ISSN 1330–0016
CODEN FIZBE7

STABILITY OF GRAVITATIONALLY-BOUND MANY-BODY SYSTEMS

ANTUN RUBČIĆ and JASNA RUBČIĆ

Department of Physics, Faculty of Science, University of Zagreb, Bijenička cesta 32, POB 162, 41 001 Zagreb, Croatia

Received 21 December 1994

Revised manuscript received 17 March 1995

UDC 523.2, 531.35

PACS 95.10.Ce, 95.10.Fh, 96.30.-t

The semimajor axes of planetary orbits and of major satellites of the planets in the solar system are described by a simple parabolic law, $r_n = \text{const} \times n^2$, where n is an integer. The orbital periods T_n are proportional to n^3 , thus obeying the third Kepler's law. The radical change, compared with the previous approaches, is that $n = 1$ is assigned to all terrestrial planets, $n = 2$ to Jupiter, etc. This is strongly suggested by the analysis of astronomical data. Hence, terrestrial planets are considered as a subgroup of Jovian planets, and have been formed between the Sun and Jupiter in place of one giant planet of the Jovian group. The reason seems to be the temperature limit of about 200 K, corresponding to a distance of about 5×10^{11} m (3.4 a.u.), that causes similar consequences as the well-known Roche limit for satellites of a planet. Relationships for r_n, T_n and other relevant quantities, which also depend on the integer n, are related to the discretization of angular momentum per unit mass of orbiting body. The mass of a central body appears as a scaling factor giving a unique approach to all systems. The mean deviation of observed orbital radii from the parabolic law for r_n is from 3.5% to 7.6% , depending on the system. On the basis of the analysis, we propose the hypotheses on stability of gravitationally-bound many-body systems.

1. *Introduction*

In the past period of over two centuries, several attempts were made to express the distribution of the planetary orbits and other relevant quantities using integer numbers. Titius (1772) and Bode (1776) [1] proposed the law describing the mean distances of planets from the Sun of the general form

$$r_n = A + BC^n, \tag{1}$$

where r_n is a mean distance characterized by an integer number n. The constants A, B and C have no convincing physical meaning, neither have the empirical correlations with definite parameters for a given system. Therefore, this law has raised many discussions. Nevertheless, it played a positive role not only in predicting the unknown planets, but also in stimulating many researches to further work in this direction.

Tomley [1] applied the law to fit the mean distances of the satellites of Jupiter, Saturn, and Uranus. In some cases, he obtained negative values of the constant A, which is a physically unreasonable result. Nieto [2] and also Sapp [3] argue that the Titius-Bode law cannot be considered a "law", but rather a coincidence.

Blagg [4] applied a pure geometric progression

$$r_n = AB^n \mathrm{f}(n) \quad (n = -2, -1, 0, 1...7), \tag{2}$$

where $\mathrm{f}(n)$ is a function oscillating about 1.

Dermott [5] considered the periods T_n of planets and found a relation $T_n = T_c j^{n/2}$, where n is an integer, $j = 6$ for the solar system, and T_c a constant of proportionality. Planetary distances may then be calculated using the Kepler's law.

Ovenden [6] tried to explain the Titius–Bode law by a computer simulation of gravitational evolution of the present planetary distribution, using an appropriate set of initial conditions. His analysis indicates that a planet of about 90 Earth masses was located at the place of the asteroid belt and was later disrupted, and the asteroids are assumed to be the remnants.

Using computer simulation of planetary accretion, Isaacman and Sagan [7] showed that distances of planets from the Sun, in all generated configurations, obeyed some sort of Titius–Bode law. Specifically, they extended Dermott's approach using noninteger values for j and obtained somewhat better agreement with the observed data.

In an attempt to explain the origin of asteroids, van Flandern [8] analyzed the idea of a planetary breakup event, and the "planet explosion" hypothesis and consequently the Titius–Bode law. However, the hypothesis of the broken-up planet has not been generally accepted, and the asteroid belt is regarded as the remnant of masses of a planet that failed to form [9-11].

More recently, Llibre and Piñol [12] proposed a gravitational explanation of the Titius–Bode law, by observing the motion of the solar system around the center of mass of the galaxy. Using a simplified model of the solar system, they found that the distances of planets that are far from the Sun, roughly follow a geometric progression of ratio equal to 2.

Gulak [13] proposed that the orbital distances are given by $r_n = (n + 1/2)r_0$ or $r_n = nr_0$, where r_0 is a characteristic of a given system. Here, n need not increase by 1 in going from one satellite to another one. For example, for satellites of Uranus, n is 7 for Miranda, 12 for Ariel, 15 for Umbriel, 25 for Titania and 34 for Oberon. Later, Gulak [14] found a theoretical support to his previous results by constructing an equation of the Schrödinger type. In this way, he tried to introduce the macroquantization of orbits in a gravitational field.

Recently, Kramer nad Gorbanev [15] demonstrated that the idea of macroquantization of orbits was not acceptable, i.e., not confirmed by the observed data, using elementary numerical examples.

This Introduction is only the scanty review of diverse ideas related to the permanently intriguing problem of the dynamics of planetary and satellite motion. In the present work only the planets and major satellites of Jupiter, Saturn, and Uranus are considered. Small satellites of planets, planetary rings, asteroids, and comets are not included in our model.

2. *Semimajor axes and periods of planets and satellites*

Astronomical data on semimajor axes r of orbits and periods T of revolution for planets and major satellites of Jupiter, Saturn and Uranus are listed in Table 1 [16a,17]. Cited semimajor axes of orbits are equal to the mean orbital radii [18].

The idea of discretization of planetary and satellite orbits is to devise the functions $r_n = \mathrm{f}_r(n)$ and $T_n = \mathrm{f}_T(n)$, n being integers, for orbital radii and periods of revolution, respectively. They must satisfy the third Kepler's law, i.e. the ratio r_n^3/T_n^2 should not depend on n. One of the simplest pairs of functions is $r_n = k_1 n^2$ and $T_n = k_2 n^3$. These relations are relatively well obeyed for solar planets. In Fig. 1 dependences of the form $r^{1/2} = \mathrm{const} \times n$ are given, because of more convenient presentation. The straight lines in the figure are the best fits, using equal weights. In the Jovian group of planets we locate Jupiter in the orbit at $n = 2$, Saturn at $n = 3$, Uranus at $n = 4$, Neptune at $n = 5$ and Pluto at $n = 6$.

If one associates $n = 1$ or $n = 3$ to Jupiter, then the resulting straight–lines in Fig. 1 will be translated, giving positive or negative intercepts at $n = 0$. With Jupiter at $n = 2$, the intercept of the straight line, obtained by the rms method, is less than its standard deviation. Therefore, we apply the constraint $r = 0$ at $n = 0$.

All terrestrial planets collect at about $n = 1$, as shown in Fig. 1. Thus, we consider the terrestrial planets to be a subgroup of the Jovian group. This is a radical change compared with previous approaches.

TABLE 1.
Orbital radii r and periods of revolution T of planets and major satellites in the solar system, with the assigned integers n[1]

	r/m	T/s	n
SUN			
Mercury	0.579×10^{11}	0.760×10^{7}	3
Venus	1.082×10^{11}	1.941×10^{7}	4
Earth	1.496×10^{11}	3.156×10^{7}	5
Mars	2.280×10^{11}	5.936×10^{7}	6
Jupiter	7.783×10^{11}	0.374×10^{9}	2
Saturn	14.27×10^{11}	0.930×10^{9}	3
Uranus	28.71×10^{11}	2.651×10^{9}	4
Neptune	44.97×10^{11}	5.200×10^{9}	5
Pluto	59.13×10^{11}	7.836×10^{9}	6
JUPITER			
Metis	1.280×10^{8}	0.251×10^{5}	2
Andrastea	1.285×10^{8}	0.259×10^{5}	2
Amalthea	1.813×10^{8}	0.432×10^{5}	2
Thebe	2.220×10^{8}	0.576×10^{5}	2
Io	4.216×10^{8}	1.529×10^{5}	3
Europa	6.710×10^{8}	3.069×10^{5}	4
Ganymede	10.700×10^{8}	6.178×10^{5}	5
Callisto	18.830×10^{8}	14.420×10^{5}	6
SATURN			
Prometheus	1.394×10^{8}	0.530×10^{5}	6
Epimetheus	1.514×10^{8}	0.600×10^{5}	6
Janus	1.514×10^{8}	0.600×10^{5}	6
Mimas	1.855×10^{8}	0.814×10^{5}	7
Enceladus	2.380×10^{8}	1.184×10^{5}	8
Tethys	2.947×10^{8}	1.631×10^{5}	9
Dione	3.774×10^{8}	2.365×10^{5}	10
Rhea	5.270×10^{8}	3.904×10^{5}	11
(Titan)	12.218×10^{8}	13.774×10^{5}	(19)
URANUS			
Puck	0.860×10^{8}	0.659×10^{5}	3
Miranda	1.294×10^{8}	1.222×10^{5}	4
Ariel	1.910×10^{8}	2.178×10^{5}	5
Umbriel	2.663×10^{8}	3.581×10^{5}	6
Titania	4.359×10^{8}	7.523×10^{5}	7
Oberon	5.835×10^{8}	11.632×10^{5}	8

[1] In the system of Jupiter, Saturn and Uranius, many smaller satellites are present. Some largest ones among them (between the planet and the first major satellite) are also included in the table. Data are taken from. Refs. 16a and 17.

Some regularities within the subgroup of terrestrial planets have also been found. We assign $n = 3$ to the orbit of Mercury, $n = 4$ to Venus, $n = 5$ to Earth, ending with the orbit of Mars at $n = 6$, that can be seen in Fig. 1. We assign consecutive integers to the planets of each subgroup of the solar system, i.e. all intermediate orbits are occupied.

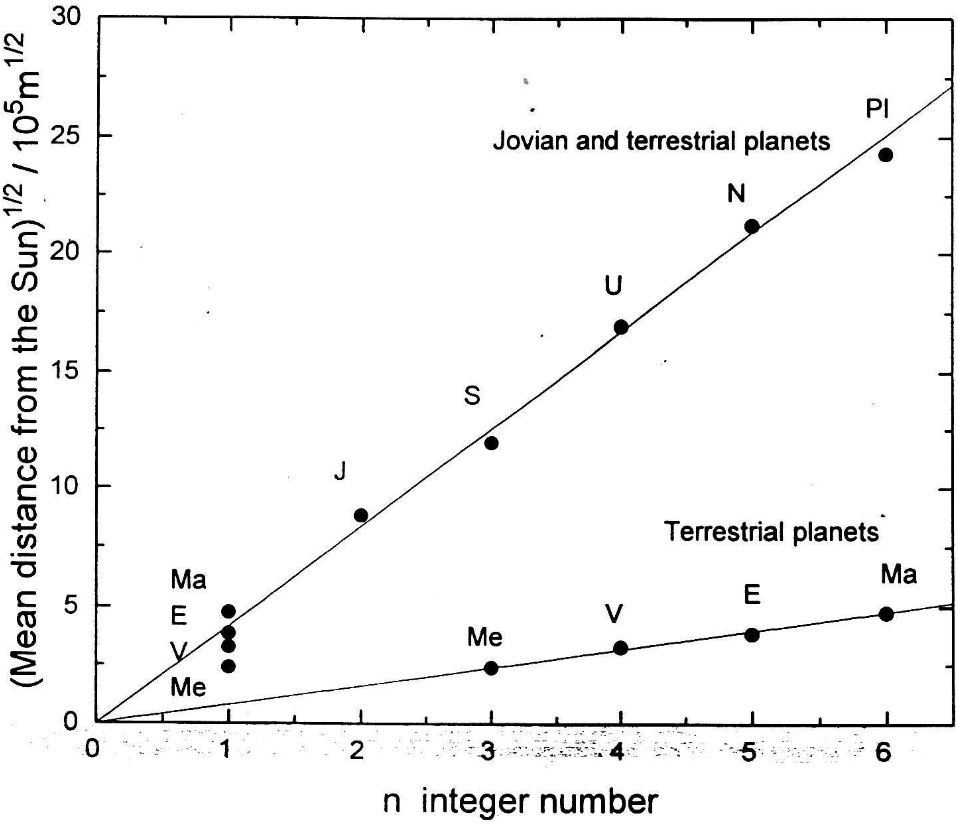

Fig. 1. Correlation of the square roots of the mean radial distances of the planets from the Sun with an integer number n. Points denoted by Me, V, E, Ma represent terrestrial planets in order from Mercury to Mars.

Orbits at $n = 1$ and $n = 2$ are not occupied. Some physical processes, due to the Sun have not permitted the existence of the first two planets. The planets at $n > 6$ have failed to be formed due to the perturbation of Jupiter and consequently the asteroid belt was formed [11].

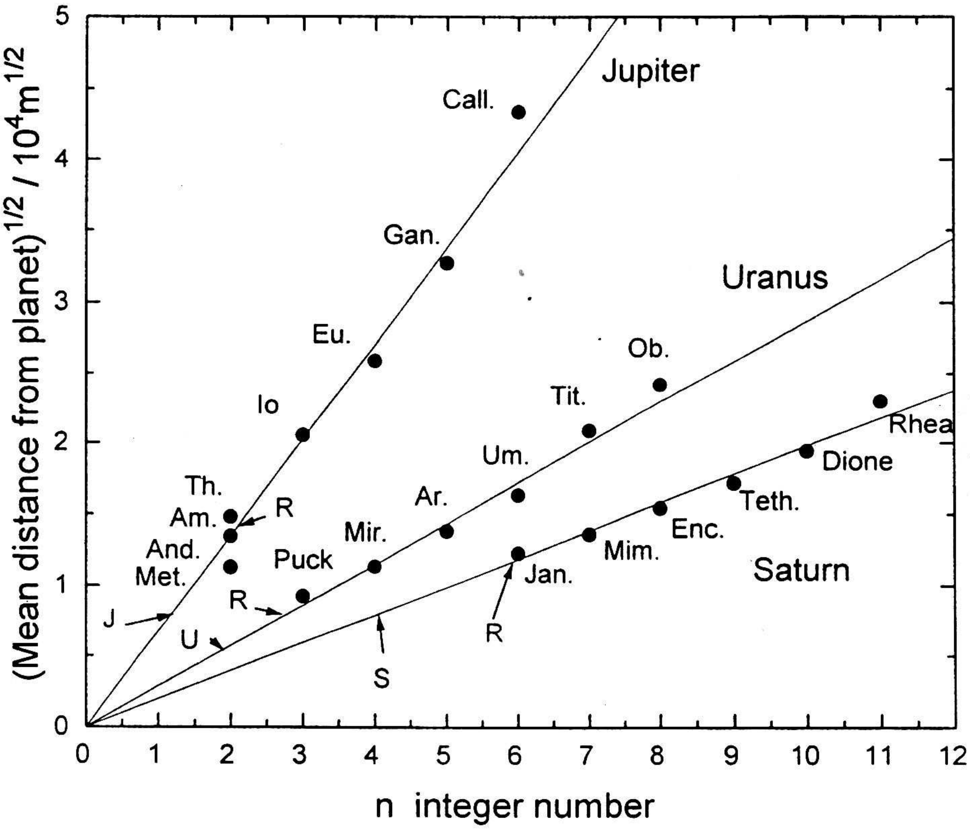

Fig. 2. Correlation of the square roots of the radial distances with an integer number n, for satellite systems. Arrows with the letter R denote the square roots of the Roche limits and arrows denoted by letter J, S, and U represent the square roots of the planet radii.

The correlations $r^{1/2} = \text{const} \times n$ for satellite systems, are shown in Fig. 2. The numbering for Jupiter satellites starts with $n = 2$ for small ones (Metis, Andrastea, Amalthea and Thebe), as one subgroup, folowed by Io at $n = 3$, Europa at $n = 4$, Ganymede at $n = 5$ and ending with Callisto at $n = 6$. The arrow denoted by R indicates the square root of the Roche limit [19a], and arrow denoted by J–the square root of Jupiter radius. Similar relations have been found for systems of Saturn and Uranus, also shown in Fig. 2. The square roots of radii of Saturn and Uranus are denoted by S and U, respectively. Between the surface of the planet and the Roche limit only small satellites and rings are found. Above the Roche limit only major satellites build up a regular orbital spacing.

The correlations $T^{1/3}$ = const × n for planets and satellites result in similar linear plots, as those in Figs. 1 and 2, respectively, due to the third Kepler's law.

TABLE 2

Values of orbital radii r_{cal}, calculated using the parabolic law for planets, with deviations from the observed values in % , and comparison with the results of the Titius–Bode law of selected references. The equations are given in the lower part of the table[2].

	Present work			Ref.16b Titius–Bode (1772-76)		Ref. 5 Dermott (1968)		Ref. 1 Tomley (1979)	
Planet	$r_{cal}/10^{11}$ m	n	%	n	%	n	%	n	%
Mercury	0.574	3	0.9	$-\infty$	3	1	19.0	(1	-33.0)
Venus	1.020	4	6.0	0	-3	2	-16.5	2	0.4
Earth	1.594	5	-6.1	1	0	2	15.3	3	-0.5
Mars	2.295	6	-0.7	2	5	3	-3.3	4	-3.5
(Ceres)	–	–	–	3	1	4	-3.4	5	1.2
Jupiter	7.014	2	11.0	4	0	5	0.0	6	3.0
Saturn	15.78	3	-9.6	5	5	6	0.9	7	-1.8
Uranus	28.05	4	2.4	6	2	7	11.7	8	0.6
Neptune	43.84	5	2.6	7	22	8	-3.7	(9	20.7)
Pluto	63.12	6	-6.3	8	49	8	26.6	(10	-47.8)

Present work: Terrestrial planets $r_{cal} = (0.638 \pm 0.022) \times 10^{10} n^2$ (m)
Jovian Planets $r_{cal} = (1.753 \pm 0.114) \times 10^{11} n^2$ (m)
Ref. 16b: All planets $r_{cal} = (0.60 + 0.45 \times 2^n) \times 10^{11}$ (m)
Ref. 5: All planets $r_{cal} = 3.930 \times 6^{n/3} \times 10^{10}$ (m)
Ref. 1: All planets $r_{cal} = [(0.6538 + 0.1047 \times (2.010)^n] \times 10^{11}$ (m)

We may compare our results for calculated orbital radii and the results of some selected references [1,5,16b], with the observed radii. These results and the deviations of the calculated values (in %) are listed in Table 2 for the solar planetary system. The equations are given at the bottom of the table. Dermott's results were presented for the periods of revolution. Using the third Kepler's law we deduced the equation and results cited under Ref. 5 in Table 2. Dermott assigned the same number n to some planets. That introduces large deviations from the observed data. Tomley [1] has not taken the planets Mercury, Neptune and Pluto into his fit. That results with large errors for these three planets.

The results for the satellite systems are presented in Table 3. The values of the Roche limit, assuming an equal density of a satellite and its parent planet, are given at the bottom of Table 3.

[2]Planets with the results in parentheses were not included in the fit.

TABLE 3

Values of orbital radii r_{cal}, calculated using the parabolic law, for major satellites, with deviations from the observed values in % , and comparison with the results of the Titius–Bode law of selected references. The equations are given in the lower part of the table[3,4].

	Present work			Ref. 5 Dermott (1968)		Ref. 1 Tomley (1979)	
System	$r_{cal}/10^8$ (m)	n	%	n	%	n	%
JUPITER							
Amalthea	1.834	2	-1.0	(0	-31.9)	(1	-38.4)
Io	4.126	3	2.3	1	-0.4	2	-0.3
Europa	7.334	4	-8.4	2	-0.1	3	2.1
Ganymede	11.46	5	-6.6	3	0.3	4	-1.6
Callisto	16.50	6	14.2	4	11.2	5	0.4
SATURN							
Mimas	1.916	7	-3.2	1	-0.9	2	-0.1
Enceladus	2.502	8	-4.9	2	1.0	3	-0.3
Tethys	3.167	9	-6.9	3	-0.8	4	-3.6
Dione	3.980	10	-5.2	4	0.9	5	-4.7
Rhea	4.731	11	11.2	5	11.8	6	2.2
Titan	(13.898)	(19	-0.9)	9	2.8	9	5.3
Hyperion						10	-2.9
URANUS							
Miranda	1.317	4	-1.7	1	-2.1	1	3.4
Ariel	2.057	5	-7.1	2	0.2	2	0.2
Umbriel	2.963	6	-10.1	3	-3.8	3	-7.0
Titania	4.033	7	8.1	4	8.4	4	5.5
Oberon	5.267	8	10.8	5	-0.1	5	-2.1

Present work:	Jupiter	$r_{cal} = (4.580 \pm 0.300) \times 10^7 n^2$ (m)	
	Saturn	$r_{cal} = (0.391 \pm 0.024) \times 10^7 n^2$ (m)	
	Uranus	$r_{cal} = (0.823 \pm 0.062) \times 10^7 n^2$ (m)	
Ref. 5:	Jupiter	$r_{cal} = 2.666 \times 2^{2n/3} \times 10^8$ (m)	
	Saturn	$r_{cal} = 1.485 \times 2^{n/3} \times 10^8$ (m)	
	Uranus	$r_{cal} = 0.9039 \times (1.75)^{2n/3} \times 10^8$ (m)	
Ref. 1:	Jupiter	$r_{cal} = [(1.4100 + 0.8383 \times (1.833)^n] \times 10^8$ (m)	
	Saturn	$r_{cal} = [(0.2505 + 0.9190 \times (1.322)^n] \times 10^8$ (m)	
	Saturn	$r_{cal} = [(-0.6320 + 1.6410 \times (1.227)^n] \times 10^8$ (m)	
		if Titan $n = 10$ and Hyperion $n = 11$	
	Uranus	$r_{cal} = [(-0.3570 + 1.1420 \times (1.408)^n] \times 10^8$ (m)	
Present work:		Roche limit	n_R
	Jupiter	$R = 1.75 \times 10^8$ (m)	2
	Saturn	$R = 1.48 \times 10^8$ (m)	6
	Uranus	$R = 0.62 \times 10^8$ (m)	3

[3] Only Amalthea, the largest one of small satellites of Jupiter, is included. The Roche limits R (calculated by assuming an equal density for a given planet and its satellites) and corresponding approximate integers n_R are given at the bottom of the table.

[4] Satellites with the results in parentheses were not included in the fit.

3. Lower bounds on integers n

The starting numbers $n > 1$ in the planetary and satellite systems may be due to different causes: the Roche limit, temperature of the central body, the "rotational limit" and possibly other causes. From the Roche limits, using equations in Table 3, one can calculate the integer lower bounds n_R. They are equal 2, 6, and 3 for Jupiter, Saturn, and Uranus, respectively. Thus, a stable systems of major satellites can exist only for $n > n_R$, as is indicated in Fig. 2.

However, the Roche limit of the Sun is approximately equal to 2×10^9 m, whereas the orbital radius for missing Jovian planet at $n = 1$ (according to our equation in Table 2) is 1.75×10^{11} m (1.17 a.u.). There must be a "limit", caused by some other physical conditions, below which the Jovian type of planet was not allowed. Due to the high temperature of the Sun, icy components of the solar nebula have been dispersed and rocky terrestrial planets were formed as a consequence of the chemical condensation sequence [20]. This "temperature limit" is about 200 K. It corresponds roughly to 5×10^{11} m (3.4 a.u.) and coincides with the upper bound of the main part of the asteroid belt.

In the subgroup of the terrestrial planets the numbering of orbits starts with Mercury at $n = 3$, for which several reasons may be responsible. We introduce "the rotational limit" which can be estimated by considering the periods of revolution of satellites and rotational period of the central body. In the primordial nebula, from which the Sun, the planets and their satellites formed, some rotational motion of gasses and dust were taking place. A large fraction of mass was captured by the central body (Sun versus the planets, planets versus their satellites). Angular velocity of the central body is related to the angular velocity of the farthest masses that were not captured. Therefore, one can expect no satellites with a shorter period of revolution T_r than is the rotational period T_{rot} of the central body, i.e $T_r > T_{rot}$. Thus, the minimum orbital radius for major satellite is approximately given by

$$r_{min} = [(GMT_{rot}^2/4\pi^2)]^{1/3}. \tag{3}$$

For Jupiter, Saturn, and Uranus r_{min} is equal to 1.59×10^8 m, 1.09×10^8 m, and 0.83×10^8 m, respectively.

Above r_{min} the major satellites may exist, but not within that limit. Of course, later on the planet could have captured some asteroids, remnants of comets, or the parts of broken satellites, making thus the planetary rings and small satellites within that limit and with periods shorter than the period of rotation of the planet. Also, planetary rotational periods could have varied in the subsequent evolution. The limit r_{min} may be called "rotational cut–off" for the major satellites. In the planetary satellite systems these cut–offs are unimportant because they are near the respective Roche limits. However, for the terrestrial planets the "rotational limit" seems to be of major importance. The Roche limit for the Sun is $R = 1.7 \times 10^9$ m, while $r_{min} = 2.51 \times 10^{10}$ m. Orbital radii for two missing terrestrial planets at $n = 1$ and $n = 2$, according to our parabolic law (see Table 2), should have had the values $r_1 = 0.638 \times 10^{10}$ m and $r_2 = 2.55 \times 10^{10}$ m, i.e. below the limit r_{min}.

Therefore, the planets at $n = 1$ and $n = 2$ did not form. Indeed, the present period of rotation of the Sun is 25 days, whereas the periods of revolution of the first two planets would have been 3.2 ± 0.2 and 25.5 ± 1.3 days, respectively. It was supposed that the small planet Vulcan exists between the Sun and Mercury, but it has not been observed [16c]. Our present analysis makes its existence unlikely, unless solar rotational period had changed considerably after solar formation. Mercury escapes the cut–off with its period of 88 days. This explains why the enumeration of orbits for terrestrial planets starts at $n = 3$. Some other effects could have been of equal importance, like the high temperature and tidal forces due to the Sun.

4. *Angular momenta*

From Newton's law of gravity, assuming a circular orbit, one can obtain orbital radius in the form

$$r = \frac{1}{GM}(vr)^2 \tag{4}$$

where G is the gravitational constant, M is the mass of a central body and the quantity vr is the angular momentum per unit mass of orbiting body, v being the speed of revolution.

If r is proportional to n^2, and T to n^3, as introduced in Sec. 2, then the angular momentum per unit mass J/m is proportional to n. A constant of proportionality between $vr = 2\pi r^2/T$ and n does not depend on the mass m. The ratio $2\pi r^2/Tn = C$ is nearly the same for a particular system. The mean value of the constant C for the each subsystem is given in Table 4. Therefore, straight lines defined by $J/m = vr = Cn$ for planets and satellites will principally correspond to those in Figs.1 and 2, respectively. The values of the ratio C/M are roughly independent of the system. They are of the same order of magnitude, as can be seen in Table 4. Therefore, we define $C/M = fA$, expecting that A could be determined by some fundamental constants, while the variations of C/M will be described by the dimensionless factor f. Therefore, we may write angular momentum per unit mass for a planet or satellite in orbit n as

$$\frac{J_n}{m} = \frac{2\pi r_n^2}{T_n} = Cn = (fA)Mn. \tag{5}$$

From Eqs. (4) and (5), a discrete set of radii is given by

$$r_n = \frac{1}{G}(fA)^2 Mn^2, \tag{6}$$

while periods are given by

$$T_n = \frac{2\pi}{G^2}(fA)^3 M n^3. \tag{7}$$

In Eqs. (5) to (7), the mass M of central body appears as the scaling factor, while the factor f becomes an adjusting parameter.

According to Eq. (5), the constant A which has the dimension of angular momentum per squared mass (Js kg^{-2}), might be understood as the fundamental constant characterizing the discretization in the gravitational field. Dimensionally, the most natural "gravitational" guess may be connected with receptor factor G/c [21], i.e.

$$A = KG/c \tag{8}$$

c being the speed of light. A dimensionless constant K cannot be determined, of course, unambigously. However, one may speculate by considering the similarity between the gravitational force F_g and the electrostatic force F_e between two identical particles of mass m_0 and charge e, respectively. The absolute ratio of the forces is $Fg/Fe = Gm_0^2/(e^2/4\pi\epsilon_0)$, where ϵ_0 is the permitivity of vacuum. By introducing the well–known fine–structure constant defined by $\alpha = e^2/(4\pi\epsilon_0\hbar c$ [22] (where $\hbar = h/2\pi$ and h is the Planck constant), the ratio may be expressed by

$$\frac{F_g}{F_e} = \frac{1}{\alpha}\left(\frac{Gm_0^2}{\hbar c}\right) = \frac{\alpha_g}{\alpha} \tag{9}$$

where α_g is dimensionless gravitational fine–structure constant [21]. Now, $(2\pi/\alpha)(G/c) = (\alpha_g/\alpha)(h/m_0^2)$ appears, dimensionally, again as an angular momentum per square unit mass. Thus, in Eq. (8) one may choose the constant $K = 2\pi/\alpha$ as possible one among many other dimensionless constants, including the purely gravitational ones.

Therefore, we introduce

$$A = \frac{2\pi G}{\alpha c} = 1.9157 \times 10^{-16} \ \ (\text{Js kg}^{-2}) \tag{10}$$

which may be considered as the "gravitational Planck constant per square unit mass", without an exact explanation. Resulting values of the factor f are given in Table 4. Of course, one can adapt factor f to be equal to 1.00, specially for Jovian planets, by choosing $K = 8\pi^2/\alpha$. Some other choices are also possible.

TABLE 4
Mean values of constant C and C/M, and the factor $f = C/MA$ for planetary and satellite systems.

SYSTEM	$C/\ \mathrm{m^2s^{-1}}$	$(C/M)/\ \mathrm{m^2s^{-1}kg^{-1}}$	f
Terrest. planets	$(0.920 \pm 0.016) \times 10^{15}$	$(0.460 \pm 0.008) \times 10^{-15}$	(2.40 ± 0.04)
Jovian planets	$(4.824 \pm 0.155) \times 10^{15}$	$(2.412 \pm 0.078) \times 10^{-15}$	(12.59 ± 0.41)
Jupiter	$(2.407 \pm 0.078) \times 10^{12}$	$(1.268 \pm 0.041) \times 10^{-15}$	$(6.62 + 0.21)$
Saturn	$(3.852 \pm 0.120) \times 10^{11}$	$(0.678 \pm 0.021) \times 10^{-15}$	(3.54 ± 0.11)
Uranus	$(2.207 \pm 0.090) \times 10^{11}$	$(2.540 \pm 0.104) \times 10^{-15}$	(13.26 ± 0.54)

By introducing A from Eq. (8) into Eqs. (5) to (7), the general relations follow:

$$\frac{J_n}{m} = \left(\frac{Kf}{c}\right) GMn, \tag{11}$$

$$r_n = \left(\frac{Kf}{c}\right)^2 GMn^2, \tag{12}$$

$$T_n = 2\pi \left(\frac{Kf}{c}\right)^3 GMn^3. \tag{13}$$

For $K = 2\pi/\alpha$, the associated values of f are given in Table 4.

From Eqs. (12) and (13), one can deduce the speed of revolution and energy per unit mass of an orbiting body. They are proportional to n^{-1} and n^{-2}, respectively, but independent on the mass M. For example, the satellites of Uranus have approximately the same value of f as Jovain planets, and consequently the orbital speed of Miranda ($n = 4$) and of Uranus (also $n = 4$) are closely equal.

The factor f, as a characteristic of the particular system, probably depends on the conditions under which the system was formed, i.e. f may be a consequence of the mass density distribution at the time of accretion, and of later perturbations that caused changes in the inclination of the orbital plane of satellites to the orbital plane of the central body. However, it is not easy to find sound physical arguments capable to explain the values of f cited in Table 4.

Eq. (12) may be written in the form

$$r_n = \frac{(Kf)^2}{2}\frac{2GM}{c^2}n^2$$

in which one recognizes the Schwartzschild's radius $R_S = 2GM/c^2$ [16d]. With $K_S = (Kf)^2/2$, one can write preceding equation as

$$r_n = K_S R_S n^2. \tag{14}$$

For $K = 2\pi/\alpha$ and $f = 1$, the value of the constant K_S is 3.7×10^5. This might be interesting in future comparison with other planetary systems.

5. *Discussion on the stability of planetary and satellite systems*

The relations $r_n = k_1 n^2$ and $T_n = k_2 n^3$ for orbital radii and periods of revolutions, respectively, are found in each of the five sybsystems of the solar system, which consist of a massive central body and four to six major bodies in orbits. The actual distances and periods show deviations from the laws that we attribute to the perturbations of the solar system since its formation. Close–by passage of celestial bodies and their impacts have certainly shifted the system away from the secular equilibrium.

It is important to note that in each subsystem we find a sequence of four to six consecutive integers corresponding to the orbits in the subsystem, starting from an initial value n_{min}. In each of the five subsystems one can find a cause for n_{min}. In our opinion three causes limit the existence of planets and/or satellites in orbits close to the central body: the Roche limit, the temperature limit and the "rotational" limit. The temperature limit seems to have prevented the formation of the Jovian planet at $n = 1$. Instead, the terrestrial group of planets were formed. In the terrestrial subsystem the starting number n is 3 (Mercury), because of the "rotational" limit, as discussed in Sec. 3.

Lower limits of the values of n in the satellite systems of the planets are due to the Roche limit. For example, the first three orbits in Saturnian system are inside the planet; the orbit at $n = 4$ is near surface of the planet and the rings C, B, and A [19a] are up to $n = 6$. The ring G is between $n = 6$ and $n = 7$. The Roche limit is near $n = 6$ and the orbit of the first major satellite Mimas is at $n = 7$.

In Sec. 3 we considered the temperature limit of about 200 K for the existence of Jovian type of planets as consequence of the chemical condensation sequence, which is responsible for the difference between rocky terrestrial planets and icy Jovian ones [20,23a]. As the temperature falls down with the distance from the Sun, one may expect a similar behaviour of the mass density. Indeed, in Fig. 3, the density of planets and the largest asteroids Ceres, Pallas, and Vesta [16a,24] is shown versus distance from the Sun. Two densities are given for the Sun: one is 1.4

gcm^{-3} for the standard radius of the Sun $r_s = 7 \times 10^7$ m and the other is 6.5 gcm^{-3} for the effective radius $r_{eff} = 0.6 r_s$ [23b]. As shown in Fig. 3, by an extrapolation, the density decreases to that of Jupiter at a distance $r_T = (5.5 \pm 0.9) \times 10^{11}$ m, in agreement with the limit of 200 K in the Lewis temperature–to–distance diagram [19c].

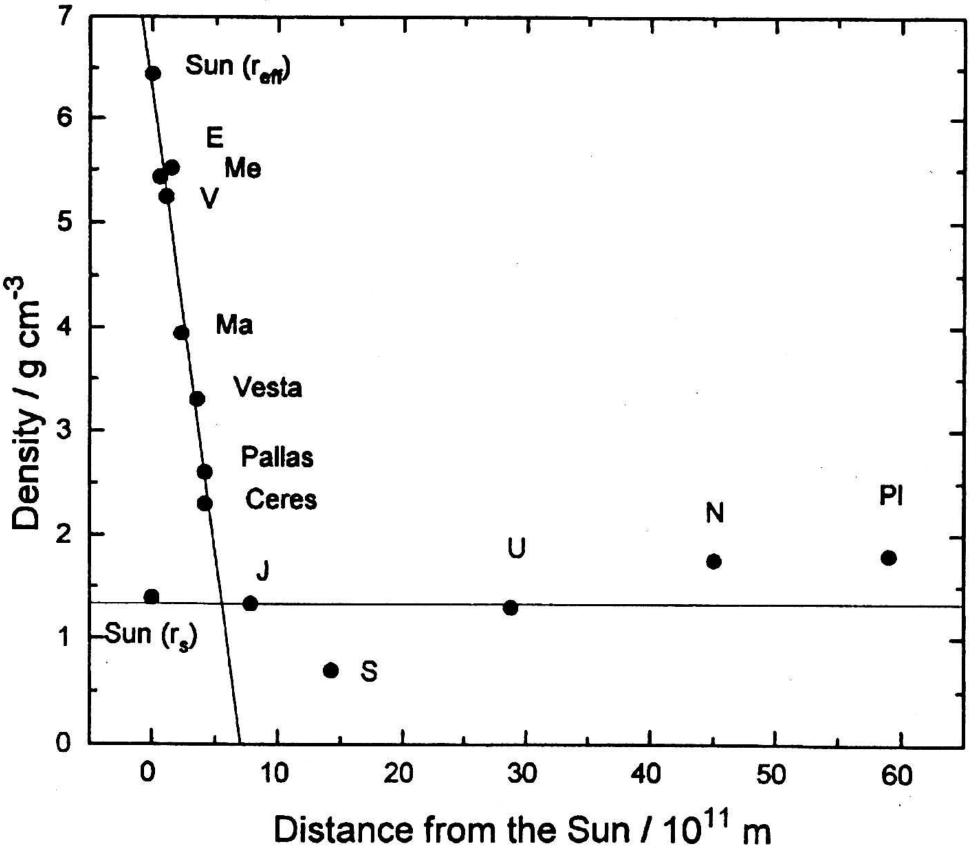

Fig. 3. Density of planets and of the largest asteroids versus the distance from the Sun. The intersection of the two straight lines at the distance 5.5×10^{11} m corresponds to the temperature limit of about 200 K. (Density of the Sun for the standard radius r_s and for the effective radius $r_{eff} = 0.6 r_s$ is also indicated.)

The mean deviation of the calculated orbital radii from the observed ones is somewhat higher (7.6%) for Uranian satellites, than for the other systems. It seems that Titania and Oberon at orbits $n = 7$ and $n = 8$, respectively, in Fig. 2, could be translated to $n = 8$ and $n = 9$ in order to have a better fit. Indeed, the deviation decreases to 4.7% . However, such a refinement is not convincing since it would introduce an empty orbit between the occupied ones.

Saturnian satellites from Mimas to Rhea are distributed between $n = 7$ and $n = 11$. However, the largest satellite Titan appears in the orbit at $n = 19$, thus allowing seven empty orbits. Such a large discontinuity suggests a division of the complete system of satellites to subsystems, with a distinct enumeration of orbits. One may suppose that Titan was captured by Saturn in order to justify its peculiarity. There are, however, some arguments against that hypothesis [19d]. We assume that Titan is not the member of the system with discrete orbits and this massive satellite does not fit into our scheme. It is a single exception.

Major satellites have generally been accreted at the time of the formation of the planetary system [16e], whereas small satellites, rings, asteroids and comets evolved later on. Therefore, due to various origin of small bodies, their motion and distribution of orbits escape a definite rule, and may rather be considered as random events. This supposition is based on the broad span of eccentricities and orbital inclinations [16a]. Obviously, it is not possible to treat them within the scheme of major satellites.

A distribution of planetary orbits is, according to Table 2, better described by introduced parabolic law, than by the laws of Titius-Bode type (Eq.1). However, from the results on satellites orbits one may conclude contrary (c.f. Table 3). Particularly, the last satellite in all three systems deviate considerably from our law, as can be seen in Fig. 2. In our oppinion, these deviations may be the outcome of the various events in the history of a system. For example, Callisto in the Jupiter system has a larger orbit than expected. Impact of an asteroid on Callisto could be responsible for such a deviation (very probably the Valhalla region appeared after such an event [23c]). Saturnian satellite Rhea has also an enlarged orbit, which may be due to perturbation of Titan. In the case of Uranian satellites, deviations from the parabolic law could be a consequence of the tilting of the rotation axis of Uranus and of all its satellites for 98° to the ecliptic [16a].

Furthermore, the ratio of orbital radius to the radius of a central body is between 3 and 5 for the nearest major satellites, and about 25 for the most remote ones. However, in the solar planetary system, this ratio is 83 for Mercury, 1110 for Jupiter and even 8430 for Pluto. From these numbers one may conclude that the Sun is a point–like center of the force for Jovian planets and partially for terrestrial ones. For all satellite systems, the central body is not a point-like center, which may have a strong influence to the nearest satellites through the tidal forces, thus changing the dynamics of the whole system. It could be a reason of the slight deviation from the parabolic law [Eq. (12)] to the Titius–Bode type one [Eq. (1)]. Therefore it is not surprising that the accuracy of the parabolic law is a few per cent less for satellite system compared with planetary one. With all these comments in mind, one can accept the parabolic law as adequate for the distribution of orbital radii in the solar system. This law seems to be the consequence of formation rather than of the later evolution, what might be of a significant cosmogonic importance.

6. *Conclusions*

On the basis of the above considerations we state the following hypotheses:

1) Orbital radii of planets and major satellites, for each of the five subgroups in the solar system, the Jovian and terrestrial group of planets and the satellite systems of Jupiter, Saturn and Uranus, are given by the parabolic law $r_n = \text{const} \times n^2$. The values of n are consecutive integer numbers.

2) Actual state of the system does not exactly agree with the law stated in 1) because the system is not in the state of secular equilibrium.

3) The planets of the Jovian group, Jupiter, Saturn, Uranus, Neptune and Pluto, are in orbits corresponding to $n = 2, 3, 4, 5$ and 6, respectively. The Jovian planet at $n = 1$ did not form because of the high temperature of the Sun.

4) The terrestrial planets are a separate subgroup that was formed between the Sun and Jupiter instead of one large Jovian planet at $n = 1$.

5) The terrestrial planets, Mercury, Venus, Earth and Mars, are in orbits corresponding to $n = 3, 4, 5$ and 6, respectively. The planets at $n = 1$ and 2 did not form because of the "rotational" limit.

6) The orbits of major satellites of Jupiter, Saturn and Uranus also follow the law $r_n = \text{const} \times n^2$, where values of n are consecutive integers. The initial values of n are 3, 7 and 4, respectively. They are due to the Roche limit that prevented formation of satellites for lower values of n.

7) The angular momentum per unit mass of the orbiting body can be expressed by the relation $J_n/m = (fA)Mn$, where f is a dimensionless adjusting parameter between 2.4 and 13.3, associated to the selected fundamental constant $A = 2\pi G/\alpha c = 1.9157 \times 10^{-16}$ Js kg^{-2}, where G, α and c are the gravitational constant, the fine–structure constant and the speed of light, respectively, and M is the mass of central body. M is the scaling factor for orbital radii, periods and angular momenta per unit mass of the orbiting bodies. However, according to our model, the speed of revolution and energy per unit mass of the orbiting body are not dependent on the mass M.

In conclusion our analysis indicates that the proposed model can be accepted as a first approximation. Although very simple, the model is consistent and provides some new aspects and a phenomenological basis for further research on stability of gravitationally–bound many–body systems.

Acknowledgements

The authors are indebted to Prof. K. Ilakovac for his interest, encouragement and valuable discussions and suggestions for the final version of the manuscript. We wish to thank also Dr. K. Pavlovski for discussions and providing us with some reprints. We are grateful to Prof. A. Bjeliš for valuable comments.

References

1) L. J. Tomley, Am. J. Phys. **47** (1979) 396;

2) M. M. Nieto, Icarus **25** (1974) 171;

3) R. C. Sapp, Am. J. Phys. **48** (2) (1980) 138;

4) M. A. Blagg, Mont. Not. Roy. Astron. Soc. **73** (1913) 414;

5) S. F. Dermott, Mont. Not. Roy. Astron. Soc. **141** (1968) 363;

6) M. W. Ovenden, Nature **239** (1972) 508;

7) R. Isaacman and C. Sagan, Icarus **31** (1977) 510;

8) T. C. van Flandern, Icarus **36** (1978) 51;

9) H. Alfven, Icarus **3** (1964) 52;

10) W. M. Kaula, Icarus **26** (1975) 1;

11) R. P. Binzel, M. A. Barucci and M. Fulchignoni, Sci. Am. **265** (4) (1991) 66;

12) J. Llibre and C. Piñol, Astron. J. **93** (5) (1987) 1272;

13) Yu. K. Gulak, Astrometria i Astrofizika **16** (1972) 92 (in russian);

14) Yu. K. Gulak, Astron. Zhurn. **57** (1980) 142 (in russian);

15) E. N. Kramer and Yu. M. Gorbanev, Astron. Zhurn. **68** (1991) 1030 (in russian);

16) M. Zeilik and J. Gaustad, *Astronomy-The Cosmic Perspectives*, J. Wiley (1990) a) pp.787-790, b) p.331, c) p.97, d) p.565, e) p.366;

17) *The Astronomical Almanac–1955*, Washington, U. S. Government Printing Office, pp. E88, F2;

18) J. E. Prussing, Am. J. Phys. **45** (12) (1977) 1216;

19) J. K. Beatty, B. O' Leary and A. Chaikin (Editors), *The New Solar System*, Cambridge Univ. Press & Sky Publishing Comp., Cambridge, Ms. (1982) a) J.A.Burns, p.131 and b) p.133, c) J. S. Lewis, p.210, d) J. B. Pollack, p.161;

20) W. R. Ward, in: *Frontiers in Astrophysics*, E. H. Avrett (Editor) Harvard Univ. Press, Cambridge, Ms., and London (1976) pp.1-40;

21) C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, Freeman and Comp., San Francisco, (1973) p.412;

22) J. Avery, *The Quantum Theory of Atoms, Molecules and Photons*, McGraw Hill Book Comp., London (1972), p.65;

23) W. J. Kaufmann III, *Discovering the Universe*, W. H. Freeman and Comp. New York (1990) a) p.102, b) p.216 c) p.286;

24) V. Vujnović, *Astronomija (in Croatian)*, Tome I, Školska knjiga, Zagreb (1989) p.262.

STABILNOST GRAVITACIJSKI–VEZANIH SISTEMA VIŠE TIJELA

ANTUN RUBČIĆ i JASNA RUBČIĆ

Fizički zavod, Prirodoslovno–matematički fakultet, Sveučilišta u Zagrebu, p.p. 162, 41 001 Zagreb, Hrvatska

UDK 523.2, 531.35

PACS 95.10.Ce, 95.10.Fh, 96.30.-t

Velike poluosi putanja planeta i glavnih satelita planetnih sistema u Sunčevom sustavu dane su kvadratičnim zakonom $r_n = \mathrm{const} \times n^2$, gdje je n cijeli broj. Ophodni periodi T_n proporcionalni su s n^3 u skladu s trećim Keplerovim zakonom. Bitna promjena u usporedbi s prijašnjim pristupima je da se terestrički planeti izdvajaju od jovijanskih kao zasebna podgrupa, te im je pridružen $n = 1$, a slijedi Jupiter s $n = 2$, i posljednji Pluton s $n = 6$. Ovaj zaključak je izveden na osnovi astronomskih podataka. Granica između terestričkih i jovijanskih planeta je oko 200 K, što odgovara udaljenosti od Sunca približno 5×10^{11} m (3.4 a.u.). Unutar temperaturne granice, slično s Roche–ovom granicom, ne mogu opstati relativno velika tijela za dani sistem. Relacija za r_n i T_n, kao i sve druge relevantne relacije, ovisne o n, povezane su s diskretizacijom momenta impulsa po jedinici mase tijela na putanji. Masa centralnog tijela je faktor skaliranja i određuje veličinu sistema. Srednja odstupanja opaženih poluosi prema izračunatim radijusima kružnih putanja su od 3.5% do 7.5% zavisno o sistemu. Na osnovi ove analize predložene su hipoteze o stabilnosti gravitacijskih sistema s više tijela.