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Violation of the string hypothesis and the Heisenberg *XXZ* spin chain

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In this paper we count the numbers of real and complex solutions to Bethe constraints in the two-particle sector of the *XXZ* model. We find the exact number of exceptions to the string conjecture and total number of solutions that is required for completeness. [S0163-1829(99)10733-1]

I. INTRODUCTION

Integrable spin chains have proven to be useful in studying various theoretical ideas in field theory and statistical physics. In the continuum limit, one can relate spin chains to the massive Thirring model, the sine-Gordon theory, the Liouville theory and others.[1,2] Faddeev and Korchensky[3] suggested their possible relevance for QCD. A connection to matrix models was also suggested.[4] A very successful method in solving spin chains and in general integrable models both on a lattice and in the continuum is the Bethe ansatz.[5,6] Despite the fact that a lot is known about this method there is still one set of open questions concerning the so-called string conjecture.[7,8]

The Bethe ansatz method leads to a set of transcendental equations (called Bethe constraints) for momenta of quasiparticles. In the usual search for solutions of these equations a simplifying assumption is made, the already mentioned string conjecture. This conjecture, which we shall afterwards formulate more precisely, classifies the complex solutions for momenta of quasiparticles. It is well known that there are exceptions to the string conjecture near the antiferromagnetic ground state.[9–11] Recently, exceptions have been found[12,13] already in the two-particle sector of the *XXX* spin chain. Similar results have been found for the Hubbard model.[14] In the case of the *XXX* spin chain, the number of missing solutions (compared to the string-conjecture prediction) was found to be $\sqrt{N}$, where N is the number of degrees of freedom. A certain class of real solutions not allowed by the string conjecture was observed by Jüttner and Dörfel[15] in the *XXZ* chain. However, a systematic investigation of complex solutions and thus of exceptions to the string conjecture is missing.

There are several reasons why it would be desirable to understand the limits of validity of the string conjecture or equivalently to have a clear understanding of nature and number of real and complex solutions for momenta of quasiparticles. One reason is that it was used in literature as a tool to obtain various results. One example, for instance, is the completeness proofs of Bethe states.[5,14,16–20] Another example is the investigations that use a lattice regularization of field theoretical models.[21] In such cases the results at orders that are lower than N may depend on modifications of even a single root as these authors stress. As is well known, the string conjecture was also used by Bergknoff and Thacker[22] in deriving breather states of the massive Thirring model. This was recently criticized on the basis of numerical analysis of Bethe equations. A recent numerical calculation, independent of Bethe ansatz and based on the lattice regularization,[23] led to the usual bound-state spectrum of the massive Thirring model, thus suggesting that the question raised by previously mentioned authors could be related to understanding of the string conjecture and its violations. This result is based on the assumption of equivalence of sine-Gordon and the massive Thirring model. Another calculation for the massive Thirring model itself is in progress.[24]

In this paper we shall classify all solutions (both complex and real) in the two-particle sector of the *XXZ* model. That will allow us to find the number of exceptions to the string conjecture for a given coupling constant and a given number of lattice sites N. We shall, in particular, find that the number of exceptions to the string conjecture in thermodynamical limit is finite, except for the value of the coupling constant in which it coincides with the *XXX* model and what is consistent with previously found result.[12,13]

We shall consider the *XXZ* spin chain defined with the following Hamiltonian:

$$H=-\frac{1}{2}\sum_{n=1}^{N}\left(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\Delta\sigma_n^z\sigma_{n+1}^z\right),\quad \vec{\sigma}_{N+1}\equiv\vec{\sigma}_1. \tag{1.1}$$

This Hamiltonian acts in N^2 dimensional Hilbert space $\mathcal{H}=(\otimes\mathcal{C}^2)^{\mathcal{N}}$. In the Bethe ansatz method one introduces the basis states $|n_1\cdots n_M\rangle$ with M spins down, where the numbers $n_1,\ldots,n_M$ denote the lattice positions of the down spins. With $|0\rangle$ we denote the state with all spins up. A general element of the above-defined Hilbert space, and thus in particular the eigenstates of the Hamiltonian, can then be written in the sector with M spins down as

$$|\psi_M\rangle=\sum_{1\leqslant n_1\leqslant n_2\leqslant\cdots\leqslant n_M\leqslant N}\psi_M(n_1\cdots n_M)|n_1\cdots n_M\rangle. \tag{1.2}$$

The Bethe method consists in searching for Hamiltonian eigenstates in the form

$$\psi_M(n_1\cdots n_M)=\sum_P\exp\left\{i\left(\sum_{j=1}^{M}k_{P_j}n_j+\frac{1}{2}\sum_{1\leqslant j\leqslant l\leqslant M}\phi_{Pj,Pl}\right)\right\}, \tag{1.3}$$

where the sum runs over elements of the permutation group S_M. The momenta k_i, $i=1,\ldots,M$ and phase shifts $\phi_{j,i}$ have to be determined from the eigenvalue equation and periodicity requirement on functions $\psi_M(n_1\cdots n_M)$. The well-known procedure gives following expressions for phase shift $\phi_{j,i}$, energy E, and momentum P in terms of pseudomomenta k_i, $i=1,\ldots,M$

$$\phi_{j,i}=2\ \arctan\frac{\Delta\ \sin[(k_j-k_i)/2]}{\cos[(k_j+k_i)/2]-\Delta\ \cos[(k_j-k_i)/2]}, \tag{1.4}$$

$$E=-\frac{N\Delta}{2}+2\sum_{i=1}^{M}(\Delta-\cos k_i), \tag{1.5}$$

$$P=\sum_{i=1}^{M}k_i. \tag{1.6}$$

The periodicity requirement leads to the following constraints for the momenta of quasiparticles:

$$Nk_i+\sum_{j=1}^{M}\phi_{i,j}=2\pi\lambda_i,\quad i=1,\ldots,M. \tag{1.7}$$

The M Bethe numbers λ_i, $i=1,\ldots,M$ are half integers (integers) for M even (odd). Thus for M even, we can chose $\lambda_i\in-(N-1)/2,\ldots,(N-1)/2\}$ for N even and $\lambda_i\in -N/2,\ldots,N/2-1\}$ for N odd. Sometimes it is useful to introduce a transformation from pseudomomenta k_i, $i=1,\ldots,M$ to rapidity variables x_i, $i=1,\ldots,M$ with the following relation:

$$\cot\frac{k_i}{2}=\cot\frac{\theta}{2}\tanh\frac{\theta x_i}{2},\quad \Delta=\cos\theta. \tag{1.8}$$

In this parametrization Bethe constraints read

$$\left\{\frac{\sinh\frac{\theta}{2}(x_k+i)}{\sinh\frac{\theta}{2}(x_k-i)}\right\}^N=-\prod_{l=1}^{M}\left\{\frac{\sinh\frac{\theta}{2}(x_k-x_l+2i)}{\sinh\frac{\theta}{2}(x_k-x_l-2i)}\right\},$$

$$k=1,\ldots,M. \tag{1.9}$$

The string-conjecture states that solutions of these equations form string configurations with rapidities that are forming strings of length n. Rapidities in string have common real parts and equidistant imaginary parts. More precisely, a string of order (length) n and parity $+$ or $-$ is a set of n rapidities

$$x_{a,+}^{n,k}=x_a^n+(n+1-2k)i+O[\exp(-\delta N)]\left(\mathrm{mod}\frac{2\pi}{\theta}\right), \tag{1.10}$$

$$x_{a,-}^{n,k}=x_a^n+(n+1-2k)i+\frac{i\pi}{\theta}+O[\exp(-\delta N)]\left(\mathrm{mod}\frac{2\pi}{\theta}\right), \tag{1.11}$$

where $\delta\geqslant 0$, $k=1,\ldots,n$ and x_a^n is real. Insertion of these assumed forms in Eq. (1.9) gives equations for real parts of strings, which are similar to Eq. (1.7) with one common Bethe number (integer) I for each string. In addition, a part of the string conjecture was that no two strings of the same length can have same integers I. These assumptions together with inequalities derived in Ref. 7 for numbers I allow one to count the number of string solutions of equations (1.9). In this paper we shall not use equations that are a consequence of string conjecture. However, for future comparison we mention that in the sector $M=2$ the following number of solutions for strings of length 2 can be obtained:

$$N_S=2\left[\frac{1}{2\pi}(N-4)(\pi-2\theta)\right]+1, \tag{1.12}$$

where $[x]$ denotes integer part of x.

II. TWO-PARTICLE SECTOR AND COMPLEX SOLUTIONS

We want now to analyze Bethe equations without assuming the string conjecture. For simplicity we shall treat the two-particle sector. In this sector Bethe constraints (1.7) read

$$Nk_1+\phi_{1,2}=2\pi\lambda_1, \tag{2.1}$$

$$Nk_2-\phi_{1,2}=2\pi\lambda_2. \tag{2.2}$$

Here we want, in particular, to look for complex solutions. Due to the reality of energy and momentum, k_1 and k_2 have to be complex conjugates of each other

$$k_1=k_r+ik_i, \tag{2.3}$$

$$k_2=k_r-ik_i. \tag{2.4}$$

We can express k_r and k_i by taking the sum and difference of Eqs. (2.1) and (2.2)

$$k_r=\frac{\pi}{N}(\lambda_1+\lambda_2), \tag{2.5}$$

$$iNk_i=\pi(\lambda_1-\lambda_2)-2\ \arctan\frac{\Delta\ \sin(ik_i)}{\cos k_r-\Delta\ \cos(ik_i)}. \tag{2.6}$$

Further straightforward manipulation allows us to introduce a simple equation for k_i in terms of k_r. So the final set of equations that we shall consider is Eq. (2.5) for k_r and the equations for k_i,

$$\frac{\sinh\left[k_i\left(\frac{N}{2}-1\right)\right]}{\sinh\left(k_i\frac{N}{2}\right)}=\frac{\cos k_r}{\cos\theta},\quad \lambda_1+\lambda_2\ \ \text{odd}, \tag{2.7}$$

$$\frac{\cosh\left[k_i\left(\frac{N}{2}-1\right)\right]}{\cosh\left(k_i\frac{N}{2}\right)}=\frac{\cos k_r}{\cos\theta},\quad \lambda_1+\lambda_2\ \ \text{even}. \tag{2.8}$$

We shall distinguish solutions of Eq. (2.7) and call them s solutions (strings) from those of Eq. (2.8), which we shall call c solutions (strings). In fact, these equations will give a basis for a natural classification of solutions. Any solution in

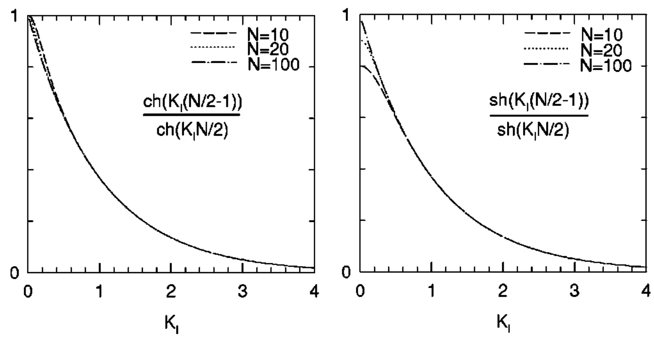

FIG. 1. Graphical description of the left-hand sides of Eqs. (2.7) and (2.8) for some values of N.

the two-particle sector will depend only on the sum of Bethe numbers and its parity. A choice of different Bethe numbers that gives the same sum [e.g., $(\frac{3}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2})$] corresponds to taking different branches of the phase shift in Eqs. (2.1) and (2.2). As we shall see later, a number of solutions will be different in these two classes and exceptions to the string conjecture will be due to the class s only. As the sum of Bethe numbers can be taken between $-N+1$ and $N-1$ for N even and between $-N$ and $N-2$ for N odd, we see that k_r can take $2N-1$ different equidistant values between $-\pi$ and π. From Eqs. (2.7) and (2.8) (and Fig. 1), we see that admissible interval for $\cos k_r$ is

$$s \text{ strings: } 0 \leqslant \cos k_r < \Delta\left(1-\frac{2}{N}\right), \tag{2.9}$$

$$c \text{ strings: } 0 \leqslant \cos k_r < \Delta \tag{2.10}$$

for $\Delta \geqslant 0$ $(0 \leqslant \theta \leqslant \pi/2)$ and

$$s \text{ strings: } \Delta\left(1-\frac{2}{N}\right) < \cos k_r \leqslant 0, \tag{2.11}$$

$$c \text{ strings: } \Delta < \cos k_r \leqslant 0 \tag{2.12}$$

for $\Delta \leqslant 0$ $(\pi/2 \leqslant \theta \leqslant \pi)$. The energy of complex solutions, according to Eq. (1.5), will be given with

$$E = 4\Delta - 4\cos k_r \cosh k_i\,. \tag{2.13}$$

Here, we measure the energy from the referent state with all spins up. Due to relations (2.8) and (2.7) one can see that energy intervals for complex solutions are

$$0 < E(c \text{ strings}) \leqslant 2\Delta, \tag{2.14}$$

$$\frac{8\Delta}{N} < E(s \text{ strings}) \leqslant 2\Delta. \tag{2.15}$$

Now the left side of both Eqs. (2.7) and (2.8) are monotonously decreasing functions so we shall have a solution for k_i for any k_r whose $\cos k_r$ is in the previously mentioned interval. For large N we can approximate admissible interval for s strings with that for c strings. In that case, complex solutions will exist if their real parts satisfy inequality $0 \leqslant \cos k_r \leqslant \Delta$. As we have $(2N-1)/2\pi$ solutions per unit k_r interval, we conclude that

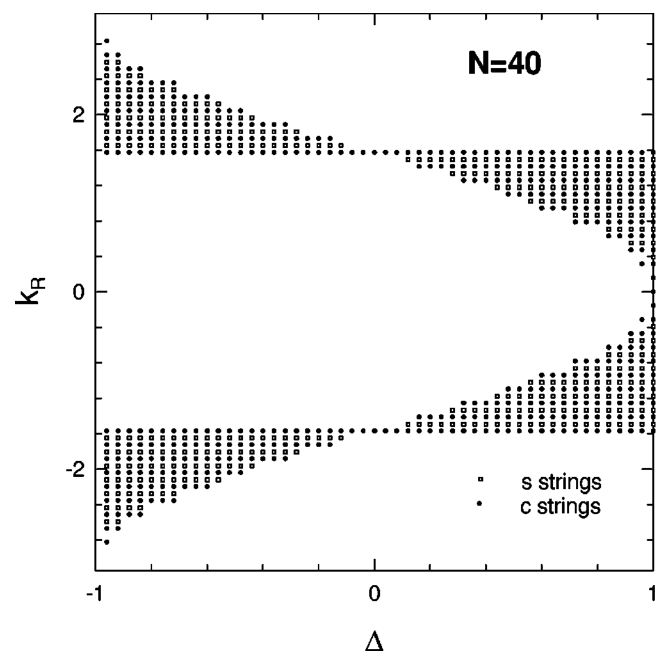

FIG. 2. Real parts of complex solutions for $-1 \leqslant \Delta \leqslant 1$ and the number of sites $N=40$. Empty squares denote solutions of Eq. (2.7) (s strings) and full circles denote solutions of Eq. (2.8) (c strings).

$$\frac{1}{2\pi}(2N-1)(\pi-2\theta) \tag{2.16}$$

solutions in form of strings can be obtained. This is consistent (up to at most two solutions) with the string conjecture and result (1.12).

III. NUMBER OF BOUND STATES (COMPLEX SOLUTIONS) AND VIOLATION OF THE STRING HYPOTHESIS

We want to determine the number of bound states as a function of the coupling constant Δ and the number of sites N. We shall first consider complex solutions for fixed N and different Δ. In Fig. 2 the case $N=40$ is presented. For each (calculated) Δ real parts of possible complex solutions are given. We see that in the region of negative coupling constant the complex solutions are present for $\pi/2 \leqslant |k_r| \leqslant \pi$ and in the region of positive coupling constant for $0 \leqslant |k_r| \leqslant \pi/2$. As k_r tends to $\pi/2$, k_i increases and so the localization of two spins down increases [notice that the ratio of probability amplitudes for finding spins down on lattice sites n_1 and n_2 is proportional to $\exp-|k_i(n_1-n_2+1)|$]. As we decrease the coupling constant $|\Delta|$, bound states with $|k_r| \geqslant |\theta|$ for $\Delta \leqslant 0$ and with $|k_r| \leqslant |\theta|$ for $\Delta \geqslant 0$ disappear. These are states with the smallest localization. The bound states with high localization (k_i high, $k_r \approx \pi/2$) exist in almost all the region of coupling constant and disappear near the free theory point $(\Delta=0)$. In Fig. 3 and 4 we present numerical analysis of N dependence of string solutions for $\Delta \neq 1$ and $\Delta=1$. In the $\Delta=1$ case, c strings are allowed for all values of $-\pi/2 \leqslant k_r \leqslant \pi/2$ and so their number rises linearly with N as predicted by Eq. (2.16) and the string conjecture. However, the number of s strings rises also linearly with N until the real parts do not reach the region where $1-2/N \leqslant \cos k_r/\Delta \leqslant 1$

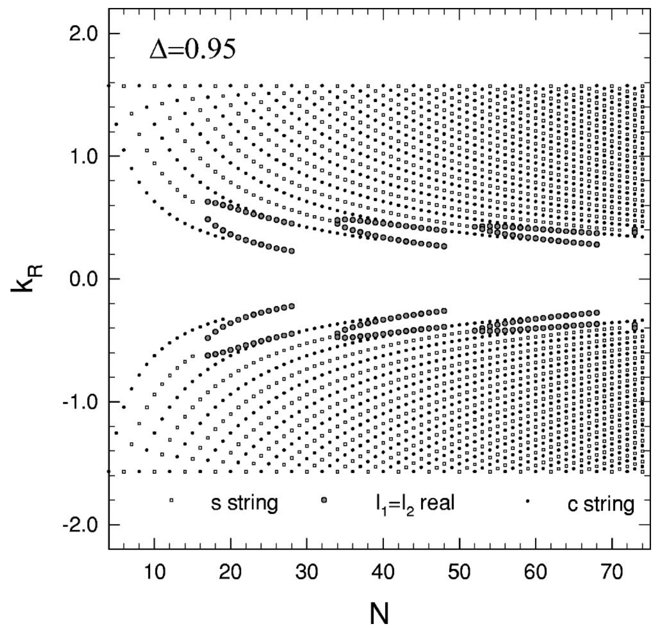

FIG. 3. This figure shows dependence of the real part of complex solutions on numbers of sites N for $\Delta=0.95$. It clearly illustrates transmutation of one complex solution in real solution (two quasimomenta) for a given critical N. These two real quasimomenta correspond to the same Bethe numbers and are obtained by the numerical iteration of Eqs. (2.1) and (2.2).

when such s string is no longer a solution of Bethe equations. So the first two strings disappear for $N=22$, next for 62, 121, etc. Simultaneously with the disappearance of s strings two real solutions with the same (odd) sum of Bethe numbers appear. An odd sum of Bethe numbers can be accomplished by two equal Bethe numbers (which is found by numerical calculation, which favors the choice of the principal branch of the phase shift) and both properties (disappearance of the complex solution and appearance of the real solution with two same Bethe numbers) represent violations of the string conjecture. These results are consistent with the results of Refs. 12 and 13. For $\Delta\neq 1$, however, we shall find a different result. From Fig. 3 we can see that again for certain values of N s strings will disappear and evolve in two close real momenta for which we find identical Bethe numbers. These exceptional real solutions will disappear again after some N, when followed by the numerical iteration method. This is in contrast to the $\Delta=1$ case. In fact, when the solutions are described by more natural classification [Eqs. (2.7) and (2.8) for complex solutions and Eqs. (4.2) and (4.3) for real solutions], one could follow their further development. However, here we were interested specifically in the choice of equal Bethe numbers when solving Eqs. (2.1) and (2.2) directly.

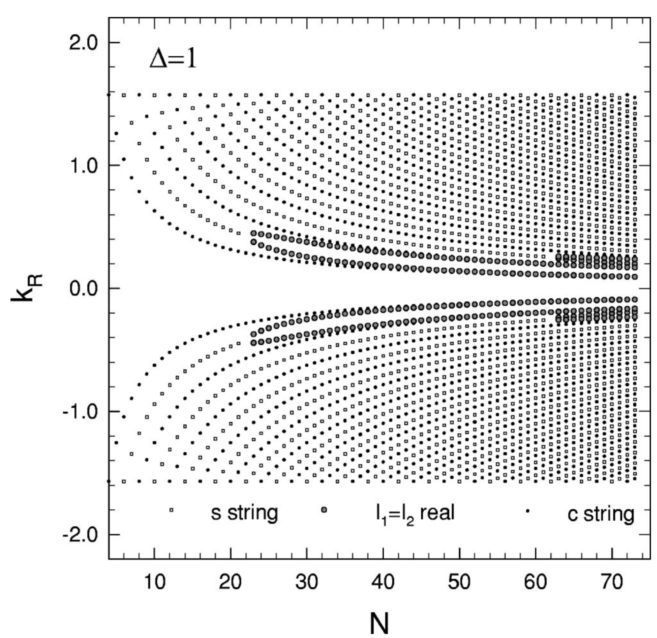

FIG. 4. Same as Fig. 3, but for $\Delta=1$. We see that exceptional real solutions with the same Bethe numbers that appear at some critical N persist for all $N\geqslant N_{\text{crit}}$.

We proceed now to give an analytical expression for the number of exceptions to the string conjecture. Due to the previous discussion we find that the exceptions arise only due to the Eq. (2.7), which has no solutions in the following interval for the sum of Bethe numbers

$$\left(1-\frac{2}{N}\right)\leqslant\frac{\cos\left[\frac{\pi}{N}(\lambda_1+\lambda_2)\right]}{\Delta}\leqslant 1,\quad (\lambda_1+\lambda_2)\ \text{ odd.} \tag{3.1}$$

Now consider inequality (3.1) first for $\Delta=1$. The maximal k_r for which s solutions would still not be possible can be found by expanding $\cos[(\pi/N)(\lambda_1+\lambda_2)]$ around zero. We find

$$(\lambda_1+\lambda_2)^2<\left(\frac{2}{\pi}\sqrt{N}\right)^2, \tag{3.2}$$

where N is number of sites after which two complex solutions (for $+k_r$ and $-k_r$) disappear and become solutions with two real momenta and $\lambda_1=\lambda_2$. For $\lambda_1+\lambda_2=1,3,5,7,\ldots$, we get $N=3,22,62,121,\ldots$. As previously said, this is consistent with Refs. 12 and 13.

Now we turn to general $\Delta\neq 1$. Consider k_r^1 and k_r^2, which are just on the edges of the interval (3.1). They satisfy

$$2\ \sin\left(\frac{k_r^1+k_r^2}{2}\right)\sin\left(\frac{k_r^1-k_r^2}{2}\right)=\frac{2\Delta}{N}. \tag{3.3}$$

From this relation the interval δk_r for which s strings are missing is given with

$$\delta k_r=2\ \arcsin\frac{\Delta}{N\ \sin\left(\frac{\arccos\Delta+\arccos[\Delta(1-2/N)]}{2}\right)}. \tag{3.4}$$

The number of s strings per unit interval of k_r is

$$\frac{1}{2}\ \frac{2N-1}{2\pi}2. \tag{3.5}$$

Here $(2N-1)/2$ is due the fact that we have to count the number of odd values of $\lambda_1+\lambda_2$. As solutions come in pairs (positive and negative total momenta) we need the last factor of 2. Finally, the number of missing strings is an integer part of

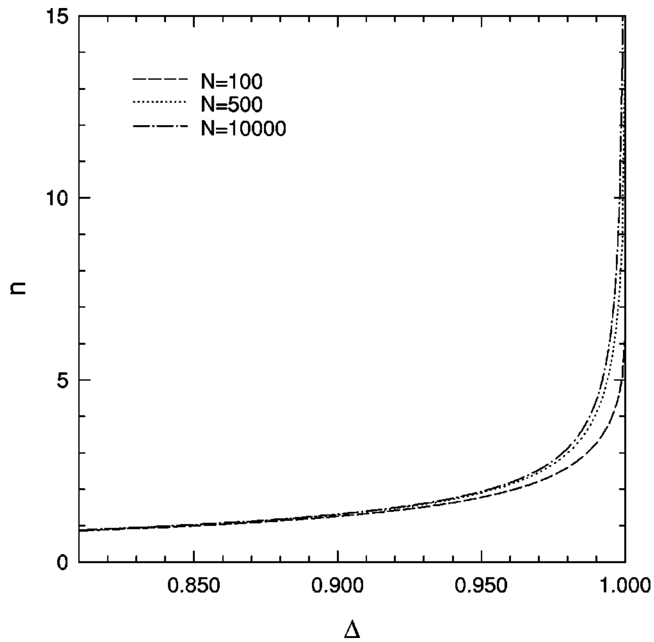

FIG. 5. The number of missing strings is an integer part of real number n, which is given as a function of the coupling constant Δ for three different values of N.

$$n=\frac{2N-1}{\pi}\arcsin\frac{\Delta}{N\sin\left(\frac{\arccos\Delta+\arccos[\Delta(1-2/N)]}{2}\right)}. \tag{3.6}$$

The function (3.6) is shown in Fig. 5 for few values of N. We can see that the number of missing strings is finite for $\Delta\neq 1$ and that there is no violation of the string hypothesis below some value of coupling constant Δ. These strings would have energies (2.13) in the forbidden interval $0\leqslant E\leqslant 8\Delta/N$ that is near the energy of the state with all spins up.

IV. NUMBER OF REAL SOLUTIONS AND COMPLETENESS PROBLEM

In this section we shall search for real solutions of Bethe equations. We start again from Eqs. (2.1) and (2.2). After manipulating their difference and sum we obtain following equations for $k=k_1+k_2$ and k_1-k_2:

$$\frac{k}{2}=\frac{\pi}{N}(\lambda_1+\lambda_2), \tag{4.1}$$

$$\frac{\sin\left[\frac{(k_1-k_2)}{2}\left(\frac{N}{2}-1\right)\right]}{\sin\left[\frac{(k_1-k_2)}{2}\,\frac{N}{2}\right]}=\frac{\cos\frac{k}{2}}{\cos\theta},\quad \lambda_1+\lambda_2\ \text{ odd}, \tag{4.2}$$

$$\frac{\cos\left[\frac{(k_1-k_2)}{2}\left(\frac{N}{2}-1\right)\right]}{\cos\left[\frac{(k_1-k_2)}{2}\,\frac{N}{2}\right]}=\frac{\cos\frac{k}{2}}{\cos\theta},\quad \lambda_1+\lambda_2\ \text{ even}. \tag{4.3}$$

From condition (4.1) we can find number of different momenta k. As was already mentioned, there are $2N-1$ different values of $\lambda_1+\lambda_2$. But changing $\lambda_1+\lambda_2$ by N is equivalent with changing one quasimomentum by 2π that gives the same solution. This reduces the number of possible values $\lambda_1+\lambda_2$ to N, e.g., $\lambda_1+\lambda_2=0,1,\ldots,N-1$. The left-hand sides of Eqs. (4.2) and (4.3) are periodic functions. Thus in principle, for each of N different fixed values of the right-hand side one can count the number of solutions by counting the number of intersections. For a given value of the $\lambda_1+\lambda_2$ and Δ we can find following number X of intersections for N even

$$\left|\frac{\cos\frac{k}{2}}{\Delta}\right|>1,\ X=\binom{N}{2};\quad \left|\frac{\cos\frac{k}{2}}{\Delta}\right|<1,\ X=\binom{N-2}{2}\ , \tag{4.4}$$

$$\left|\frac{\cos\frac{k}{2}}{\Delta}\right|>1-\frac{2}{N},\ X=\binom{N-2}{2};\quad \left|\frac{\cos\frac{k}{2}}{\Delta}\right|<1-\frac{2}{N},$$
$$X=\binom{N-4}{2}, \tag{4.5}$$

for $\lambda_1+\lambda_2$ even and odd, respectively. For N odd

$$\left|\frac{\cos\frac{k}{2}}{\Delta}\right|>1,\ X=\binom{N-1}{2};\quad \left|\frac{\cos\frac{k}{2}}{\Delta}\right|<1,\ X=\binom{N-3}{2}, \tag{4.6}$$

$$\left|\frac{\cos\frac{k}{2}}{\Delta}\right|>1-\frac{2}{N},\ X=\binom{N-3}{2};\quad \left|\frac{\cos\frac{k}{2}}{\Delta}\right|<1-\frac{2}{N},$$
$$X=\binom{N-3}{2}, \tag{4.7}$$

for $\lambda_1+\lambda_2$ even and odd, respectively. When right-hand side of Eqs. (4.3) and (4.2) becomes smaller than 1 and $1-2/N$, respectively, corresponding the real solution (in fact a pair with $\pm k$) disappears and we get a pair of complex solutions with positive and negative real parts. Now we can proceed to obtain the full number of solutions. For N even we have $N/2$ $(N/2)$ possible values for $\lambda_1+\lambda_2$ even (odd). For N odd there are $(N-1)/2$ $[(N+1)/2]$ possible values for $\lambda_1+\lambda_2$ even (odd). Together with the results from the previous section on complex solutions one can count the total number of real and complex solutions. It is important to realize that the disappearance of pair of real solutions results in the formation of a two complex solution and vice versa. Let us count number of solutions in two extreme cases $\Delta\to 0$ and $\Delta\geqslant N/(N-2)$. For $\Delta\to 0$ there are no complex solutions and the number of real solutions is $N_{real}=N^2/2-N/2=\binom{N}{2}$. For $\Delta\geqslant N/(N-2)$ the number of complex solutions is N $(\cos k_r/\Delta\leqslant 1-2/N)$ and from Eqs. (4.4) and (4.6) number of real solutions is $N_{real}=N^2/2-3N/2=\binom{N}{2}-\binom{N}{1}$. Again total number of solutions is $\binom{N}{2}$. We conclude that we find $\binom{N}{2}$ solutions of Bethe Eqs. (2.1) and (2.2) for every value of Δ. We stress that this result is obtained without assuming string conjecture, which is usually assumed in completeness proofs.

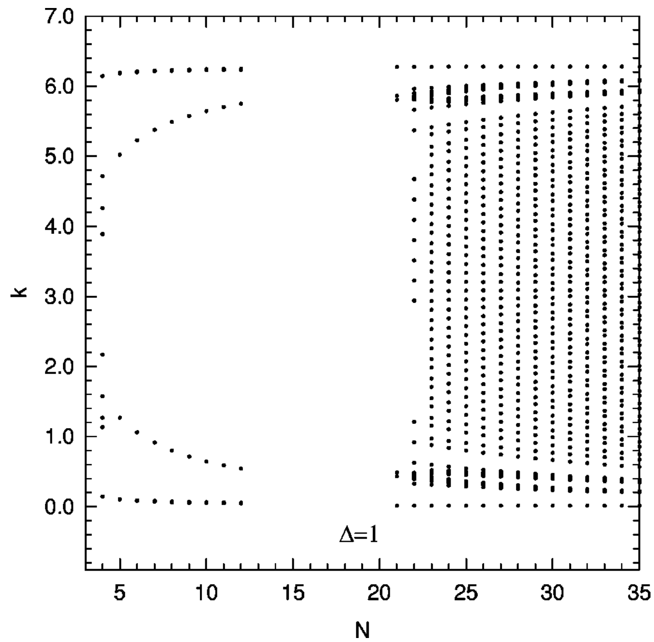

FIG. 6. All real solutions (triplets of quasimomenta) in the $M=3$ sector with at least two identical Bethe numbers are given for different numbers of sites and $\Delta=1$. They are obtained by the numerical iteration of Bethe Eqs. (1.7).

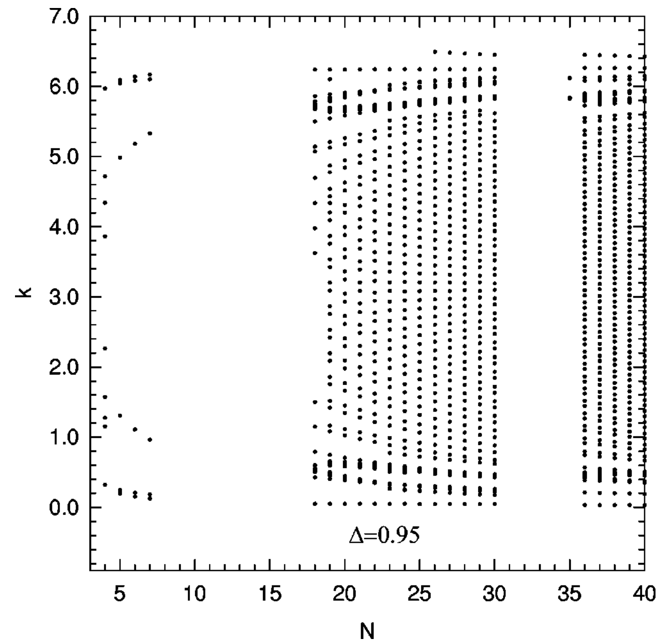

FIG. 7. Same as Fig. 6, but for $\Delta=0.95$.

In fact, as we discussed in this paper, string conjecture has exceptions. However, they do not affect completeness proofs because the disappearance of complex solution (bound state) results in the appearance of real solution and vice versa, so that the total number is unchanged.

As we have explained already, analysis was done for simplicity reasons in the two-particle sector. Of course a systematic analysis for higher sectors may be desirable but it is much more complicated. However, we will mention some preliminary results for the $M=3$ sector. By numerical analysis we search for exceptions to string conjecture among real solutions with coinciding two or all Bethe numbers. They are exhibited in Fig. 6 and Fig. 7 for $\Delta=1$ and $\Delta=0.95$. They show similar regularities as the $M=2$ case. In particular, with the appearance of real solutions violating string conjecture in the two-particle sector in the three-particle sector such exceptions arise in the form of perturbed pair of near momenta (of identical Bethe numbers in $M=2$ sector) and a third almost independent momenta with a distinct Bethe number. For instance, Fig. 6 for $\Delta=1$ shows that the appearance of such solutions around $N=22$ similar to the Fig. 4 for $M=2$ and $\Delta=1$. On Fig. 7 for $\Delta=0.95$ we see that such solutions are found in finite intervals of N. This is again the same as in the $M=2$ case (Fig. 3). Finally this preliminary investigation for larger M raises hope that a simple pattern for the exception to string conjecture could arise.

V. CONCLUSION

In this paper we count all complex and real solutions of Bethe equations in the two-particle sector. The complex solutions are classified in two classes. For one of them (s class) the sum of Bethe numbers is odd and for the other (c class) it is even. We are able to count the number of solutions in each class for a given coupling constant Δ and the number N of lattice sites. In such a way we are able to check the validity of usual string conjecture. We find that there are exceptions to string conjecture and that they are entirely due to the s class of solutions. In particular, in the thermodynamic limit we show that number of these exceptions is finite for $\Delta\neq1$ contrary to the $\Delta=1$ case, where it was previously known that it is infinite. Finally, we also show independently of the string conjecture that the number of all solutions is $\binom{N}{2}$ and that is required for completeness. The usual proofs of completeness rely on string hypothesis. Some preliminary numerical results have been presented also for the three-particle sector. These results suggest that a similar pattern observed in the two-particle sector persists also for larger sectors.

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