η and η' in a coupled Schwinger-Dyson and Bethe-Salpeter approach

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Extending our earlier treatments of π^0, η_c, and η_b, we study the η-η' system and its $\gamma\gamma$ decays using a model which is a leading version of the consistently coupled Schwinger-Dyson (SD) and Bethe-Salpeter (BS) approaches. The electromagnetic interactions are incorporated through a (generalized) impulse approximation consistent with this bound-state approach, so that the Ward-Takahashi identities of QED are preserved when quarks are dynamically dressed. To overcome some of the limitations due to the ladder approximation, we introduce a minimal extension to the bound-state approach employed, so that the $U_A(1)$ problem is avoided. Pointing out which of our predictions hold in the coupled SD-BS approach in general, and which are the consequences of the specific, chosen model, we present the results for the axial-current decay constants of η_8, η_0, and of their physical combinations η and η', the results for the $\gamma\gamma$-decay constants of η_0 and η_8, for the two-photon decay widths of η and η', and for the mixing-independent R ratio constructed from them. [S0556-2821(98)03217-2]

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I. INTRODUCTION

A particularly interesting example of the applications of Schwinger-Dyson equations to hadronic physics (reviewed in, e.g., Refs. [1,2]) is the approach through consistently coupled Schwinger-Dyson (SD) equations for quark propagators and Bethe-Salpeter (BS) equations for bound states of quarks. Among various studies of this kind, those of Jain and Munczek [3–5] are judged by many as ‘‘the most extensive and phenomenologically successful spectroscopic studies in the rainbow-ladder approximation’’ [6] and therefore are often chosen [1,2,6–8] as a representative, paradigmatic example of such studies. The essence of such a treatment of $q\bar{q}$ bound states is the solving of the ladder Schwinger-Dyson (SD) equation for the dressed quark propagator $S(q)$, and then solving in the consistent approximation, with this resulting dressed quark propagator and with the same interaction kernel, the Bethe-Salpeter (BS) relativistic bound-state equation for a $q\bar{q}$ meson. This procedure is crucial for obtaining the mesons from the light pseudoscalar octet as Goldstone bosons when the chiral symmetry is spontaneously broken. Thanks to this, a coupled SD-BS approach (notably, Refs. [3–5]) can reproduce the correct chiral limit behavior (crucial in the light sector) simultaneously with the realistic results for heavy mesons. In Refs. [3–5], the interaction kernel is given by a modeled gluon propagator consisting of (a) the well-known perturbative part, reproducing correctly the ultraviolet (UV) asymptotic behavior unambiguously required by QCD in its high-energy perturbative regime and (b) the nonperturbative part, which should describe the infrared (IR) behavior. Since the IR behavior of QCD is still more or less unknown, this latter nonperturbative part of the gluon propagator is modeled. In Refs. [3–5], several forms for this IR part have been used and their parameters varied, with the outcome that results are not very sensitive to such variations. Jain and Munczek [3–5] have succeeded in reproducing the leptonic decay constants of pseudoscalar mesons and, even more importantly, a very large part of the meson spectrum, except for such elusive cases as the η-η' system.

Such an up till now successful and reputable referent model should be tested further by calculating other quantities (e.g., electromagnetic processes) to see how well it will do. This was our motivation for calculating $\pi^0, \eta_c, \eta_b \to \gamma\gamma$, and $\gamma^\star \pi^0 \to \gamma$ in Refs. [9,10], and Jain and Munczek's model passed this test very well. Other applications are also under investigation, and still many others are possible. However, for the full assessment of a model and for getting useful insight in how to improve it, it is also very interesting to see how it performs at the very edges of its applicability. Although Jain and Munczek's model is cleverly constructed so that it works well for most pseudoscalar and vector mesons below, above, and even on the mass scale of η and η', the limitations of the (‘‘improved’’ [2] or ‘‘generalized’’ [7]) ladder approximation employed by the model put the η-η' system on such an ‘‘applicability edge’’ of this model—although not beyond it, contrary to what a pessimist could have concluded. This will be clarified below, where we analyze the η-η' system and its $\gamma\gamma$ decays in Jain and Munczek's model [3–5], demonstrate the abilities and limitations of this model, and anticipate in which way it can be extended to improve further the description of η and η'.

II. SOLVING THE CONSISTENTLY COUPLED SD AND BS EQUATIONS

Dressed quark propagators $S_f(q)$ for various flavors f,

$$S_f^{-1}(q) = A_f(q^2)\not{q} - B_f(q^2) \quad (f = u,d,s, \ldots), \qquad (1)$$

are obtained by solving the SD equation, which in the ladder approximation (i.e., with the true quark-gluon vertex replaced by the bare one, namely, $\gamma^\nu \lambda^j/2$) becomes

$$S_f^{-1}(p)=\not p-\tilde m_f-ig_{\rm st}^2C_F\int\frac{d^4k}{(2\pi)^4}\gamma^\mu S_f(k)\gamma^\nu G_{\mu\nu}(p-k), \tag{2}$$

where $\tilde m_f$ is the bare mass of the quark flavor f, breaking the chiral symmetry explicitly, and C_F is the second Casimir invariant of the quark representation, here 4/3 for the case of the (halved) Gell-Mann matrices $\lambda^j/2$ $(j=1,\dots,8)$ of $SU(3)_c$. Neglecting ghosts, the product of the strong coupling constant $g_{\rm st}$ and the Landau-gauge gluon propagator can be approximated by the *Ansatz* often described as the ''Abelian approximation'' [11]:

$$g_{\rm st}^2C_FG^{\mu\nu}(k)=G(-k^2)\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right). \tag{3}$$

As explained in the Introduction, the function G is given by the sum of the known perturbative part $G_{\rm UV}$, and the modeled nonperturbative part $G_{\rm IR}$:

$$G(Q^2)=G_{\rm UV}(Q^2)+G_{\rm IR}(Q^2)\quad (Q^2=-k^2). \tag{4}$$

In $G_{\rm UV}$, we employ, following Ref. [5], the two-loop asymptotic expression for $\alpha_{\rm st}(Q^2)$

$$\begin{aligned}G_{\rm UV}(Q^2)&=4\pi C_F\frac{\alpha_{\rm st}(Q^2)}{Q^2}\\ &\approx\frac{4\pi^2C_Fd}{Q^2\ln(x_0+Q^2/\Lambda_{\rm QCD}^2)}\\ &\quad\times\left\{1+b\frac{\ln[\ln(x_0+Q^2/\Lambda_{\rm QCD}^2)]}{\ln(x_0+Q^2/\Lambda_{\rm QCD}^2)}\right\},\end{aligned} \tag{5}$$

where $d=12/(33-2N_f)$, $b=2\beta_2/\beta_1^2=2(19N_f/12-51/4)/(N_f/3-11/2)^2$. As in Ref. [5], we set the number of flavors $N_f=5$, $\Lambda_{\rm QCD}=228$ MeV, and $x_0=10$. We adopt the modeled $G_{\rm IR}$, together with its parameters $a=(0.387\ \text{GeV})^{-4}$ and $\mu=(0.510\ \text{GeV})^{-2}$, from Ref. [5]:

$$G_{\rm IR}(Q^2)=4\pi^2C_FaQ^2e^{-\mu Q^2}. \tag{6}$$

Solving Eq. (2) for the propagator functions $A_f(q^2)$ and $B_f(q^2)$ also yields the constituent quark masses, defined (at $q^2=0$ for definiteness) as $\mathcal{M}_f\equiv B_f(0)/A_f(0)$ for the flavor f.

The case $\tilde m_f=0$ corresponds to the chiral limit, where the current quark mass $m_f=0$, and where the constituent quark mass stems exclusively from dynamical chiral symmetry breaking (DχSB) [3]. For u and d quarks, the chiral limit is a very good approximation. Solving Eq. (2) with $\tilde m_u=\tilde m_d=0$ leads to $\mathcal{M}_{ud}=B_u(0)/A_u(0)=B_d(0)/A_d(0)=356$ MeV for the gluon propagator (3)–(6) with parameters quoted above and used in Ref. [5].

When $\tilde m_f\neq 0$, the SD equation (2) must be regularized by a UV cutoff Λ [4,5], and $\tilde m_f$ is in fact a cutoff-dependent quantity. We adopted the parameters of Ref. [5], where (for $\Lambda=134$ GeV) the bare mass $\tilde m_f(\Lambda^2)=3.1$ MeV—chosen to ultimately lead to the realistically massive pion—yields the light nonstrange isosymmetric constituent quark mass $\mathcal{M}_{ud}=375$ MeV, just 5% above its value in the chiral limit. For s quarks, $\tilde m_f(\Lambda^2)$ is 73 MeV, giving us the strange quark constituent mass $\mathcal{M}_s\equiv B_s(0)/A_s(0)=610$ MeV [5].

In the chiral limit, solving of Eq. (2) with $\tilde m_f=0$ is already sufficient to give us the Goldstone pion bound-state vertex[1] Γ_π that is of zeroth order in the pion momentum p,

$$\begin{aligned}\Gamma_{\pi^0}(q;p^2=M_\pi^2=0)&=\frac{\lambda^3}{\sqrt2}\Gamma_{f\bar f}(q;p^2=0)_{m_f=0}\\ &=\frac{\lambda^3}{\sqrt2}\gamma_5\frac{\sqrt2B_f(q^2)_{m_f=0}}{f_\pi},\end{aligned} \tag{7}$$

leading [12,7] to the famous result [Eqs. (26) and (28) below] for the $\pi^0\to\gamma\gamma$ amplitude due to the Abelian Adler-Bell-Jackiw (ABJ), or axial, anomaly.

Of course, for heavier $q\bar q$ composites one cannot circumvent solving the BS equation by invoking the chiral-limit (and the soft-limit, $p^\mu\to 0$) result (7). This is obvious when they contain c or b quarks, for which the whole concept of the chiral limit is of course useless even qualitatively. When strange quarks are present, Eq. (7) can be regarded only as an ''exploratory'' [8] expression and is useful for considering the chiral limit, since this limit is *qualitatively* meaningful for the s quarks. Nonetheless, we need the *quantitative* predictions of Jain and Munczek's model for the $s\bar s$ pseudoscalar bound state, which is not physical, but enters as the heaviest component in the pseudoscalars η and η', introduced in the next section.

Therefore, we must obtain the bound-state vertex $\Gamma_{s\bar s}$ by explicitly solving

$$\begin{aligned}\Gamma_{s\bar s}(q,p)=ig_{\rm st}^2C_F\int\frac{d^4q'}{(2\pi)^4}\gamma^\mu S_s\left(q'+\frac p2\right)\Gamma_{s\bar s}(q',p)\\ \times S_s\left(q'-\frac p2\right)\gamma^\nu G_{\mu\nu}(q-q'),\end{aligned} \tag{8}$$

the homogeneous BS equation again in the ladder approximation, consistently with Eq. (2). For pseudoscalar (P) quarkonia, the complete decomposition of the BS bound state vertex Γ_P in terms of the scalar functions Γ_i^P is

[1] In Eq. (7), we explicitly included the (elsewhere suppressed) flavor factor $\lambda^3/\sqrt2$, appropriate for π^0, to emphasize the change of our convention with respect to Refs. [9,10]: we now adopt the convention of Jain and Munczek's papers [3–5] for the *flavor factors*, but not their conventional color factor of $1/\sqrt{N_c}$. Hence, we have the additional factor of N_c multiplying the integral in Eq. (2.8) of Ref. [4], the formula which otherwise specifies our normalization.

$$\Gamma_P(q,p)=\gamma_5\{\Gamma_0^P(q,p)+\not p\Gamma_1^P(q,p)+\not q\Gamma_2^P(q,p)+[\not p,\not q]\Gamma_3^P(q,p)\}. \tag{9}$$

[The flavor structure is suppressed again. For neutral pseudoscalars, Γ_P is decomposed into $f\bar{f}$ components $\Gamma_{f\bar{f}}$ according to Eq. (15) below.] The BS equation (8) leads to a coupled set of integral equations for the functions Γ_i^P ($i=0,\ldots,3$), which we find to be most easily solved numerically in the Euclidean space by following the procedure of Jain and Munczek [3–5], who formulate the problem in terms of the BS amplitudes $\chi_{f\bar{f}}(q,p)\equiv S_f(q+p/2)\Gamma_{f\bar{f}}(q,p)S_f(q-p/2)$.

In order to avoid the angular integration, we also adopt the momentum expansion (in the Chebyshev polynomials) [3–5] of the four scalar functions appearing in the decomposition of the BS amplitudes. Reference [5] often kept only the lowest order moment in the Chebyshev expansion, because they found it adequate for most of the meson spectrum. In contrast, while using the kernel and parameters of Ref. [5], we always retain all four functions when solving the BS equation (8) and the first two moments in the Chebyshev expansion. The accuracy of this procedure has recently received an independent confirmation—especially for the presently interesting charge conjugation eigenstates—from Maris and Roberts [11]. In their study of π- and K-meson BS amplitudes, they employed *both* the Chebyshev decomposition and straightforward multidimensional integration. Their comparison of these two techniques showed the very quick convergence of the Chebyshev expansion: in the case of equal quark and antiquark masses, such as in the pion, the zeroth and the first Chebyshev moment are enough for an accurate representation of the solution. Even for the kaon, still just one more is needed [11], in spite of the difference in the masses of its constituents. (Of course, the limitations of the ladder approximation would in the end lead to increasing difficulties if one of the fermion messes became much larger still, as recognized also by Ref. [4]. However, if the mass ratio of the constituents is not too large, various contributions beyond ladder approximation largely cancel out in the flavor-nonsinglet pseudoscalar, vector, and axial channels [13,14,11], explaining the success of the ladder approximation in these channels.)

Our procedure, already successfully used in Refs. [9,10] for M_{η_c} and M_{η_b}, gives us $M_{s\bar{s}}=721$ MeV for the unphysical pseudoscalar $s\bar{s}$ bound state entering in the η-η' system in the fashion discussed in the next section. Naturally, when we abandon the chiral limit approximation in Eq. (2), we can also obtain the (isosymmetric) pion bound-state vertex $\Gamma_{\pi^0}=\Gamma_{u\bar{u}}=\Gamma_{d\bar{d}}$, replacing $s\to u$ in Eq. (8). Although we stress that the chiral limit is an excellent approximation for many purposes in the case of pions, including the computation of $\pi^0\to\gamma\gamma$, it is also very important that the experimental π^0 mass $M_{\pi^0}=135$ MeV is reproduced [5] through Eq. (8) as $M_{u\bar{u}}$ ($=M_{d\bar{d}}$) with the small explicit chiral symmetry breaking $\widetilde{m}_{ud}(\Lambda^2)=3.1$ MeV, corresponding to (isosymmetric) current u- and d-quark masses $m=8.73$ MeV, close to the empirical values extracted by current algebra. Such a small m cannot jeopardize the relevance of Eq. (7) for the computation of $\pi^0\to\gamma\gamma$, as shown also by Ref. [15], which found (in an approach closely related to ours) that the amplitude decreased with respect to the analytic chiral-limit axial anomaly result only by less than 1% when they introduced the nonvanishing but small u,d-quark mass $m=6.7$ MeV.

III. $\eta-\eta'$ COMPLEX AND ITS AXIAL-CURRENT DECAY CONSTANTS

The $SU(3)_f$ octet and singlet isospin zero states, η_8 and η_0, are in the $q\bar{q}$ basis given by

$$|\eta_8\rangle=\frac{1}{\sqrt{6}}(|u\bar{u}\rangle+|d\bar{d}\rangle-2|s\bar{s}\rangle), \tag{10}$$

$$|\eta_0\rangle=\frac{1}{\sqrt{3}}(|u\bar{u}\rangle+|d\bar{d}\rangle+|s\bar{s}\rangle). \tag{11}$$

In our phenomenologically successful model choice [5], the flavor $SU(3)_f$ symmetry is broken by the s-quark mass being realistically larger than the u,d masses. Nevertheless, the isospin symmetry for u and d quarks is assumed exact throughout this paper. As is most commonly done, Eqs. (10) and (11) both employ the same quark basis states $|f\bar{f}\rangle$ ($f=u,d,s$) to define η_8 and η_0. As pointed out by Gilman and Kauffman [16] (following Chanowitz, their Ref. [8]), this usual procedure implicitly assumes nonet symmetry. However, it is ultimately broken by nonabelian (''gluon'') axial anomaly, which will be discussed in Sec. V.

η_8 and η_0 cannot be physical as they are not the mass eigenstates. However, that are their mixtures η and η':

$$|\eta\rangle=\cos\theta|\eta_8\rangle-\sin\theta|\eta_0\rangle, \tag{12}$$

$$|\eta'\rangle=\sin\theta|\eta_8\rangle+\cos\theta|\eta_0\rangle. \tag{13}$$

The determination of the specific value that the mixing angle θ should take is a difficult issue which will be handled separately in Sec. V. We will keep our discussion general until we evaluate those of our results which are independent of the mixing and θ—such as the decay constants of the unmixed states η_8 and η_0—and point out those quantities for evaluation of which we need a concrete value of θ.

For the light neutral pseudoscalar mesons $P=\pi^0,\eta_8,\eta_0$, their axial-current decay constants $f_P=f_{\pi^0}$, f_{η_8} and f_{η_0}, are defined by the matrix elements

$$\left\langle 0\left|\bar{\psi}(0)\gamma^\mu\gamma_5\frac{\lambda^j}{2}\psi(0)\right|P(p)\right\rangle=i\delta^{jP}f_Pp^\mu, \tag{14}$$

where $\psi=(u,d,s)$ is the fundamental representation of $SU(3)_f$, while $P=\pi^0,\eta_8,\eta_0$ simultaneously has the meaning of the respective $SU(3)_f$ indices 3,8,0. This picks out the diagonal ($j=3,8$) $SU(3)_f$ Gell-Mann matrices λ^j and $\lambda^0\equiv(\sqrt{2/3})\mathbf{1}_3$ in Eq. (14).

The neutral pseudoscalars P are expressed through the quark basis states $|f\bar{f}\rangle$ by

$$|P\rangle=\sum_f \left(\frac{\lambda^P}{\sqrt{2}}\right)_{ff}|f\bar{f}\rangle\equiv\sum_f a_f^P|f\bar{f}\rangle \quad (f=u,d,s), \quad (15)$$

where the nonvanishing coefficients $a_f^P\equiv(\lambda^P/\sqrt{2})_{ff}$ for $P=\pi^0$ are $a_u^{\pi^0}=-a_d^{\pi^0}=1/\sqrt{2}=(\lambda^3/\sqrt{2})_{11}$, whereas for η_8 they are $a_u^{\eta_8}=a_d^{\eta_8}=1/\sqrt{6}=(\lambda^8/\sqrt{2})_{11}=(\lambda^8/\sqrt{2})_{22}$, $a_s^{\eta_8}=-2/\sqrt{6}=(\lambda^8/\sqrt{2})_{33}$, and for η^0, $a_u^{\eta_0}=a_d^{\eta_0}=a_s^{\eta_0}=(\lambda^0/\sqrt{2})_{ff}=1/\sqrt{3}$.

The axial-current decay constants defined in Eq. (14) can be expressed as

$$f_P=\sum_{f=u,d,s}\frac{(\lambda^P{}_{ff})^2}{2}f_{f\bar{f}}$$

$$(P=\pi^0\leftrightarrow 3,\ \ \eta_8\leftrightarrow 8,\ \text{and}\ \ \eta_0\leftrightarrow 0), \quad (16)$$

where we have for convenience introduced the auxiliary decay constant $f_{f\bar{f}}$, defined as the decay constant of the $f\bar{f}$-pseudoscalar bound state which has the mass $M_{f\bar{f}}$ and is described by the BS vertex $\Gamma_{f\bar{f}}(q,p)$, so that using the definitions of Bethe-Salpeter bound-state amplitudes or vertices in the matrix elements (14) as in, e.g., Refs. [3–6], leads to

$$f_{f\bar{f}}=i\frac{N_c}{\sqrt{2}}\frac{1}{M_{f\bar{f}}^2}\int\frac{d^4q}{(2\pi)^4}\mathrm{tr}\left[\not{p}\gamma_5 S_f\left(q+\frac{p}{2}\right)\Gamma_{f\bar{f}}(q,p)\right.$$
$$\left.\times S_f\left(q-\frac{p}{2}\right)\right]. \quad (17)$$

It turns out that this equation can also be applied for $M_{f\bar{f}}=0$, as the limit exists.

In the isospin limit, we get $f_{\pi^0}\equiv f_{u\bar{u}}=f_{d\bar{d}}=f_{u\bar{d}}\equiv f_\pi=93.2$ MeV in our chosen model [5]. For the axial decay constant of the $s\bar{s}$ pseudoscalar bound state, we obtain $f_{s\bar{s}}=136.5$ MeV$=1.47f_\pi$. (This factor with respect to f_π is very reasonable and even expected, since the model [5] also predicts the decay constant of the charged kaon $f_{K^+}=f_{u\bar{d}}=114$ MeV$=1.23f_\pi$.) Equation (16) then yields $f_{\eta_8}=122.1$ MeV and $f_{\eta_0}=107.6$ MeV. Note that $f_{\eta_8}=1.31f_\pi$, which is rather close to the result $f_{\eta_8}=1.25f_\pi$ [17], obtained in the chiral perturbation theory (χPT). Evaluating the matrix elements of the pertinent mixtures yields the η and η' decay constants

$$f_\eta=\left(\frac{1}{\sqrt{3}}\cos\theta-\frac{\sqrt{2}}{\sqrt{3}}\sin\theta\right)^2 f_\pi+\left(-\frac{\sqrt{2}}{\sqrt{3}}\cos\theta-\frac{1}{\sqrt{3}}\sin\theta\right)^2 f_{s\bar{s}}, \quad (18)$$

$$f_{\eta'}=\left(\frac{\sqrt{2}}{\sqrt{3}}\cos\theta+\frac{1}{\sqrt{3}}\sin\theta\right)^2 f_\pi+\left(\frac{1}{\sqrt{3}}\cos\theta-\frac{\sqrt{2}}{\sqrt{3}}\sin\theta\right)^2 f_{s\bar{s}}. \quad (19)$$

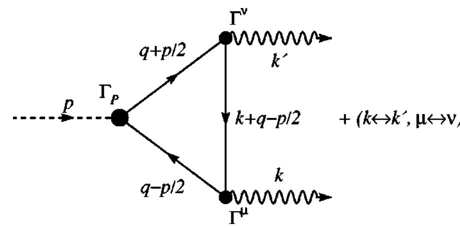

FIG. 1. The diagram for $P\to\gamma\gamma$ decays ($P=\pi^0,\eta,\eta',\ldots$). Within the scheme of generalized impulse approximation, the propagators and vertices are dressed.

IV. $\pi^0,\eta_8,\eta_0\to\gamma\gamma$, AND $\eta,\eta'\to\gamma\gamma$ PROCESSES

The transition amplitudes for $\eta,\eta'\to\gamma\gamma$ can be obtained from the $\gamma\gamma$-transition amplitudes for η_8 and η_0 by forming the appropriate mixtures, in line with Eqs. (10)–(13). The $\eta_8,\eta_0\to\gamma\gamma$ amplitudes are in turn calculated in the same way as $\pi^0,\eta_c,\eta_b\to\gamma\gamma$ in Refs. [9,10].

This means that we assume that these decays proceed through the triangle graph (depicted in Fig. 1), and that we calculate the pertinent amplitudes [18]

$$T_P^{\mu\nu}(k,k')=\varepsilon^{\alpha\beta\mu\nu}k_\alpha k'_\beta T_P(k^2,k'^2), \quad (20)$$

and the corresponding on-shell ($k^2=0$ and $k'^2=0$) decay widths

$$W(P\to\gamma\gamma)=\frac{\pi\alpha_{\mathrm{em}}^2}{4}M_P^3|T_P(0,0)|^2 \quad (P=\pi^0,\eta,\eta',\ldots), \quad (21)$$

using the framework advocated by (for example) Refs. [12,7,15,8,19] in the context of electromagnetic interactions of BS bound states, and often called the generalized impulse approximation (GIA), e.g., by Refs. [15,8]. To evaluate the triangle graph, we therefore use the *dressed* quark propagator $S_f(q)$, Eq. (1), and the pseudoscalar BS bound-state vertex $\Gamma_P(q,p)$ instead of the bare γ_5 vertex. Another ingredient, crucial for the GIA's ability to reproduce the correct Abelian anomaly result, is employing an appropriately dressed *electromagnetic* vertex $\Gamma_f^\mu(q',q)$, which satisfies the vector Ward-Takahashi identity (WTI):

$$(q'-q)_\mu\Gamma_f^\mu(q',q)=S_f^{-1}(q')-S_f^{-1}(q) \quad (f=u,d,s,\ldots). \quad (22)$$

Namely, assuming that photons couple to quarks through the bare vertex γ^μ would be inconsistent with our quark propagator, which, dynamically dressed through Eq. (2), contains the momentum-dependent functions $A_f(q^2)$ and $B_f(q^2)$. The bare vertex γ^μ obviously violates Eq. (22), implying the nonconservation of the electromagnetic current and of the electric charge. Since solving the pertinent SD equation for the dressed quark-photon vertex Γ_f^μ is a difficult problem that has only recently begun to be addressed [20], it is customary to use realistic *Ansätze*. Following, e.g., Refs. [15,8,7,19], we choose the Ball-Chiu [21] vertex

$$\Gamma_f^\mu(q',q)=A_+^f(q'^2,q^2)\frac{\gamma^\mu}{2}+\frac{(q'+q)^\mu}{(q'^2-q^2)}\times\left\{A_-^f(q'^2,q^2)\frac{(\not q'+\not q)}{2}-B_-^f(q'^2,q^2)\right\}, \tag{23}$$

where $H_\pm^f(q'^2,q^2)\equiv[H_f(q'^2)\pm H_f(q^2)]$, for $H=A$ or B. This *Ansatz* (i) satisfies the WTI (22), (ii) reduces to the bare vertex in the free-field limit as must be in perturbation theory, (iii) has the same transformation properties under Lorentz transformations and charge conjugation as the bare vertex, (iv) has no kinematic singularities, and (v) does not introduce any new parameters as it is completely determined by the quark propagator (1).

For the meson P whose flavor content is given by Eq. (15), the GIA yields the amplitude

$$\begin{aligned}T_P^{\mu\nu}(k,k')=&\sum_{f=u,d,s}a_f^P Q_f^2 N_c(-)\int\frac{d^4q}{(2\pi)^4}\\&\times\mathrm{tr}\left\{\Gamma_f^\mu\left(q-\frac{p}{2},k+q-\frac{p}{2}\right)S_f\left(k+q-\frac{p}{2}\right)\right.\\&\times\Gamma_f^\nu\left(k+q-\frac{p}{2},q+\frac{p}{2}\right)\\&\left.\times S_f\left(q+\frac{p}{2}\right)\Gamma_{f\bar f}(q,p)S_f\left(q-\frac{p}{2}\right)\right\}\\&+(k\leftrightarrow k',\mu\leftrightarrow\nu).\end{aligned} \tag{24}$$

The coefficients a_f^P of various flavor components $|f\bar f\rangle$ in $P=\pi^0,\eta_8,\eta_0$, are given below Eq. (15). Q_f denotes the charge of the quark flavor f. The dependence on the flavor f has been indicated on the BS vertices, dressed propagators and electromagnetic vertices in the loop integral for each quark flavor. It is convenient to separate out the a_f^P and Q_f^2 dependence by denoting each integral [times $(-N_c)$] in Eq. (24) the ''reduced $\gamma\gamma$ amplitude'' $\widetilde T_{f\bar f}^{\mu\nu}$. The ''reduced scalar amplitude'' $\widetilde T_{f\bar f}$ for the flavor f is then

$$\widetilde T_{f\bar f}^{\mu\nu}(k,k')=\varepsilon^{\alpha\beta\mu\nu}k_\alpha k'_\beta\widetilde T_{f\bar f}(k^2,k'^2). \tag{25}$$

A. $\gamma\gamma$ amplitudes and $\gamma\gamma$-decay constants

Regardless of what the chiral-limit solutions for the propagator (1) and the bound-state vertex (7) are in detail, $\widetilde T_{f\bar f}^{\mu\nu}(0,0)$ can be evaluated analytically in the chiral (and soft) limit [7,12], which is perfectly adequate for $f=u,d$, i.e., for a Goldstone $P=\pi^0$. There,

$$\widetilde T_{\pi^0}(0,0)\equiv\widetilde T_{u\bar u}(0,0)=\widetilde T_{d\bar d}(0,0)=\frac{N_c}{2\sqrt2\pi^2 f_\pi}, \tag{26}$$

to which we stick throughout. In terms of

$$T_P(k^2,k'^2)\equiv\sum_f a_f^P Q_f^2\widetilde T_{f\bar f}(k^2,k'^2), \tag{27}$$

this leads to the standard form of the successful axial-anomaly result[2] for $\pi^0\to\gamma\gamma$:

$$T_{\pi^0}(0,0)=\frac{N_c}{2\sqrt2\pi^2 f_\pi}\sum_f a_f^{\pi^0}Q_f^2=\frac{1}{4\pi^2 f_\pi}. \tag{28}$$

Note that this reproduction of the chiral limit relation between the $\pi^0\to\gamma\gamma$ decay amplitude and the pion axial-current decay constant f_π, is not dependent on the pion's internal structure (or the interaction kernel that produces it) in any way [7,12] and this is an important advantage of the coupled SD-BS approach over most other bound-state approaches, since the axial anomaly is on fundamental grounds known to be independent of the structure. (Those calculations of $\pi^0\to\gamma\gamma$ which rely on the details of the hadronic structure, be it in the context of the BS equation without dynamical chiral symmetry breaking (DχSB), nonrelativistic quarks, or otherwise, have problems describing this decay accurately even when the model parameters are fine-tuned for that purpose; e.g., see Refs. [22–25], and references therein. The most successful of these model fits, Ref. [24], numerically obtains the width of 7.6 eV at the expense of fine-tuning constituent quark masses to unusually small values.) Of course, f_π itself *is* structure dependent. It is a *calculated quantity* in the SD-BS approach. Our model choice [5] successfully reproduces the experimental value of f_π, and this is obviously of utmost importance for the theoretical description of anomalous processes.

The implications thereof for the η_8,η_0 and their mixtures η and η' are now clear, because those parts of their $\gamma\gamma$-decay amplitudes which stem from their $u\bar u$ and $d\bar d$ components are (just as in π^0) accurately given by the Abelian anomaly [i.e., Eq. (26)] for *any* interaction kernel which leads to the correct f_π—be it the present one, or some improved one. In other words, Eq. (26) implies that the uncertainty (in the $\gamma\gamma$ amplitudes) due to modeling of the interaction kernel and the resulting bound state, is to a large extent cornered into the $s\bar s$ sector, since only $\widetilde T_{s\bar s}(0,0)$, the $\gamma\gamma$-decay amplitude of the relatively heavy $s\bar s$-pseudoscalar, has to be evaluated numerically. From Eqs. (24)–(25), we find numerically that in the model of Ref. [5], $\widetilde T_{s\bar s}(0,0)=0.62\widetilde T_{u\bar u}(0,0)$.

The $\pi^0\to\gamma\gamma$ decay amplitude $T_{\pi^0}(0,0)$ at any pion mass can be used as a definition of pionic $\gamma\gamma$-decay constant $\overline{f}_\pi$ through $T_{\pi^0}(0,0)\equiv1/4\pi^2\overline{f}_\pi=(N_c/2\sqrt2\pi^2\overline{f}_\pi)\Sigma_f a_f^{\pi^0}Q_f^2$. Equation (28) then reveals that $\overline{f}_\pi=f_\pi$ in the chiral limit, which result is well-known from the axial anomaly analysis. Although the chiral limit formula (28) can be applied without reservations only to pions, it is for historical reasons custom-

[2]We can also get, in the fashion of Ref. [19], the anomalous ''box'' amplitude for $\gamma^\star\to\pi\pi\pi$.

ary to write the amplitudes for $\eta_8, \eta_0 \to \gamma\gamma$ in the same form as Eq. (28), defining thereby the $\gamma\gamma$-decay constants $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$:

$$T_{\eta_8}(0,0) \equiv \frac{N_c}{2\sqrt{2}\pi^2 \overline{f}_{\eta_8}} \sum_f a_f^{\eta_8} Q_f^2 = \frac{f_\pi}{\overline{f}_{\eta_8}} \frac{T_{\pi^0}(0,0)}{\sqrt{3}}, \quad (29)$$

$$T_{\eta_0}(0,0) \equiv \frac{N_c}{2\sqrt{2}\pi^2 \overline{f}_{\eta_0}} \sum_f a_f^{\eta_0} Q_f^2 = \frac{f_\pi}{\overline{f}_{\eta_0}} \frac{\sqrt{8} T_{\pi^0}(0,0)}{\sqrt{3}}. \quad (30)$$

As pointed out by Ref. [26], $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$ are not *a priori* simply connected with the usual axial-current decay constants f_{η_8} and f_{η_0}, in contradistinction to the pion case, where $f_\pi = \overline{f}_\pi$ because the chiral limit is such a good approximation for pions.

Equations (27)–(30) reveal that in the present approach $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$ are naturally expressed through f_π [i.e., through $\widetilde{T}_{u\bar{u}}(0,0) = \widetilde{T}_{d\bar{d}}(0,0)$ evaluated in the chiral limit], and $\widetilde{T}_{s\bar{s}}(0,0)$, the $\gamma\gamma$-decay amplitude of the unphysical pseudoscalar $s\bar{s}$ bound state, calculated for nonvanishing m_s. Our predictions for $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$ are thus

$$\overline{f}_{\eta_8} = \frac{3 f_\pi}{5 - 4\pi^2 \sqrt{2} f_\pi \widetilde{T}_{s\bar{s}}(0,0)/N_c}, \quad (31)$$

$$\overline{f}_{\eta_0} = \frac{6 f_\pi}{5 + 2\pi^2 \sqrt{2} f_\pi \widetilde{T}_{s\bar{s}}(0,0)/N_c}. \quad (32)$$

Derivation of Eqs. (31) and (32) shows that irrespective of any specific model choice, any $q\bar{q}$ bound-state approach (such as our coupled SD-BS approach in conjunction with the GIA) *which has the merit* of reproducing the anomalous $\pi^0 \to \gamma\gamma$ amplitude in the chiral limit, Eq. (26) or (28), should give the relations (31) and (32) for $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$, when pions are approximated by the chiral limit. The concrete numerical values of $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$ depend on what the model predictions for f_π and $\widetilde{T}_{s\bar{s}}$ are.

Since in the coupled SD-BS approach we can numerically evaluate $\widetilde{T}_{s\bar{s}}(0,0)$ for arbitrary values of the s-quark mass, Eqs. (31) and (32) give our predictions for the effects of the SU(3)$_f$ breaking on $\gamma\gamma$ decays in the η-η' system. In the SU(3)$_f$ limit (where $\widetilde{T}_{s\bar{s}} = \widetilde{T}_{u\bar{u}}$) and the chiral limit applied also to s quarks, we obviously recover $\overline{f}_{\eta_8} = f_\pi$, but also $\overline{f}_{\eta_0} = f_\pi$, since nonet symmetry in the sense of Ref. [16] is a starting assumption of ours.

For our present model choice [5], where $\widetilde{T}_{s\bar{s}}(0,0) = 0.62 \widetilde{T}_{u\bar{u}}(0,0)$, Eqs. (31) and (32) give

$$\overline{f}_{\eta_8} = 73.64 \text{ MeV} = 0.797 f_\pi, \quad (33)$$

$$\overline{f}_{\eta_0} = 98.58 \text{ MeV} = 1.067 f_\pi. \quad (34)$$

While this $\overline{f}_{\eta_0}$ agrees with the results of chiral perturbation theory (χPT) [17,27], there is a difference concerning $\overline{f}_{\eta_8}$, since $\overline{f}_{\eta_8} > f_\pi$ in χPT. This is important because the value of $f_\pi / \overline{f}_{\eta_8}$ has impact on the possible values of the η-η' mixing angle. We therefore devote the following subsection to the discussion of this result and the meaning of this difference.

B. In the SD-BS approach, $\overline{f}_{\eta_8} < f_\pi$ generally

The $\overline{f}_{\eta_8}$ value (33) is a result of a specific model. However, for the s-quark mass realistically heavier than the u,d-quark masses, $\overline{f}_{\eta_8} < f_\pi$ holds in the coupled SD-BS approach generally, i.e., independently of chosen model details. To see this, let us start by noting that $\overline{f}_{\eta_8} < f_\pi$ is equivalent to $T_{\eta_8}(0,0) > T_{\pi^0}(0,0)/\sqrt{3}$, and since we can rewrite Eq. (27) for η_8 as

$$T_{\eta_8}(0,0) = \frac{T_{\pi^0}(0,0)}{\sqrt{3}} + \frac{1}{9} \frac{2}{\sqrt{6}} [\widetilde{T}_{d\bar{d}}(0,0) - \widetilde{T}_{s\bar{s}}(0,0)], \quad (35)$$

the inequality $\overline{f}_{\eta_8} < f_\pi$ is in our approach simply the consequence of the fact that the (''reduced'') $\gamma\gamma$ amplitude of the $s\bar{s}$-pseudoscalar bound state $\widetilde{T}_{s\bar{s}}$ is smaller than the corresponding nonstrange $\gamma\gamma$ amplitude $\widetilde{T}_{d\bar{d}}$ ($= \widetilde{T}_{u\bar{u}} = \widetilde{T}_{\pi^0}$ in the isosymmetric limit), for any realistic relationship between the nonstrange and much larger strange quark masses.

Only in the chiral limit (and close to it), subtle cancellations between the bound-state vertices, WTI-preserving $qq\gamma$ vertices and dynamically dressed propagators lead to the large anomalous amplitude (26), or its slight modification (the size of which is controlled by Veltman-Sutherland theorem) for small u and d masses. Significantly away from the chiral limit, what happens is basically a simple suppression of $\widetilde{T}_{f\bar{f}}(0,0)$ by the large quark mass in the propagators in the triangle loop of Fig. 1. Essentials and generality of the suppression mechanism can be understood in basic terms in two (related) ways, through the *simple free quark loop* (QL) model and the Goldberger-Treiman (GT) relation.

(i) In a QL model (e.g., see Ref. [28], and references therein), the strength of the Yukawa point couplings of the free quarks of the flavor f to the pseudoscalar P is given by the constant g_f, and quarks have constant constituent masses $\mathcal{M}_f$ [in contradistinction to the momentum-dependent mass functions $\mathcal{M}_f(q^2) = B_f(q^2)/A_f(q^2)$ in our framework]. Up to some arcsine-type dependence unessential here, each flavor f then contributes simply $(g_{Pff}/\mathcal{M}_f) Q_f^2 \equiv (g_f/\mathcal{M}_f) a_f^P Q_f^2$ to the triangle-loop $\gamma\gamma$ amplitude [28]. In the case of the strictly SU(3)$_f$-symmetric coupling, the Yukawa couplings would be the same for all flavors, $g_f = g$. The broken SU(3)$_f$ symmetry implies that g_f *can* differ for various flavors f, but not by much, so that relative strengths of the factors $g_f/\mathcal{M}_f$

for various flavors is essentially determined by $1/\mathcal{M}_f$. Actually, this is what we find in our SD-BS framework, where the pseudoscalar bound-state vertices $\Gamma_{f\bar{f}}$ are analogous to the coupling g_f in the QL model, and $g_f/\mathcal{M}_f$ is analogous to our ''reduced'' amplitude $\widetilde{T}_{f\bar{f}}(0,0)$. Obviously, our approach allows for the flavor dependence of our BS $Pq\bar{q}$ vertices $\Gamma_{f\bar{f}}$, but due to the fact that the broken SU$(3)_f$ is still an approximate symmetry, their variation with the breaking, given in terms of strange-to-nonstrange constituent mass ratio, is rather weak and cannot influence much the suppression occurring as the constituent mass in the denominator grows significantly. Hence, essentially the same mechanism is at work as in the QL model. That this parallel works very well, can be seen from the fact that the inverse of the strange-to-nonstrange constituent mass ratio in our SD-BS model, namely, 1.63, quite accurately reproduces the suppression of the $s\bar{s}$ decay amplitude $\widetilde{T}_{s\bar{s}}(0,0)=0.62\widetilde{T}_{u\bar{u}}(0,0)$, found numerically from Eqs. (24) and (25).

(ii) A related way to see the same effect is to apply the quark-level GT relation, $g_f/\mathcal{M}_f=1/f_{f\bar{f}}$, to the pseudoscalars with the $f\bar{f}$ quark content in the QL model. Then, roughly the same suppression factor occurs again, due to $f_{s\bar{s}}=1.47f_\pi$. This is only roughly, since $s\bar{s}$ is further away from the chiral limit than $u\bar{u}$ and $d\bar{d}$ constituting the pion. Nevertheless, invoking the GT relation is in fact a very robust way to show that $\widetilde{T}_{s\bar{s}}(0,0)<\widetilde{T}_{u\bar{u}}(0,0)$ must surely hold, even though s quarks are much lighter than c or b quarks (where the suppression is by orders of magnitude [9]), and DχSB is for s quarks of importance similar to that for u,d quarks. Precisely because the chiral limit makes sense for s quarks *qualitatively* (as the pseudoscalars containing s quarks can still be considered pseudo-Goldstone bosons), the GT relation must continue to hold *approximately* in the $s\bar{s}$ sector, regardless of any specific interaction kernel and of the resulting hadronic structure. $\widetilde{T}_{s\bar{s}}(0,0)<\widetilde{T}_{u\bar{u}}(0,0)$ is therefore obligatory simply due to $f_{s\bar{s}}>f_\pi$.

The GT relation is useful also for demonstrating the robustness even of our model-dependent result on $\widetilde{T}_{s\bar{s}}(0,0)$, namely, that in spite of the model dependence in the s-quark sector, the model kernel of our choice [5] should not lead to $\gamma\gamma$-amplitude $\widetilde{T}_{s\bar{s}}(0,0)$ excessively different than the ones which would result from an improved kernel. The usage of the GT relation at the quark level is especially transparent in the context of the simple free quark loop model in which the GT relation $g_f/\mathcal{M}_f=1/f_{f\bar{f}}$ is necessary for reproducing the $\gamma\gamma$ anomaly amplitudes (26) and (28). In the context of the coupled SD-BS approach, with its dynamically dressed quarks and BS vertices, the GT relation for quarks and Goldstone bosons is given by the chiral-limit relation (7) (see, e.g., Ref. [29]). The chiral-limit analytic derivation of Eq. (26) from (7), with its subtle interplay and cancellations between the bound-state vertex, WTI-preserving $qq\gamma$ vertices, and dynamically dressed propagators, transparently demonstrates the way the GT relation works for $\gamma\gamma$ decays in this context. For massive pions, the $\gamma\gamma$ amplitude must be evaluated numerically, but in fact changes very little, implying that the GT relation continue to hold very accurately. As just argued above, in (ii), the GT relation should still hold as a rough approximation for the $s\bar{s}$ pseudoscalar bound state. This, together with Eq. (26), implies that $\widetilde{T}_{s\bar{s}}\sim N_c/(2\sqrt{2}\pi^2 f_{s\bar{s}})$. For $f_{s\bar{s}}=1.47f_\pi$ which we obtained in the model [5], this gives the GT relation-based estimate $\widetilde{T}_{s\bar{s}}\sim 0.68\widetilde{T}_{u\bar{u}}$. This is indeed in expected rough agreement with the accurate, numerically obtained prediction of the model [5], that $\widetilde{T}_{s\bar{s}}(0,0)=0.62\widetilde{T}_{u\bar{u}}(0,0)=N_c/(2\sqrt{2}\pi^2 1.61 f_\pi)$. This shows that our result is quite reasonable. We remark that any model which is successful enough to reproduce empirical values of f_π and f_{K^+}, should give a value for $f_{s\bar{s}}$ close to ours, since anything very different than the estimate $f_{s\bar{s}}\sim f_\pi+2(f_{K^+}-f_\pi)$ would be unreasonable. Bound state descriptions that would be obtained by using kernels supposedly better than ours (improved beyond the ladder approximation by, say, including fully the *gluon* anomaly), must retain the good feature of agreeing approximately with the GT relation. This means that improving interaction kernels and, consequently, $f\bar{f}$ bound states, would not change very much the $\gamma\gamma$ amplitudes with respect to our $\widetilde{T}_{f\bar{f}}$ even in the s-quark sector.

Regardless of any specific model realization of the coupled SD-BS approach, Eqs. (31) and (32) with $\widetilde{T}_{s\bar{s}}(0,0)\leqslant\widetilde{T}_{\pi^0}(0,0)$ imply the following bounds on $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$. The equality holds when the chiral limit is applied to all three flavors, implying that f_π is the upper bound for $\overline{f}_{\eta_8}$ and lower bound for $\overline{f}_{\eta_0}$. As the s-quark mass grows, $\widetilde{T}_{s\bar{s}}(0,0)$ gradually diminishes so that the lower bound for $\overline{f}_{\eta_8}$ is $0.6f_\pi$ and the upper bound for $\overline{f}_{\eta_0}$ is $1.2f_\pi$.

Let us now address the meaning of the apparent contradiction between the results of the coupled SD-BS approach on $\overline{f}_{\eta_8}/f_\pi$ and the corresponding results of χPT, as well as possible ways to, at least in principle, overcome it. To this end, let us recall Pham's paper [30] on $q\bar{q}$-loop corrections δ to the χPT result on $\overline{f}_{\eta_8}/f_\pi$. In the notation of our Eq. (29), he gets

$$T_{\eta_8}(0,0)=\frac{f_\pi}{f_{\eta_8}}(1-\delta)\frac{T_{\pi^0}(0,0)}{\sqrt{3}}, \tag{36}$$

where the axial-current decay constant f_{η_8} appears. From the standpoint of our approach [see Eq. (29) relating the $\gamma\gamma$ amplitudes of η_8 and the chiral pion], $f_{\eta_8}/(1-\delta)$ obviously corresponds to *our* $\overline{f}_{\eta_8}$. Can δ resolve the discrepancy between our $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_8}=f_{\eta_8}=1.25f_\pi$ of χPT [17]? Pham's rough estimate is $-0.28<\delta<-0.19$. This would practically reduce the χPT result from $1.25f_\pi$ down to f_π. In addition, his (following Ref. [31]) result

$$\delta=-\frac{8}{3}\left(1+\ln\frac{\Lambda^2}{M^2}\right)\left(\frac{m_s M}{\Lambda^2}\right) \tag{37}$$

can be even more negative than -0.28, because there are large uncertainties in the value of his cutoff Λ (which can be lower than the lowest value of 1.2 GeV used by Ref. [30]) and in his effective s-quark mass M. In addition, the current s-quark mass m_s can be higher [32] than $m_s=175$ MeV used by Pham. Each of these possibilities would make δ more negative. On the other hand, the χPT results [17] $\overline{f}_{\eta_8}=f_{\eta_8}$ on the equality of the axial and $\gamma\gamma$ decay constants and $f_{\eta_8}=(1+\alpha_8)f_\pi=1.25f_\pi$ on $SU(3)_f$ breaking effects in f_{η_8}, are obtained in the one-loop approximation. Higher loops can introduce significant changes, including polynomial terms in meson masses, which may be both large with respect to chiral logarithms, and also not unambiguous [33,17]. It is thus possible that $\alpha_8<0.25$ after all, which could make even easier for the correction factor $(1-\delta)$ to reduce $\overline{f}_{\eta_8}$, as defined by us in Eq. (29), even below f_π.

Large uncertainties in the quantities entering in the estimate for $q\overline{q}$ correction (37), obviously leave plenty of room for usage of $q\overline{q}$ bound state models such as ours, which properly embed the chiral behavior of the underlying theory. Including meson loops in $q\overline{q}$ bound state approaches is a difficult task and implies going beyond the ladder approximation, but it would help further diminish the gap that Pham [30] started closing from the side of χPT by incorporating into it the corrections due to quark loops.

C. $\eta,\eta'\to\gamma\gamma$ decay widths

The $\eta,\eta'\to\gamma\gamma$ amplitudes are given in terms of the $\gamma\gamma$ amplitudes of η_8 and η_0 as

$$T_\eta(0,0)=\cos\theta\, T_{\eta_8}(0,0)-\sin\theta\, T_{\eta_0}(0,0), \tag{38}$$

$$T_{\eta'}(0,0)=\sin\theta\, T_{\eta_8}(0,0)+\cos\theta\, T_{\eta_0}(0,0). \tag{39}$$

Expressing $T_{\eta_8}(0,0)$ and $T_{\eta_0}(0,0)$ through the $\gamma\gamma$-decay constants $\overline{f}_{\eta_8}$ (31) and $\overline{f}_{\eta_0}$ (32), we arrive at the standard (e.g., see Ref. [26]) formulas for the η and η' decay widths

$$W(\eta\to\gamma\gamma)=\frac{\alpha_{\rm em}^2}{64\pi^3}\frac{M_\eta^3}{3f_\pi^2}\left[\frac{f_\pi}{\overline{f}_{\eta_8}}\cos\theta-\sqrt{8}\frac{f_\pi}{\overline{f}_{\eta_0}}\sin\theta\right]^2, \tag{40}$$

$$W(\eta'\to\gamma\gamma)=\frac{\alpha_{\rm em}^2}{64\pi^3}\frac{M_{\eta'}^3}{3f_\pi^2}\left[\frac{f_\pi}{\overline{f}_{\eta_8}}\sin\theta+\sqrt{8}\frac{f_\pi}{\overline{f}_{\eta_0}}\cos\theta\right]^2. \tag{41}$$

The version of Eqs. (40) and (41) in which the axial-current decay constants f_{η_8} and f_{η_0} appear in place of $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$, requires a derivation where PCAC (partial conservation of axial vector current) and the soft meson technique is applied to η and η' [34]. These assumptions are impeccable for pions (leading to $f_\pi=\overline{f}_\pi$), but *not* for the η-η' complex. In fact, the latter is quite dubious for the heavy η' [34]. We do not need and do not use these assumptions since we *directly* calculate the $\eta_8,\eta_0\to\gamma\gamma$ amplitudes, i.e., $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$. We also calculate f_{η_8} and f_{η_0} independently of the $\gamma\gamma$ processes.

For the values of $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$ obtained in our model choice [5], namely, Eqs. (33) and (34), the best achievable consistency with the present overall fit [35] to the experimental widths

$$W^{\rm exp}(\eta\to\gamma\gamma)=(0.46\pm0.04)\text{keV} \quad \text{and}$$

$$W^{\rm exp}(\eta'\to\gamma\gamma)=(4.26\pm0.19)\text{keV} \tag{42}$$

then occurs for $\theta\equiv\theta^{\rm exp}=-12.0°$ (obtained through χ^2 minimization). Then

$$W(\eta\to\gamma\gamma)=0.54 \ \text{keV}, \tag{43}$$

$$W(\eta'\to\gamma\gamma)=5.0 \ \text{keV}. \tag{44}$$

However, the present approach is capable of *predicting* the mixing angle θ, and it remains to be seen if the predicted θ can be close to the angle favored by the experimental $\gamma\gamma$ widths.

The issue of predicting θ will be addressed in the next section. There is another mixing-independent quantity related to $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$, which we can predict before predicting θ. It is the R ratio, which is in fact measurable because it is the combination of π^0, η, and η' widths:

$$R\equiv\left[\frac{W(\eta\to\gamma\gamma)}{M_\eta^3}+\frac{W(\eta'\to\gamma\gamma)}{M_{\eta'}^3}\right]\frac{M_\pi^3}{W(\pi^0\to\gamma\gamma)} = \frac{1}{3}\left(\frac{f_\pi^2}{\overline{f}_{\eta_8}^2}+8\frac{f_\pi^2}{\overline{f}_{\eta_0}^2}\right), \tag{45}$$

which is presently not known with satisfactory precision; Ref. [36] quotes $R_{\rm exp}=2.5\pm0.5({\rm stat})\pm0.5({\rm syst})$. A more precise value of the R ratio (45) should come with DAΦNE's operation at its higher energy $\sqrt{s}=0.15$ GeV, as this will enable the measurement of $\gamma\gamma\to\eta'$ [36].

Since R is independent of θ, it will most cleanly test our predictions (31),(32). A precisely determined $R_{\rm exp}$ can also help with finding out whether (a) $\overline{f}_{\eta_8}<f_\pi$, as follows in the coupled SD-BS approach from $\widetilde{T}_{s\bar{s}}(0,0)<\widetilde{T}_{u\bar{u}}(0,0)$ or (b) $\overline{f}_{\eta_8}=f_{\eta_8}\geqslant f_\pi$ as in χPT [17,26]. The present model gives us $\widetilde{T}_{s\bar{s}}(0,0)=0.62\widetilde{T}_{u\bar{u}}(0,0)$, so that $R=2.87$ is obtained, which is well within the error bars of the present experimental average [36]. (Taking the chiral limit also for s quarks would give $R=3$, which is the upper bound for Eq. (45) in the present approach. $R=3$ is still consistent with $R_{\rm exp}$ [36] within the present experimental accuracy.)

We should note (i) $\widetilde{T}_{s\bar{s}}$ is a quantity which can be especially practically used in conjunction with a more accurate $R_{\rm exp}$ to narrow down the choice of models suitable for describing η_8 and η_0 (and ultimately η and η') and (ii) precise experimental determinations of R can help to find out if there are other admixtures $|X\rangle$ [e.g., gluonium $|gg\rangle$, $\eta(1295)$,

$\eta(1440)$ (former ι), $\eta_c, \ldots$] to η and η', on top of the mixture of η_8 (10) and η_0 (11). We can see both (i) and (ii) if we use our predictions for $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$, Eqs. (31) and (32), in Eq. (45), yielding

$$R=\frac{25}{9}+\frac{2}{9}\left[\frac{2\sqrt{2}\pi^2 f_\pi}{N_c}\widetilde{T}_{s\bar{s}}(0,0)\right]^2, \tag{46}$$

which shows that in our approach R depends only on one *variable*[3] model-dependent quantity $\widetilde{T}_{s\bar{s}}$. This is because $\overline{f}_{\eta_8}$ (31) and $\overline{f}_{\eta_0}$ (32) follow from the fact that, for *any* interaction kernel and resulting propagator and bound state solutions, $\widetilde{T}_{\pi^0}(0,0)\equiv\widetilde{T}_{u\bar{u}}(0,0)=N_c/(2\sqrt{2}\pi^2 f_\pi)$ in the chiral limit, and that this result remains an excellent approximation for realistic m_u and m_d leading to empirical M_π. Therefore, important variations in our predictions for $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$, and thus R (46), can come only from $\widetilde{T}_{s\bar{s}}(0,0)$. The accuracy of $\widetilde{T}_{s\bar{s}}(0,0)$ depends on the quality of the bound-state solution, but regardless of concrete model choices and results, the general inequality $\widetilde{T}_{u\bar{u}}(0,0)>\widetilde{T}_{s\bar{s}}(0,0)>0$ enables Eq. (46) to provide the bounds $3>R>25/9=2.777\ldots$. Hence, if experiments establish $R<25/9$ by a significant amount, this will most probably indicate that in η and η' there are admixtures (e.g., glueballs) to η_8 (10) and η_0 (11) which are "inert" with respect to the interactions with photons, because this can lower the bound $R>25/9$ most efficaciously. We will be able to address this in more detail after the discussion of the mixing, presented in the next section.

V. COPING WITH MIXING OF ETAS IN COUPLED SD-BS APPROACH

The mixing angle θ is often inferred from the empirical $\gamma\gamma$ decay widths of η and η'. This is how we established, in Sec. IV, that $\theta^{\mathrm{exp}}=-12.0°$ is the empirically preferred mixing angle for the values of $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$ obtained in our model, namely, Eqs. (33) and (34). On the other hand, the angle θ is predicted by diagonalizing the η-η' mass matrix evaluated in the η_8-η_0 basis, such as the one predicted by our SD-BS approach and given in Eq. (47) below. Obviously, for a satisfactory model description of the η-η' complex, the latter procedure should give the mixing angle close to the angle θ^{exp} required by the $\eta,\eta'\to\gamma\gamma$ widths.

In the η-η' complex, subtleties arise from the interplay of the mixing due to the $SU(3)_f$ breaking with the $U_A(1)$ *gluon* axial ABJ anomaly, which couples to the flavor-singlet η_0 and removes the nonet symmetry. (For a simple introduction, see Sec. 12.8 of Ref. [2] and Secs. III-3, VII-4, and X-3 of Ref. [26]). Namely, in the coupled SD-BS approach, where the states with good $SU(3)_f$ quantum numbers are constructed from the $f\bar{f}$ bound states ($f=u,d,s$) obtained in

[3]As clarified above, getting a reasonable f_π first is obligatory for applications to $\gamma\gamma$ processes.

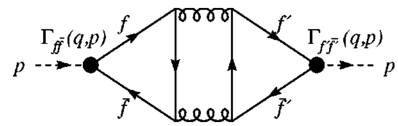

FIG. 2. The annihilation of the $f\bar{f}$ bound-state vertex $\Gamma_{f\bar{f}}$ into two gluons and their recombination into the quark-antiquark bound state consisting of possibly different flavors $f'\bar{f}'$.

Sec. II, the eta (mass)^2 matrix $\hat{M}^2$ in the η_8-η_0 basis (10),(11) is given by

$$\hat{M}^2=\begin{bmatrix} M_{88}^2 & M_{80}^2 \\ M_{08}^2 & M_{00}^2\end{bmatrix}=\begin{bmatrix}\frac{2}{3}\left(M_{s\bar{s}}^2+\frac{1}{2}M_\pi^2\right) & \frac{\sqrt{2}}{3}(M_\pi^2-M_{s\bar{s}}^2) \\ \frac{\sqrt{2}}{3}(M_\pi^2-M_{s\bar{s}}^2) & \frac{2}{3}\left(\frac{1}{2}M_{s\bar{s}}^2+M_\pi^2\right)\end{bmatrix} \tag{47}$$

if we neglect the gluon anomaly for the moment. In agreement with other cases when the gluon ABJ anomaly is not included, or is turned off, as in $N_c\to\infty$ limit (e.g., Refs. [37,26]), the diagonalization of Eq. (47) yields an η degenerate with the pion, $M_\eta^2=M_\pi^2$, and without the $s\bar{s}$ component, $\eta=(1/\sqrt{2})(u\bar{u}+d\bar{d})$, whereas η' is a pure $s\bar{s}$ pseudoscalar, with $M_{\eta'}^2=M_{s\bar{s}}^2$. This happens at $\theta=-54.74°$ and is obviously analogous to the "ideal" or "extreme" mixing which is known to be a very good approximation for the mixing of the vector mesons ω and ϕ. We can thus note in passing that the present approach works well for the mixing of ω and ϕ. Nevertheless, this scenario is obviously catastrophic for the η-η' system [the $U_A(1)$ problem], so that gluon anomaly must be incorporated into our SD-BS framework. Doing this on the fundamental level represents a formidable task in any case, a task noone has accomplished yet. Moreover, an interaction kernel in the ladder approximation, such as the simple gluon-exchange one that we have in the present model, is inadequate for this task even in principle. Namely, by definition it does not contain even the simplest annihilation graph of a quark-antiquark pseudoscalar into two gluons (and their recombination into another quark-antiquark pair) contributing to the processes such as the one in Fig. 2. The contribution of the gluon ABJ anomaly operator $\epsilon^{\alpha\beta\mu\nu}F^a_{\alpha\beta}F^a_{\mu\nu}$ to the η_0 mass M_{00} therefore cannot be captured through a ladder kernel even in the roughest approximation (leaving alone the issue of nonperturbative gluon configurations such as instantons).

Therefore, some additional ingredients or assumptions must be introduced into the present model in order to cope with the η-η' system. Since going beyond the ladder approximation is not within the scope of the present work, the following scheme is the most sensible at this level: note that there is a standard way (see, e.g., Refs. [2,26]) to account for the anomaly effect by *parametrizing* it through the term λ_η added to the η_0 mass, since only this singlet combination (11) is coupled to the gluon anomaly, so that only its mass is affected by it. This corrects the $U_A(1)$ problem arising in the mass matrix evaluated with the nonet $SU(3)_f$ states η_8 and η_0. Let us do the same in our mass matrix (47):

$$M_{00}^2 \to \frac{2}{3}\left(\frac{1}{2}M_{s\bar{s}}^2 + M_\pi^2\right) + \lambda_\eta. \tag{48}$$

Of course, parametrizing the effect of the gluon anomaly is far from actually calculating it unambiguously. In particular, the quantities we calculated for the η_0 under the assumption of nonet symmetry, f_{η_0} and $\overline{f}_{\eta_0}$, are in fact also affected by the coupling of the gluon anomaly to η_0. However, due to the large N_c arguments, it makes sense to break nonet symmetry only on the level of the mass-shift parameter λ_η while keeping our η_0 built of the same $f\bar{f}$ bound-state vertices as η_8, to which the gluon anomaly does not couple. This is because the gluon anomaly is in the large N_c limit suppressed [37,26] as $1/N_c$, so that in our f_{η_0} and $\overline{f}_{\eta_0}$ calculated within the nonet scheme, only the contributions of the order $\mathcal{O}(1/N_c)$ are missed. Our scheme is therefore a controlled approximation on the level of large N_c arguments.

The results obtained below for the mixing-dependent $\eta,\eta' \to \gamma\gamma$ widths also turn out to be reasonable, providing an *a posteriori* justification for our scheme. In the light of large N_c arguments, such reasonable results are not accidental and can be expected beforehand.

Let us also note that our assumptions are in fact shared by many other approaches, explicitly or implicitly. For example, Gilman and Kauffman [16] employ in their analysis nonet symmetry or broken version thereof, pointing out that it is at least implicitly assumed by all who use the quark basis not differentiating between quark states belonging to the singlet from those belonging to the octet. Moreover, imposing the nonet breaking via introducing the additional parameter λ_η is basically the same way in which nonet symmetry is broken in the chiral perturbation theory (χPT). In χPT, one faces the problem of how to incorporate η_0, shifted upwards in mass by the gluon anomaly, into the scheme that should involve Goldstone pseudoscalar mesons. Bijnens, Bramon, and Cornet [27] comment on the problems encountered when working with this ninth state, but stick to what they did earlier [38], namely, including η_0 (η_1 in their notation) ''in a simple nonet-symmetry context''. Their Ref. [38] conveniently parametrized the nonet of (pseudoscalar) Goldstone bosons in terms of nine fields entering in the lowest order Lagrangian consistent with current algebra and explicit breaking by the quark masses, but the effect of the breaking of $U(1)_A$ is included only via *an extra mass term* for η_0. This is justified if one relies on large N_c arguments, since η_0 is indeed a Goldstone boson in the limit $N_c \to \infty$ [39,26], and then η_0 mass is introduced as an extra parameter on top of that, which basically corresponds to our scheme.

Precisely in the light of χPT, our result (34) for $\overline{f}_{\eta_0}$ appears very reasonable in spite of missing the contributions of $\mathcal{O}(1/N_c)$, supporting our relying on the nonet symmetry scheme. Namely, it is in excellent agreement with the results of χPT, being right between $\overline{f}_{\eta_0} \approx 1.1 f_\pi$ quoted by Ref. [27] and $\overline{f}_{\eta_0} = (1.04 \pm 0.04) f_\pi$ of Ref. [17]. Finally, the robustness of our $\gamma\gamma$ amplitudes to kernel variations, resulting from the good chiral features of the SD-BS approach (as explained in Sec. IV B), also supports our scheme.

We therefore pursue the procedure of removing the $U_A(1)$ problem by lumping the effects of the gluon anomaly into a single η_0-mass shift parameter λ_η as in Eq. (48). Then, with our result $M_{s\bar{s}} = 0.721$ GeV, and with the experimental pion mass $M_{\pi^0} = 0.135$ GeV (which the present SD-BS approach readily reproduces when the strict chiral limit is relaxed [5]), and with the choice $\lambda_\eta = 1.165$ GeV^2, we get $\theta = -12.7°$, which is very close to $\theta^{\exp} = -12.0°$ favored by the empirical $\gamma\gamma$ widths. Moreover, we then reproduce the experimental value of the η mass, $M_\eta = M_\eta^{\exp} = 0.547$ GeV. Admittedly, the η' mass is then somewhat too high, $M_{\eta'} = 1.18$ GeV. Of course, we could in principle pick such a value of λ_η, that this other mass $M_{\eta'}$ would be reproduced, or still another value ($\lambda_\eta = 0.677$ GeV^2) to reproduce empirical $M_\eta^2 + M_{\eta'}^2$. Naturally, M_η would then be spoiled, but this is not the main reason why the two latter possibilities are disfavored in the present approach. The main problem is that they yield so negative values of the mixing angle ($\theta = -21.4°$ when $M_\eta^2 + M_{\eta'}^2$ is fitted, and $\theta = -22.8°$ when $M_{\eta'}^2$ is fitted to experiment), that they are incompatible with the present approach. In the present model [5], so negative mixing angles obviously yield unacceptable $\gamma\gamma$ widths, since the empirical $\gamma\gamma$ widths favor the mixing angle $\theta^{\exp} = -12.0°$, and this speaks in favor of the first possibility, $\lambda_\eta = 1.165$ GeV^2, leading to $\theta = -12.7° \approx \theta^{\exp}$. Still, this $\theta^{\exp} = -12.0°$ is the consequence of the particular model choice [5] which led to the values (33) and (34) for $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$, respectively. Although we explained in the previous section why $\widetilde{T}_{s\bar{s}}$ (and consequently $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$) must be relatively stable to model kernel variations, it is desirable to have a criterion which is even less model dependent. And indeed, we do have a reason why the coupled SD-BS approach *in general* prefers the first procedure leading to larger values of λ_η and, consequently, less negative values of θ. Namely, it turns out that since in the coupled SD-BS approach $\overline{f}_{\eta_8} < f_\pi$ for any realistic value of strange quark mass, the consistency with the experimental $\eta,\eta' \to \gamma\gamma$ widths is possible in this approach only for mixing angles less negative than roughly $-15°$. This is easily seen, for example, in Fig. 1 of Ball *et al.* [40], where the values of $\overline{f}_{\eta_{8(0)}}/f_\pi$ consistent with experiment are given as a function of the mixing angle θ. [It does not matter that they in fact plotted $f_{\eta_{8(0)}}/f_\pi$ and not $\overline{f}_{\eta_{8(0)}}/f_\pi$. Namely, they used Eqs. (40),(41) for comparison with the experimental $\gamma\gamma$ widths, just with $f_{\eta_{8(0)}}/f_\pi$ instead of $\overline{f}_{\eta_{8(0)}}/f_\pi$, so that the experimental constraints displayed in their Fig. 1 apply to whatever ratios are used in these expressions.] On the other hand, the more negative values $\theta \lesssim -20°$ give good $\eta,\eta' \to \gamma\gamma$ widths in conjunction with the ratio $\overline{f}_{\eta_8}/f_\pi = 1.25$ obtained by Ref. [17] in χPT. However, our approach belongs among constituent quark ones. In the next section we discuss why considerably less negative angles, $\theta \approx -14° \pm 2$ [41], are natural for constituent quark approaches in general.

The procedure leading to $\theta = -12.7°$ is also corroborated by the results of some different approaches, most notably, by the results of the instanton liquid model, where one can actually calculate the gluon anomaly mass shift instead of parametrizing it. As Shuryak [42] pointed out, the instanton-induced interaction leads simultaneously to both light pion and heavy η', i.e., the dynamics provided by instantons can take care of the effects of the *gluon* axial anomaly *and* provide the light pseudoscalars as the Goldstone bosons of DχSB. While the instanton-induced interaction may therefore be the main candidate which in the future one may try to include in the interaction kernel of the coupled SD-BS equations, the results of Alkofer *et al.* [43] in the framework of the instanton liquid model have already indicated that such an inclusion could easily lead to a *calculated* λ_η similar to its present *parametrized* value. Namely, Alkofer *et al.* [43] find that due to instantons, the $U(1)_A$-anomalous contribution $(2N_f/f)N/V$ must be added to the η-η' mass matrix. This term, corresponding to our λ_η, also has the value very close to our $\lambda_\eta = 1.165$ GeV^2; it is equal to approximately 1.1 GeV^2 for their standard instanton density $N/V = 1$ fm^{-4} and their pseudoscalar decay constant $f = 91$ MeV. (Number of flavors $N_f = 3$.) This gives them $\theta \approx -11.5°$, $M_\eta \approx 0.527$ GeV, and $M_{\eta'} \approx 1.172$ GeV, which is very similar to our results.

A. Values of the mixing-dependent quantities

Once the mixing angle θ has been fixed, the predictions for the axial η and η' decay constants are found from Eqs. (18),(19). $\theta = -12.7°$ implies $f_\eta = 112.6$ MeV and $f_{\eta'} = 117.1$ MeV. This agrees almost perfectly with Scadron's [44] estimates $f_\eta \approx 1.22 f_\pi$ and $f_{\eta'} \approx 1.28 f_\pi$ obtained from the GT relations at the quark level for the strange-to-nonstrange constituent mass ratio $\mathcal{M}_s/\mathcal{M}_{ud} \approx 1.5$ (and for θ advocated by Scadron [44], which is, interestingly, the same as our favored $\theta = -12.7°$.) However, these values are somewhat higher than the experimental values $f_\eta^{\rm exp} = 94 \pm 7$ MeV [45] or 79 ± 9 MeV [46] and $f_{\eta'}^{\rm exp} = 89 \pm 5$ MeV [45] or 96 ± 8 MeV [46], deduced (under certain theoretical assumptions discussed below) by CELLO [45] and TPC/2γ [46] Collaborations from the Q^2 dependence of their measured $\eta(\eta')\gamma^\star\gamma$ transition form factors $T_{\eta(\eta')}(0,-Q^2)$ (in our notation), where $k'^2 = -Q^2 \neq 0$ is the momentum-squared of the spacelike off-shell photon $\gamma^\star$. The same TPC/2γ reference [46] quotes also another pair of experimental values $f_\eta^{(\rm exp2)} = 91 \pm 6$ MeV and $f_{\eta'}^{(\rm exp2)} = 78 \pm 5$ MeV, which were obtained from the experimental decay amplitudes into two on-shell photons under the assumption that one can write $T_{\eta(\eta')}(0,0) = 1/4\pi^2 f_{\eta(\eta')}$ in analogy with the axial anomaly result (28) for the pion. However, because of the large s-quark mass, as well as the masses of η and η' which are, respectively, 4 and 7 times larger than the pion mass, this procedure can yield only a rough qualitative estimate.

On the other hand, our value of f_η is much closer not only to Scadron's [44] estimates and to the value $f_\eta = 114$ MeV of an approach [6] somewhat related to ours, but also to the model-independent result of χPT, that $f_\eta = 1.02 f_\pi (f_K/f_\pi)^{4/3}$ [47]. For the experimental ratio $f_K/f_\pi = 1.22 \pm 0.01$, this gives $f_\eta = (1.3 \pm 0.05) f_\pi = 120 \pm 5$ MeV [47], for which both CELLO [45] and TPC/2γ [46] results are too low.

TABLE I. Comparison of the calculated $\gamma\gamma$ decay widths (in eV) of π^0, η, and η' with their average experimental widths, as well as the experimental widths $W_{\rm NEW}^{\rm exp}$ obtained when only more recent measurements are taken into account. The widths are calculated using the empirical masses in the phase-space factors in conjunction with calculated amplitudes. The tabulated η and η' calculated widths correspond to the case when their mixing adjusts the mass of η to its empirical mass.

P	$W(P \to \gamma\gamma)$	$W^{\rm exp}(P \to \gamma\gamma)$	$W_{\rm NEW}^{\rm exp}(P \to \gamma\gamma)$
π^0	7.7	7.74 ± 0.56	not applicable
η	$0.56 \times 10^{+3}$	$(0.46 \pm 0.04) \times 10^{+3}$	$(0.51 \pm 0.026) \times 10^{+3}$
η'	$4.9 \times 10^{+3}$	$(4.26 \pm 0.19) \times 10^{+3}$	$(4.53 \pm 0.59) \times 10^{+3}$

The experimental values $f_\eta^{\rm exp}$ and $f_{\eta'}^{\rm exp}$ were extracted from the CELLO [45] and TPC/2γ [46] data on the transition form factors $T_{\eta(\eta')}(0,-Q^2)$ assuming that the pole mass $\Lambda_{\eta(\eta')}$ parametrizing their fit to the data, can be identified with $2\pi\sqrt{2} f_{\eta(\eta')}$. Then, the pole fits to the data could smoothly join (as $Q^2 \to \infty$) the perturbative QCD prediction [48] for $T_{\eta(\eta')}(0,-Q^2)$, i.e., the pole fits would then agree not only with the QCD asymptotic form $1/Q^2$, but also with its coefficient. However, note that the values of $f_\eta^{\rm exp}$ and $f_{\eta'}^{\rm exp}$ quoted above, are *all* close to $m_\rho/(2\pi\sqrt{2}) = 86.4$ MeV, indicating that a connection with the vector-meson dominance interpretation (that $\Lambda_{\eta(\eta')} \approx m_\rho$) [45,46] may indeed exist at the investigated range of Q^2. On the other hand, since Gasser and Leutwyler's model-independent calculation [47], Scadron's [44] GT estimates, Burden *et al.* [6], and the present approach, all agree that $f_{\eta(\eta')}$ should be noticeably larger than f_π, the extraction of $f_\eta^{\rm exp}$ and $f_{\eta'}^{\rm exp}$ from the transition form factors $T_{\eta(\eta')}(0,-Q^2)$ probably cannot be done accurately at the ranges of Q^2 investigated so far. That this is indeed so, is indicated by the experimental value [35] $f_{\pi^0} = 84.1 \pm 2.8$ MeV [$\approx m_\rho/(2\pi\sqrt{2})$ again], extracted by the same method. The central value is 10% below well established $f_\pi^{\rm exp} = 92.4 \pm 0.3$ MeV. This cannot be explained by the small isospin violation, indicating that $f_\eta^{\rm exp}$ and $f_{\eta'}^{\rm exp}$ could have been underestimated too.

Our predictions for the η and η' two-photon widths are also totally fixed now, being given by our $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$ used in Eqs. (40) and (41), without any additional parameters to adjust. Our preferred angle $\theta = -12.7°$ leads to the predictions (displayed also in Table I)

$$W(\eta \to \gamma\gamma) = 0.561 \text{ keV}, \tag{49}$$

$$W(\eta' \to \gamma\gamma) = 4.913 \text{ keV}. \tag{50}$$

These predictions are at first sight not very successful since, according to Table I, our best predictions overshoot the present [35] experimental averages (42) for $\eta, \eta' \to \gamma\gamma$ by some 20%. However, we should not be dissatisfied with these results because of the following.

(a) Ball *et al.* [40] and, in effect, Review of Particle Properties itself [35] (referring to the note on p. 1451 of [32]), suggest that only the more recent data on $\eta, \eta' \to \gamma\gamma$ should be retained, whereby the presently "official" values (42) are modified to [32,40]

$$W^{\rm exp}_{\rm NEW}(\eta \to \gamma\gamma) = (0.510 \pm 0.026) \ \ {\rm keV}, \tag{51}$$

$$W^{\rm exp}_{\rm NEW}(\eta' \to \gamma\gamma) = (4.53 \pm 0.59) \ \ {\rm keV}, \tag{52}$$

and these experimental values agree much better with our predictions.

(b) We did not vary any model parameters, but used the parameters obtained from the broad fit of Ref. [5] to the meson spectrum and pseudoscalar decay constants. This fit did not include η-η' system in any way, so that everything we calculated for it is pure prediction.

B. A side issue: speculations about other admixtures

In the present approach, η_0 and η_8 (and consequently η and η') are constructed exclusively of the ground state pseudoscalar $q\bar{q}$ bound states. Nevertheless, additional admixtures have often been speculated, notably glueballs. Farrar [49] points out that experiments appear to indicate that there is a glueball-like pseudoscalar which is much lighter than estimated by quenched lattice calculations, thus motivating us to speculate on the consequences of such admixtures. We do *not* have at this point the ambition to include such additional admixtures in our approach. However, we can look into some of the consequences that such admixtures would have by simply *assuming* that they were present in addition to the quarkonium η_0 and η_8 as constructed in this paper.

Take, for example, the simplest and most usual assumption [26], that only the $\mathrm{SU}(3)_f$-singlet (11) can be significantly modified in this way:

$$|\eta_0\rangle \to \frac{\cos\varphi}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) + \sin\varphi|X\rangle, \tag{53}$$

where φ is the new mixing angle, a new parameter expressing the assumed strength of the unspecified admixture $|X\rangle$ into η_0.

If $|X\rangle$ is a state that does not couple to photons directly (e.g., gluonium $|gg\rangle$), the results for $\gamma\gamma$ decays will be modified in a particularly simple way: in formulas (30), (40), (41), and (45), one should just replace $1/\overline{f_0}$ by $\cos\varphi/\overline{f_0}$. This can reduce R (46) strongly, as the largest term in Eq. (46), 25/9, would then be modified to $25/27 + \cos^2\varphi\, 50/27$.

We should also note that such an admixture (53) would help to fit the masses of both η and η' to their experimental values precisely—thanks to the new free parameter φ, of course. Equation (53) modifies elements of the mass matrix to

$$M^2_{80} \to \cos\varphi \frac{\sqrt{2}}{3}(M^2_\pi - M^2_{s\bar{s}}), \tag{54}$$

$$M^2_{00} \to \cos^2\varphi \frac{2}{3}\left(\frac{1}{2}M^2_{s\bar{s}} + M^2_\pi\right) + \tilde{\lambda}_\eta, \tag{55}$$

where $\tilde{\lambda}_\eta \equiv \lambda_\eta + \sin^2\varphi M_X^2$ takes the place that λ_η has for $\varphi = 0$. If $|X\rangle$ is not a single state, but a mixture of various states, its mass M_X has the meaning of an effective mass.

The experimental masses $M_\eta = 547$ MeV and $M_{\eta'} = 958$ MeV, as well as the η-η' mixing angle $\theta = -17.1°$, are then obtained for $\varphi = 42.43°$ and $\tilde{\lambda}_\eta = (0.873 \ \ {\rm GeV})^2$. Nevertheless, it turns out that the fit to the data is still not improved as much as one would expect when an additional free parameter is introduced, so that we did not detect indications for the need for an admixture of such states to what we have in the present model. For example, our R-ratio then drops to $R = 1.80$. This is much further from the present central experimental value than R predicted by our approach without glueballs, but just in case data from future precision measurements strongly violate our bound on the R ratio, it is important to point out that, at least from the standpoint of our approach, such a violation would be a strong indication of the presence of some "inert" admixture, such as gluonium. At present, however, the data are consistent with the bound $R > 25/9$ following generally from the SD-BS approach without gluonium admixture, and even favor the value $R = 2.87$ following from the present concrete model choice [5] without glueballs, over the value with the admixture quoted above. Moreover, the $\eta \to \gamma\gamma$ width with the gluonium admixture improves only marginally, by 4%, while the $\eta' \to \gamma\gamma$ width gets *spoiled* by more than a factor of 2.

We therefore conclude that we found no indication that admixtures of glueballs, or other states with similar effects on $\gamma\gamma$ decays, would be favored by the present experimental data. Consequently, there is no strong motivation for enlarging the present framework by finding solutions for pseudoscalar glueballs and treating them on the same footing as our pseudoscalar $q\bar{q}$ bound states. [It is amusing that $\varphi = 42.43°$ in conjunction with the vanishing gluon anomaly contribution $\lambda_\eta = 0$ implies $M_X = 1.294$ GeV—practically the same as the mass of $\eta(1295)$. However, this can only be viewed as accidental at this point.]

VI. SUMMARY, DISCUSSION, AND CONCLUSIONS

The relativistically covariant constituent $q\bar{q}$ bound-state model [5] used here is consistent with current algebra because it incorporates the correct chiral symmetry behavior thanks to DχSB obtained in an, essentially, Nambu–Jona-Lasinio (NJL) fashion, but the model interaction is less schematic. Notably, when care is taken to preserve WTI of QED, it reproduces (in the chiral limit even analytically and independently of the internal meson structure) the Abelian axial anomaly results, which are otherwise notoriously difficult to reproduce in bound-state approaches (as illustrated by, e.g., Ref. [22] and especially references therein). Observables such as meson masses, $f_\pi, f_K, f_\eta, f_{\eta'}$ and $\gamma\gamma$-decay amplitudes can be calculated without additional parameters after an *Ansatz* has been made for the gluon propagator entering in the SD-BS equations, which are consistently coupled in the generalized (or improved) rainbow-ladder approximation (in the terminology of, e.g., Refs. [2] or [7]). However, to avoid the $U(1)_A$ problem in the η-η' complex, we have to introduce an additional parameter λ_η representing the contribution of the gluon axial anomaly to the mass of η_0, in analogy with the similar η_0-mass parameter in the χPT Lagrangian in Ref. [38], for example. Since the gluon anomaly contribution vanishes in the large N_c limit as $1/N_c$, our $q\bar{q}$ bound-state pseudoscalar mesons behave in the $N_c \to \infty$ and chiral limits in the same way as those in χPT (e.g., see Refs. [47] or [26]): as the strict chiral limit is approached for all three flavors, the $SU(3)_f$ octet pseudoscalars *including* η become massless Goldstone bosons, whereas the η' mass is of order $1/N_c$ since it is purely due to the gluon anomaly. In the $N_c \to \infty$ limit with nonvanishing quark masses, the ''ideal'' mixing takes place so that η consists of u,d quarks only and becomes degenerate with π, whereas η' is the pure $s\bar{s}$ pseudoscalar. In our bound-state approach, f_π, $\overline{f}_{\eta_8}$, $\overline{f}_{\eta_0}$, as well as f_{η_8}, f_{η_0} and f_η and $f_{\eta'}$, are *all* calculated quantities, while most other theoretical frameworks treat at least one of them, $\overline{f}_{\eta_0}$, as a free parameter (fixed together with θ from the experimental widths of $\eta,\eta' \to \gamma\gamma$).

Our prediction $f_{\eta_8}/f_\pi = 1.31$ agrees rather well with $f_{\eta_8}/f_\pi = 1.25$ of χPT [17]. Nevertheless, this one-loop χPT calculation also lead to the identification of their axial-current and $\gamma\gamma$-decay constants, $f_{\eta_8} = \overline{f}_{\eta_8}$, which differs from our results on η_8. More precisely, the observation that for realistic s-quark masses, $\widetilde{T}_{s\bar{s}}(0,0) < \widetilde{T}_{\pi^0}(0,0)$ always holds in the coupled SD-BS approach, leads to $\frac{3}{5} f_\pi < \overline{f}_{\eta_8} < f_\pi$ and $f_\pi < \overline{f}_{\eta_0} < \frac{6}{5} f_\pi$. These inequalities hold irrespective of the model parameters or the quality of the interaction kernel. $\overline{f}_{\eta_8} = f_\pi = \overline{f}_{\eta_0}$ is realized in the chiral limit, whereas the opposite bounds are approached when the s-quark mass grows huge, leading to the decrease of $\widetilde{T}_{s\bar{s}}(0,0)/\widetilde{T}_{\pi^0}(0,0)$. There is no disagreement with χPT regarding $\overline{f}_{\eta_0}$, either concerning the general bound $f_\pi < \overline{f}_{\eta_0} < \frac{6}{5} f_\pi$ of the coupled SD-BS approach, or our result obtained using the concrete model of Ref. [5], namely, $\overline{f}_{\eta_0}/f_\pi = 1.067$. This agrees well with the values found in χPT [17,27]. The apparent contradiction between the results of the coupled SD-BS approach on $f_\pi/\overline{f}_{\eta_8}$, and the corresponding results of χPT was discussed in detail in Sec. IV B. Let us now address the intimately related issue of different preferred mixing angles in these respective approaches.

In conjunction with the updated experimental widths (51),(52), $\overline{f}_{\eta_8} < f_\pi$ implies that the coupled SD-BS approach is compatible with the mixing angles which are less negative than $\theta \approx -15°$. For our concrete model choice [5] and the resulting values (33),(34) of $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$, the favored value of θ is between the values accepted until the mid 1980s, namely, $\theta \approx -10°$ determined from the $SU(3)_f$ breaking given by the Gell-Mann–Okubo mass formula, and the lowest of the values $\theta \in [-17°, -20°]$ favored nowadays [40,16,17,47].

In order to see that the mixing angles considerably less negative than those in χPT ($\theta \sim -20°$) are a natural and expected prediction in a constituent approach such as ours, it is instructive to recall the paper of Bramon and Scadron [41] where the mixing angle of $\theta = -14° \pm 2°$ follows from a rather exhaustive set of data if the $SU(3)_f$ breaking is taken into account in terms of the *constituent* quark mass ratio $\mathcal{M}_s/\mathcal{M}_{ud} \approx 1.4-1.5$. $SU(3)_f$-breaking ratios somewhere around this interval are considered realistic because they lead to good descriptions of many hadronic properties in numerous dynamical models; notably, close to this interval is also the ratio (≈ 1.63) of the constituent masses $B_f(0)/A_f(0)$ generated by DχSB in Jain and Munczek's approach. Bramon and Scadron [41] extracted their average $\theta = -14° \pm 2°$ from the strong interaction tensor $T \to PP$ decays, and the vector $V \to \gamma P$ and pseudoscalar $P \to \gamma\gamma$ radiative decays. [When extracted just from $\eta,\eta' \to \gamma\gamma$ pertinent here, and other $SU(3)_f$ breaking-ratio-dependent *radiative* decays, their angle is even lower, $-11° \pm 2.4°$.] They point out that more negative values $\theta \sim -20°$ in the χPT framework are due to the way of implementing the $SU(3)_f$ breaking (through the values of the decay constants f_{η_8} and f_{η_0}), differing from that in the constituent-quark approaches.

Now, our $SU(3)_f$-breaking is fixed by Jain and Munczek's [5] choice of parameters, so that our calculated value of θ varies only if we vary λ_η which parametrizes the effects of the gluon anomaly. In light of Bramon and Scadron's [41] observations discussed above, and the fact that that our $SU(3)_f$ breaking leads to the ratio of strange-to-nonstrange constituent masses of 1.63, it is understandable and expected that our constituent approach should give a reasonably good description of $\eta,\eta' \to \gamma\gamma$ for angles less negative than in χPT; i.e., it is no longer surprising that our preferred angle turned out to be $\theta = -12.7°$. However, it is not only that these values are the preferred ones in our presently chosen model [5] because they are more empirically successful than other values. In addition to that, since in the coupled SD-BS approach $\overline{f}_{\eta_8} < f_\pi$ rather generally (for any realistic value of strange quark mass), the consistency with the experimental $\eta,\eta' \to \gamma\gamma$ widths is possible, in our approach, only for mixing angles less negative than roughly $-15°$, as already pointed out above.

That all this is in qualitative agreement with what was known from relatively simple-minded constituent-quark models even before the analysis of Ref. [41] can be seen, e.g., from Zieliński's review [50] on radiative decays of mesons. He observed that in the scenarios that related apparent suppressions of radiative decays of strange mesons to a larger mass of the s quark, a significant suppression of the annihilation amplitude of $s\bar{s}$ pairs into two photons was also expected, and with the latter suppression of order 0.5 relative to annihilation amplitudes of nonstrange quarks (relevant, e.g., to the model of Ref. [51]), the two-photon widths of both η and η' could (however, roughly) best be described with $\theta \sim -11°$. Remembering the limitations on mutually consistent θ and $\overline{f}_{\eta_{8(0)}}$, we see that our values of θ and $\overline{f}_{\eta_{8(0)}}$ fit with our third element $\widetilde{T}_{s\bar{s}}(0,0) = 0.62 \widetilde{T}_{u\bar{u}}(0,0)$ into a logical scheme which is consistent with the behavior of the approaches similar to ours. Zieliński [50] also discussed how θ was much more negative ($\sim -20°$) in chiral theories, but pointed out that the determination of the pseudoscalar mixing angle was model dependent, and a clean-cut choice among various schemes was rather difficult to establish. Our discussion, and the results of, e.g., Bramon and Scadron [41], Pham [30], and Ball *et al.* [40], shows that this assessment still holds, but also that there has been some progress in narrowing the interval of possible mixing angles. In this connection, recall the observation of Ref. [40], that newer experimental input [our Eqs. (51) and (52)] reduces the mixing angle even more than Pham [30] realized, to $\theta = -(17 \pm 2)°$. This is no longer so far away from our preferred θ (especially considering that the value of the correction δ, Eq. (37), can be even more negative than Pham's values [30]). If various approaches succeed in including physical mechanisms they have been missing so far, their predictions for θ will probably tend to a unique value. This will also be true for $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$. In view of Refs. [41,30,40], this final value at which θ will settle may well be roughly in between the values favored nowadays by χPT and by quark model approaches such as ours. Thus, $\theta \sim -14°$ to $-17°$ may encompass the final result. An $\overline{f}_{\eta_8}$ which would be rather close to the chiral limit value $\overline{f}_{\eta_8} = f_\pi$, because the chiral-loop contributions would be, as in Ref. [30], to some extent (over)canceled by some other contributions (such as our bound-state strange mass-breaking effects), would agree better with such a $\theta \sim -14°$ to $-17°$. In our approach, the physical mechanisms which are now absent, are those corresponding to loops in χPT. Including them obviously implies substantial enlargements beyond the present framework. However, this also holds for others, e.g., in χPT one might pose the question of what the effects of higher loops and vector mesons would be. At present, no approach can claim to have all the relevant physics included, and therefore the ultimate values for θ and $\overline{f}_{\eta_8}$.

The present experimental value of the R ratio (45) is described reasonably well by our approach. What if more precise measurements (e.g., at DAΦNE [36]) constrain $R_{\exp}$ below 25/9? A strong violation of this bound (say, $R_{\exp} < 2.5$) would indicate that important admixtures other than η_8 and η_0 are present in η and η'. If the violation is not that strong, the following possibility is also viable: some of the values in the interval $2.5 < R_{\exp} < 25/9$ can be satisfied by $\overline{f}_{\eta_8}$ and $\overline{f}_{\eta_0}$ predicted by χPT. Hence, such a smaller violation of our bound can also mean that the prediction of χPT, that $f_\pi < \overline{f}_{\eta_8}$, is favored over our prediction. This would indicate that in the case of the η-η' complex, the ladder-approximated SD-BS approach makes a larger error by neglecting meson loops than, e.g., in the case of the charge pion form factor calculated in the context of SD equations, where the contribution of meson loops was estimated to be much smaller than that of the quark core [52].

The quantities dependent on the η-η' mixing, namely, axial-current decay constants f_η and $f_{\eta'}$, masses, and $\gamma\gamma$-decay widths of η-η' are satisfactorily close to data (or other theoretical predictions such as χPT) considering that, except for parametrizing the mass shift due to coupling of η_0 to non-Abelian axial anomaly, we did not do any parameter fitting, but used the parameters obtained from Jain and Munczek's [5] broad fit to the meson spectrum and decay constants. We conclude that their model [5] again performed well.

Since the coupled SD-BS approach is, due to the key role of DχSB, akin to the NJL model conceptually, the progress we made is best illustrated through the comparison with the analysis of the $\pi^0, \eta \to \gamma\gamma$ decays and properties of the pion, kaon, and η, performed in a NJL model (extended to include three flavors and the 't Hooft determinantal instanton-induced interaction) in Ref. [53] and in parts of Refs. [54,55]. (η' was not treated in Refs. [53–55].)

For the choice of model parameters preferred by Takizawa, Oka, and Nemoto [53–55], the experimental amplitude for $\eta \to \gamma\gamma$ is reproduced, but the η mass is 7% below the experimental value. The mixing angle is $\theta = -1.25°$, showing that their $U(1)_A$ breaking is stronger than in our approach (not to mention the one in χPT), forcing their η to be an almost pure η_8. Their kaon decay constant $f_K = 96.6$ MeV is 15% below the observed one. Accordingly, the predicted η decay constant, $f_\eta \approx f_\pi$, is uncomfortably far from what the model-independent result of χPT [47], $f_\eta = 1.02 f_\pi (f_K/f_\pi)^{4/3}$, gives when the empirical f_K/f_π is plugged in.

While our results compare rather favorably with the above, the best examples of advantages both in the conceptual consistency and in the quantitative details which the coupled SD-BS approach has with respect to the NJL model, are $P \to \gamma\gamma$ decays. Namely, we must criticize the seemingly successful reproduction of the anomalous $\pi^0, \eta \to \gamma\gamma$ amplitudes by Refs. [53–55]. In contradistinction to the coupled SD-BS approach with nonlocal interactions—in particular Jain and Munczek's model, where the UV cutoff is either not needed (in the chiral limit [3]) or practically infinite compared to the relevant hadronic scales—the NJL approach contains a low cutoff. In spite of this, Refs. [53–55] leave the convergent integrals unregulated, because the triangle diagram reproduces the anomalous $\pi^0 \to \gamma\gamma$ amplitude (28) only if there is no NJL cutoff [53,56–58]— otherwise, an underestimate of, typically, 20% occurs [57]. While Refs. [53,55] claim the improvement of the $\eta \to \gamma\gamma$ amplitude and width with respect to the earlier treatment of Bernard *et al.* [59], the consistent viewpoint is that of Ref. [59]: once the cutoff is introduced, the effective theory is defined and should not be altered for the purpose of calculating various quantities. In such effective theories, the missing part of the anomalous amplitude, lost due to the cutoff, should be found in additional diagrams [56] which contribute since the cutoff cannot be let to infinity. That is, in the class of models employing only local interactions and therefore needing a low cutoff, a *simple* incorporation of the anomaly is not possible [56–58], in contrast to the coupled SD-BS approaches employing also nonlocal interactions and thus not having such a cutoff.

Reference [6] is another approach to $q\bar{q}$ substructure incorporating DχSB, and it is even closer to us than the NJL model. The interaction used in Ref. [6] is nonlocal, as in ours, allowing the generation of momentum-dependent dynamical mass and BS vertices, so that there are no problems with a low cutoff as in the NJL model. The mixing angle they favor, $\theta \sim +5$, results from its treatment as an external parameter on which the mass and other properties of η depend. However, their axial current decay constant $f_\eta = 114$ MeV is close to ours. Extending the treatment of the $\pi^0 \gamma^\star \to \gamma$ transition form factor of Refs. [9,10] to the $\eta(\eta')\gamma^\star \to \gamma$ transition form factors is presently under investigation [60].

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