and J .

The C.-G. coefficients are real for p, q real and the following relation is valid:

$$\langle j_1 m_1 j_2 m_2 | JM \rangle_{p,q} = \langle JM | j_1 m_1 j_2 m_2 \rangle_{p,q}. \tag{13}$$

Using the tensor notation $|jm\rangle = |e_{a_1 \dots a_k}\rangle$, $k = 2j$, we first calculate C.-G. coefficients for $j_1 \otimes j_2 \rightarrow j_1 + j_2$:

$$\begin{aligned}
 \langle j_1 + j_2 m_1 + m_2 | j_1 m_1 j_2 m_2 \rangle_{p,q} &= \langle e_{\{b,a\}} | e_{\{a\}} e_{\{b\}} \rangle_{p,q} = \\
 &= \sqrt{\left(\frac{f_1 \cdot f_2}{f_3} \right)_{p,q}} \cdot \left(\frac{q}{p} \right)^{\frac{1}{4}(M_1 + M_2 - M_3)} (p \cdot q)^{\frac{1}{2}(j_1 m_2 - j_2 m_1)} = \\
 &= \sqrt{\left(\frac{f_1 \cdot f_2}{f_3} \right)_Q} Q^{j_1 m_2 - j_2 m_1} = \langle j_3 m_3 | j_1 m_1 j_2 m_2 \rangle_Q
 \end{aligned} \tag{14}$$

where

$$M_i = (j_i + m_i)(j_i - m_i), \quad f_1 = \left(\frac{2j_i}{j_i + m_i} \right)_{p,q} \quad j_3 = j_1 + j_2$$

,

$$m_3 = m_1 + m_2 \quad \text{for } i = 1, 2, 3$$

We point out that these C.-G. coefficients depend effectively only on one parameter $Q = \sqrt{pq}$ and that

$$\langle j_1(\pm j_1)j_2(\pm j_2)|(j_1 + j_2) \pm (j_1 + j_2) \rangle_{p,q} = 1. \tag{15}$$

Three of the four C.-G. coefficients appearing on the right-hand side of equation (12) have the simple form (14). The fourth coefficient $\langle jmj - m|00 \rangle_{p,q}$ also has a simple form and depends only on one parameter Q . Namely, for $n = 2j$ we have

$$\begin{aligned} \langle jmj - m|00 \rangle_{p,q} &= \frac{1}{\sqrt{[n+1]_{p,q}}} \varepsilon_{a_1 b_1} \dots \varepsilon_{a_n b_n} = \\ &= (-1)^{j-m} \frac{1}{\sqrt{[n+1]_Q}} Q^m = \langle jmj - m|00 \rangle_Q. \end{aligned} \tag{16}$$

After inserting equations (16) and (14) into equation (12), we conclude that the C.-G. coefficients depend only on one parameter Q :

$$\begin{aligned} \langle j_1 m_1 j_2 m_2 | JM \rangle_{p,q} &= \langle j_1 m_1 j_2 m_2 | JM \rangle_Q = \\ &= N_Q \sum_{m=-j}^{+j} \frac{(-1)^{j-m}}{\sqrt{[2j+1]_Q}} \cdot Q^{(j_1 m_2 - j_2 m_1)} Q^{m(2J+2j+1)} \times \\ &\times \frac{\binom{2j}{j+m}_Q \cdot \binom{2j_1 - 2j}{j_1 - j + m_1 - m}_Q \cdot \binom{2j_2 - 2j}{j_2 - j + m_2 - m}_Q}{\sqrt{\binom{2J}{J+M}_Q \cdot \binom{2j_1}{j_1 + m_1}_Q \cdot \binom{2j_2}{j_2 + m_2}_Q}} \end{aligned} \tag{17}$$

with $2j = j_1 + j_2 - J$ and the norm

$$\begin{aligned} N_{p,q} &= \sqrt{\frac{[2j_1]_{p,q}! [2j_2]_{p,q}! [2J+1]_{p,q}! [j_1 + j_2 - J + 1]_{p,q}!}{[j_1 + j_2 - J]_{p,q}! [j_1 - j_2 + J]_{p,q}! [-j_1 + j_2 + J]_{p,q}! [j_1 + j_2 + J + 1]_{p,q}!}} \\ &\equiv N_Q. \end{aligned} \tag{18}$$

Finally, we mention that the C.-G. problem for the two-parameter quantum algebra $SU(2)_{p,q}$ was also analyzed in reference [5] using the projection operator technique. However, their calculation contains a few errors, for example the expression for their projection operator $p_{mm'}^j = |jm\rangle\langle jm'|$ is wrong and their C.-G. coefficients do not satisfy orthonormality relations. Hence, the conclusion that C.-G. coefficients nontrivially depend on both parameters p and q is not correct⁷⁾.

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CLEBSCH–GORDANOVI KOEFICIJENTI ZA KVANTNU ALGEBRU $SU(2)_{p,q}$

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Clebsch–Gordanovi koeficijenti $SU(2)_{p,q}$ algebre izračunati su pomoću kovarijantne tenzorske metode za kvantne grupe. Pokazano je da Clebsch–Gordanovi koeficijenti ovise o jednom parametru $Q = \sqrt{pq}$.