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Hawking fluxes, fermionic currents, $W_{1+\infty}$ algebra, and anomalies

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We complete the analysis carried out in previous papers by studying the Hawking radiation for a Kerr black hole carried to infinity by fermionic currents of any spin. We find agreement with the thermal spectrum of the Hawking radiation for fermionic degrees of freedom. We start by showing that the near-horizon physics for a Kerr black hole is approximated by an effective two-dimensional field theory of fermionic fields. Then, starting from two-dimensional currents of any spin that form a $W_{1+\infty}$ algebra, we construct an infinite set of covariant currents, each of which carries the corresponding moment of the Hawking radiation. All together they agree with the thermal spectrum of the latter. We show that the predictive power of this method is based not on the anomalies of the higher-spin currents (which are trivial) but on the underlying $W_{1+\infty}$ structure. Our results point toward the existence in the near-horizon geometry of a symmetry larger than the Virasoro algebra, which very likely takes the form of a W_∞ algebra.

I. INTRODUCTION

This paper is complementary to [1,2]. The subject of these previous papers was the calculation of the Hawking radiation and its thermal spectrum by the method of anomalies and the role played by a W_∞ algebra of currents in this derivation.

Hawking radiation [3,4] does not depend on the details of the collapse that gives rise to a black hole. Therefore one expects that the methods to calculate it should have the same character of universality. The anomaly method has these features. The first attempt to compute Hawking radiation by exploiting trace anomalies was made by Christensen and Fulling, [5] (see also [6]), and reproposed subsequently by [7,8] in a modified form. More recently a renewed attention to the same problem has been pioneered by the paper [9], which makes use of the diffeomorphisms anomaly. This paper led to a number of contributions [10–68].

Most of these papers are concerned with the derivation of the integrated Hawking radiation and do not describe its spectrum. However, one of the most interesting features of the Hawking radiation is precisely its thermal spectrum. This can be "Fourier analyzed" and expressed in terms of its higher moments or fluxes. An interesting proposal was made by the authors of [12–15], who attributed these higher fluxes to phenomenological higher-spin currents, i.e., higher-spin generalizations of the energy-momentum tensor.

In [1] it was shown that such higher currents do describe the higher-spin fluxes of the Hawking radiation. The main result of [1] was that this is not due to their trace anomalies, but rather the their underlying W_∞ algebra structure. In fact it was shown in [1] that these higher-spin currents cannot have trace anomalies and in [2] that they cannot have diffeomorphism anomalies (or, rather, that if there are anomalies they are trivial). In [1,2] the analysis was limited to bosonic higher-spin currents. In the present paper we would like to extend the analysis to fermionic currents. Our conclusion will not change: the thermal spectrum of the Hawking radiation is induced not by the anomalies of such currents, which do not exist, but by their underlying $W_{1+\infty}$ structure [the 1 stands for the extension of the W_∞ algebra to include a U(1) current]. We will also examine some aspects of the $W_{1+\infty}$ algebra which were not duly clarified in [1,2], but are basic to appreciating the central role of the W_∞ algebra. The main conclusion of our series of papers is that the Hawking radiation, and, in particular, its thermal spectrum, points toward the existence in the near-horizon region of a symmetry much larger than the Virasoro algebra, that is, a W_∞ or a $W_{1+\infty}$ algebra.

In this paper we will start (Sec. II) from a Kerr-like metric in four dimensions (4D) and consider fermionic matter coupled to it and to a background electromagnetic field. As in [1,2] we will reduce the problem to two dimensions. This can be done by using azimuthal symmetry and the near-horizon properties in the Kerr background. The spinor field $\psi(t, r, \theta, \varphi)$ will be expanded in the appropriate spherical harmonics. After integrating the action over the polar angles one is left with an infinite number of free two-dimensional spinor fields interacting with the background gravity specified by the metric

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 \tag{1}$$

as well as with the electromagnetic field. $f(r)$ near the

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horizon behaves like $f(r) \approx 2\kappa(r - r_H)$, where κ is the surface gravity. In the following we will focus on one of these complex fermion fields. The analysis for all the other fermion fields is the same; what is left out of our analysis is the resummation of all these contributions and the attainment of the relevant four-dimensional information (see for instance [69]). After Sec. II the paper is organized as follows. In Sec. III we recall the trace anomaly method, which is basic in this paper. In Sec. IV we introduce the currents of the $W_{1+\infty}$ algebra relevant to our problem. In Sec. V we construct the covariant higher-spin currents and show that their flux at infinity is in agreement with the moments of the fermionic Hawking radiation. In Sec. VI we discuss the problem of trace anomalies in higher-spin currents and, as in [1,2], show on general grounds that there cannot be trace anomalies in these currents in accord with our explicit construction in the previous section. Finally in Sec. VII we draw our conclusions. Two appendixes are devoted to some details of the calculations in Secs. II and V.

II. REDUCTION TO TWO DIMENSIONS

We start with the four-dimensional action for fermions in a curved background:

$$
\begin{aligned}
S &= \int d^4x \sqrt{-g}\, \bar{\psi}\, \slashed{\nabla}\, \psi \\
&= \int d^4x \sqrt{-g}\, \psi^\dagger \gamma^0 \gamma^a e_a{}^\mu \left(\partial_\mu - \frac{1}{8} \omega_{bc\mu}[\gamma^b, \gamma^c] \right)\psi,
\end{aligned}
\tag{2}
$$

where the vierbein $e^a{}_\mu$ satisfies $\eta_{ab}e^a{}_\mu e^b{}_\nu = g_{\mu\nu}$, and the spin connection $\omega^a{}_{b\mu}$ is given by $\omega^a{}_{b\mu} = e^a{}_\nu \nabla_\mu e_b{}^\nu$. (Indices a, b, $c = 0, 1, 2, 3$ are flat; indices μ, $\nu = t, r, \theta, \phi$ are curved.)

We consider the Kerr metric,

$$
\begin{aligned}
ds^2 &= \frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\phi)^2 - \frac{\sin^2\theta}{\Sigma}(adt - (r^2 + a^2)d\phi)^2 \\
&\quad - (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{\Delta} + d\theta^2\right),
\end{aligned}
\tag{3}
$$

and we choose the following local Lorentz frame (i.e., the vierbein) $e_a{}^\mu$:

$$
\sqrt{\Delta\Sigma}\, e_0{}^\mu \partial_\mu = (r^2 + a^2)\partial_t + a\partial_\phi,
$$

$$
\sqrt{\Delta\Sigma}\, e_1{}^\mu \partial_\mu = \Delta\partial_r, \qquad \sqrt{\Delta\Sigma}\, e_2{}^\mu \partial_\mu = \sqrt{\Delta}\partial_\theta, \tag{4}
$$

$$
\sqrt{\Delta\Sigma}\, e_3{}^\mu \partial_\mu = \sqrt{\Delta}\left(a\sin\theta\partial_t + \frac{\partial_\phi}{\sin\theta}\right),
$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = (r - r_+)(r - r_-)$, $r_+ + r_- = 2M$, $r_+ r_- = a^2$. Near the horizon we have $r \to r_+$ and consequently $\Delta \to 0$. From the third and the fourth relation of (4) we see that the terms in the action (2) which are multiplied by $\gamma^2 e_2{}^\mu$ and $\gamma^3 e_3{}^\mu$ are suppressed by a

factor of $\sqrt{\Delta}$. We can see that the term $\gamma^1 e_1{}^\mu \partial_\mu$ is not suppressed, by changing to tortoise coordinate r^* defined by $\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}$. Expressed in terms of r^*, $\sqrt{\Delta\Sigma}e_1{}^\mu \partial_\mu$ becomes $(r^2 + a^2)\partial_{r^*}$. Therefore, the leading order contribution from the term $\gamma^a e_a{}^\mu \partial_\mu$ in the action (2) is $\gamma^0 e_0{}^t \partial_t + \gamma^0 e_0{}^\phi \partial_\phi + \gamma^1 e_1{}^r \partial_r$, and is of order $1/\sqrt{\Delta}$. Furthermore, a straightforward calculation shows that the leading contribution of the term $e_c{}^\mu \omega_{ab\mu}$ comes from $e_0{}^\mu \omega_{01\mu} = -e_0{}^\mu \omega_{10\mu} = \frac{r_+ - r_-}{2\sqrt{\Delta\Sigma}}$ and is also of order $1/\sqrt{\Delta}$ (the spin coefficients are listed in the Appendix A).

In summary, on the horizon $r \approx r_+$, we obtain in the leading order

$$
\begin{aligned}
\slashed{\nabla}\psi &= \left\{ \frac{\gamma^0}{\sqrt{\Delta\Sigma}}[(r_+^2 + a^2)\partial_t + a\partial_\phi] \right. \\
&\quad \left. + \frac{\gamma^1}{\sqrt{\Delta\Sigma}}\left[(r_+^2 + a^2)\partial_{r^*} - \frac{1}{4}(r_+ - r_-) \right] \right\}\psi. \tag{5}
\end{aligned}
$$

To be able to integrate over θ and ϕ in the action (2), we expand ψ in the following way: $\psi = \sum_{lm} \psi_{lm}(t, r) S_{lm}(\theta) e^{-im\phi}$, where S_{lm} are normalized so that $\int d\theta \sqrt{\Sigma}\sin\theta S_{lm}^*(\theta) S_{l'm}(\theta) = 2\delta_{ll'}$. That produces the change $\partial_\phi \to -im$. We first integrate over ϕ, and then over θ, using the normalization condition for S_{lm} and obtain

$$
\begin{aligned}
S &= 4\pi \int dt dr \frac{r_+^2 + a^2}{\sqrt{\Delta}} \sum_{lm} \psi_{lm}^\dagger \left\{ \gamma^0 \gamma^0 \left(\partial_t - \frac{iam}{r_+^2 + a^2} \right) \right. \\
&\quad \left. + \gamma^0 \gamma^1 \left(\partial_{r^*} - \frac{r_+ - r_-}{4(r_+^2 + a^2)} \right) \right\}\psi_{lm}.
\end{aligned}
$$

We choose the following gamma matrices in 4D,

$$
\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad
\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\gamma^2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \qquad
\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},
$$

and the following gamma matrices in 2D,

$$
\sigma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6}
$$

The choice ensures that $\gamma^0\gamma^1$ and $\sigma^0\sigma^1$ look very simple. Both are diagonal, and satisfy $\gamma^0\gamma^1 = I \otimes \sigma^0\sigma^1$. Since $\gamma^0\gamma^1$ and $\sigma^0\sigma^1$ generate 01-Lorentz transformation in 4D and 2D, respectively, the upper two, as well as the lower two components of ψ_{lm}, will transform like the 2D spinors. We denote the two upper components by $\chi_{(1)lm}$ and the two lower by $\chi_{(2)lm}$:

$$\psi_{lm} = \begin{pmatrix} \chi_{(1)lm} \\ \chi_{(2)lm} \end{pmatrix}. \tag{7}$$

In terms of $\chi_{(s)lm}$ ($s = 1, 2$) the action reads

$$S = 4\pi \int dtdr \frac{r_+^2 + a^2}{\sqrt{\Delta}} \sum_{s=1}^{2} \sum_{lm} \chi_{(s)lm}^{\dagger} \left\{ \sigma^0 \sigma^0 \left(\partial_t - \frac{iam}{r_+^2 + a^2} \right) \right.$$
$$\left. + \sigma^0 \sigma^1 \left(\partial_{r^*} - \frac{r_+ - r_-}{4(r_+^2 + a^2)} \right) \right\} \chi_{(s)lm}.$$

Now we show that we can interpret the action in terms of 2D quantities: the spinors $\chi_{(s)lm}$, the metric (1), its *zweibein* $e_i^{(2)\alpha}$ and spin connection $\omega_{jk\alpha}^{(2)}$, a vector potential A_α, and a dilaton Φ. We take the letters $i, j, k = 0, 1$ to denote flat 2D indices, and $\alpha = t, r$ to denote the curved. First we calculate the 2D covariant derivative $^{(2)}\nabla_\alpha$ contracted with 2D gamma matrices $\sigma^i e_i^{(2)\alpha}$,

$$^{(2)}\slashed{\nabla}\chi = \sigma^i e_i^{(2)\alpha} \left(\partial_\alpha - \frac{1}{8} \omega_{jk\alpha}^{(2)} [\sigma^j, \sigma^k] \right) \chi$$
$$= \left\{ \frac{\sigma^0}{\sqrt{f(r)}} \partial_t + \frac{\sigma^1}{\sqrt{f(r)}} \left[\partial_{r^*} - \frac{f'(r)}{4} \right] \right\} \chi. \tag{8}$$

Next, motivated by the fact that for the 4D metric (3) the tortoise coordinate satisfies $\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}$, whereas for the 2D metric (1) it satisfies $\frac{dr^*}{dr} = \frac{1}{f(r)}$, we identify

$$f(r) = \frac{\Delta(r)}{r^2 + a^2}. \tag{9}$$

Finally, plugging this into (8), we see that in the leading order near the horizon we can write the action in the following way:

$$S = \sum_{s=1}^{2} \sum_{lm} 4\pi \int dtdr \Phi \bar{\chi}_{(s)lm} \slashed{D} \chi_{(s)lm}, \tag{10}$$

where the covariant derivative now includes the gauge part $D_\alpha = {}^{(2)}\nabla_\alpha - iqA_\alpha$, and the charge q of $\chi_{(s)lm}$ is m. This is the 2D action for an infinite number of two component fermions $\chi_{(s)lm}$ in the background given by the dilaton Φ,

$$\Phi = \sqrt{r^2 + a^2}, \tag{11}$$

the gauge field A_α,

$$A_t = \frac{e_0{}^\phi}{e_0{}^t} = \frac{a}{r^2 + a^2}, \qquad A_r = 0, \tag{12}$$

and the metric (1).

In the sequel we restrict our analysis to the near-horizon region. In this region the dilaton is approximately constant, so we may disregard it: the equations of motion are those of free fermions in two dimensions, coupled to the metric and the gauge field (but not to the dilaton).

III. THE TRACE ANOMALY METHOD

To start with let us recall the trace anomaly method to compute the integrated Hawking radiation (in the absence of a gauge field). With reference to the metric (1) we transform it into a conformal metric by means of the "tortoise" coordinate r_* defined via $\frac{\partial r}{\partial r_*} = f(r)$. Next we introduce light-cone coordinates $u = t - r_*$, $v = t + r_*$. Let us denote by $T_{uu}(u, v)$ and $T_{vv}(u, v)$ the classically nonvanishing components of the energy-momentum tensor in these new coordinates. Our black hole problem is now reduced to the background metric $g_{\alpha\beta} = e^\varphi \eta_{\alpha\beta}$, where $\varphi = \log f$. The energy-momentum tensor can be calculated by integrating the conservation equation and using the trace anomaly. The result is (see [2])

$$T_{uu}(u, v) = \frac{\hbar c_R}{24\pi} \left(\partial_u^2 \varphi - \frac{1}{2} (\partial_u \varphi)^2 \right) + T_{uu}^{(\text{hol})}(u), \tag{13}$$

where $T_{uu}^{(\text{hol})}$ is holomorphic, while T_{uu} is conformally covariant. Namely, under a conformal transformation $u \to \tilde{u} = f(u)(v \to \tilde{v} = g(v))$ one has

$$T_{uu}(u, v) = \left(\frac{df}{du} \right)^2 T_{\tilde{u}\tilde{u}}(\tilde{u}, v). \tag{14}$$

Since, under a conformal transformation, $\tilde{\varphi}(\tilde{u}, \tilde{v}) = \varphi(u, v) - \ln(\frac{df}{du} \frac{dg}{dv})$, it follows that

$$T_{\tilde{u}\tilde{u}}^{(\text{hol})}(\tilde{u}) = \left(\frac{df}{du} \right)^{-2} \left(T_{uu}^{(\text{hol})}(u) + \frac{\hbar c_R}{24\pi} \{\tilde{u}, u\} \right). \tag{15}$$

Let us pass to Kruskal coordinates, which are regular at the horizon, i.e., to (U, V) defined by $U = -e^{-\kappa u}$ and $V = e^{\kappa v}$. Under this transformation we have

$$T_{UU}^{(\text{hol})}(U) = \left(\frac{1}{\kappa U} \right)^2 \left(T_{uu}^{(\text{hol})}(u) + \frac{\hbar c_R}{24\pi} \{U, u\} \right). \tag{16}$$

Now we require the outgoing energy flux to be regular at the future horizon $U = 0$ in the Kruskal coordinate. Therefore at that point $T_{uu}^{(\text{hol})}(u)$ is given by $\frac{c_R \kappa^2}{48\pi}$. As was noticed in [2] this requirement corresponds to the condition that $T_{uu}(u, v)$ vanishes at the horizon.

Since the background is static, $T_{uu}^{(\text{hol})}(u)$ is constant in t and therefore also in r. Therefore at $r = \infty$ it takes the same value $\frac{\hbar c_R \kappa^2}{48\pi}$. On the other hand we can assume that at $r = \infty$ there is no incoming flux and that the background is trivial [so that the vacuum expectation values of $T_{uu}^{(\text{hol})}(u)$ and $T_{uu}(u, v)$ asymptotically coincide].

Therefore the asymptotic flux is (we denote by $\langle \cdot \rangle$ the value at infinity)

$$\langle T_t^r \rangle = \langle T_{uu} \rangle - \langle T_{vv} \rangle = \frac{\hbar \kappa^2}{48\pi} c_R. \tag{17}$$

This is the integrated Hawking radiation (see below).

We would like to apply a similar method to the higher-spin currents. Let us start by recalling a few notions about the thermal fermionic radiation.

The thermal fermionic spectrum of the Kerr black hole is given by the Planck distribution

$$N(\omega) = \frac{1}{e^{\beta(\omega - m\Omega)} + 1},$$

where $1/\beta$ is the Hawking temperature of the black hole, ω is the absolute value of the momentum ($\omega = |k|$), and Ω is the total angular momentum, in our case $\Omega = A_t$ evaluated at the horizon, and m is the charge.

Let us consider first the case $m = 0$. In two dimensions we can define the flux moments F_n, which vanish for n odd, while for n even they are given by [16]

$$F_{2n} = \frac{1}{2\pi} \int_0^\infty d\omega \frac{\omega^{2n-1}}{e^{\beta\omega} + 1}$$
$$= \frac{\kappa^{2n} B_{2n}}{8\pi n} (1 - 2^{1-2n})(-1)^{n+1}, \qquad (18)$$

where B_s's are the Bernoulli numbers ($B_2 = 1/6$, $B_4 = -1/30, \dots$) and $\kappa = 2\pi/\beta$ is the surface gravity of the black hole.

When $m \neq 0$ we do not have similar compact formulas; however, it makes sense to sum over the emission of a particle (with charge m) and the corresponding antiparticle (with charge $-m$). In this case the flux moments become

$$F_{n+1}^\Omega = \frac{1}{2\pi} \left(\int_0^\infty dx \frac{x^n}{e^{\beta(x - m\Omega)} + 1} - (-1)^n \right.$$
$$\left. \times \int_0^\infty dx \frac{x^n}{e^{\beta(x + m\Omega)} + 1} \right)$$
$$= \frac{(m\Omega)^{n+1}}{2\pi(n+1)} - \sum_{k=1}^{[(n+1)/2]} (-1)^k$$
$$\times \frac{n!(1 - 2^{1-2k})\kappa^{2k}}{2\pi(2k)!(n+1-2k)!} B_{2k}(m\Omega)^{n+1-2k}. \quad (19)$$

Once we know F_n^Ω we do not have enough information to reconstruct the full thermal spectrum with $m \neq 0$, but being able to reproduce the moments F_n^Ω represents anyhow an important positive test.

IV. A $W_{1+\infty}$ ALGEBRA

In order to derive the higher Hawking fluxes the same way we derived above the integrated Hawking radiation, we postulate the existence of conserved spin currents consisting of fermionic bilinears in the 2D effective field theory near the horizon. They will play a role analogous to the energy-momentum tensor for the integrated radiation (the lowest moment). To construct such currents we start from a $W_{1+\infty}$ algebra defined in an abstract flat space spanned by a local coordinate z. These currents were introduced in [70] (see also [71–74]):

$$j_{z\dots z}^{(s)}(z) = -\frac{B(s)}{s} \sum_{k=1}^s (-1)^k \binom{s-1}{s-k}^2 :\partial_z^{s-k}$$
$$\times \Psi^\dagger(z) \partial_z^{k-1} \Psi(z): \qquad (20)$$

$$B(s) \equiv \frac{2^{s-3} s!}{(2s-3)!!} q^{s-2}, \qquad s = 1, 2, 3, \dots, \qquad (21)$$

where q is a deformation parameter.

The spin s currents $j_{z\dots z}^{(s)}(z)$ are linear combinations of bilinears

$$j_{z\dots z}^{(m,n)}(z) \equiv :\partial^m \Psi^\dagger \partial^n \Psi:$$
$$\equiv \lim_{z_1, z_2 \to z} (\partial_{z_1}^m \Psi^\dagger(z_1) \partial_{z_2}^n \Psi(z_2)$$
$$- \partial_{z_1}^m \partial_{z_2}^n \langle \Psi^\dagger(z_1) \Psi(z_2) \rangle).$$

We want to relate the currents written in two different coordinate systems, connected by coordinate change $z \to w(z)$. That is, we would like to obtain a relation analogous to the one found in [1]

$$j_{z\dots z}^{(s)}(z) \to \left(\frac{1}{\kappa w}\right)^s (j_{z\dots z}^{(s)} + \langle X_s^F \rangle) \qquad (22)$$

and apply it to the transformation $w(z) = -e^{-\kappa z}$ so as to obtain the value of $j_{z\dots z}^{(s)}(z)$ at the horizon by requiring regularity.

The following transformation property of holomorphic fermionic fields will be needed:

$$\Psi(z) = (w'(z))^{1/2} \Psi(w)$$

Using it we get

$$:\partial_{z_1}^m \Psi^\dagger(z_1) \partial_{z_2}^n \Psi(z_2): = \partial_{z_1}^m \partial_{z_2}^n :\Psi^\dagger(z_1)\Psi(z_2): = \partial_{z_1}^m \partial_{z_2}^n (\Psi^\dagger(z_1)\Psi(z_2) - \langle \Psi^\dagger(z_1)\Psi(z_2) \rangle)$$
$$= \partial_{z_1}^m \partial_{z_2}^n ((w_1'(z_1))^{1/2}(w_2'(z_2))^{1/2} \Psi^\dagger(w_1)\Psi(w_2) - \langle \Psi^\dagger(z_1)\Psi(z_2) \rangle)$$
$$= \partial_{z_1}^m \partial_{z_2}^n ((w_1'(z_1))^{1/2}(w_2'(z_2))^{1/2} :\Psi^\dagger(w_1)\Psi(w_2):) + \partial_{z_1}^m \partial_{z_2}^n ((w_1'(z_1))^{1/2}(w_2'(z_2))^{1/2} \langle \Psi^\dagger(w_1)\Psi(w_2) \rangle)$$
$$- \langle \Psi^\dagger(z_1)\Psi(z_2) \rangle).$$

Let us set

$$G(z_1, z_2) \equiv -((w_1'(z_1))^{1/2}(w_2'(z_2))^{1/2} \langle \Psi^\dagger(w_1)\Psi(w_2) \rangle - \langle \Psi^\dagger(z_1)\Psi(z_2) \rangle) \qquad (23)$$

Then

$$:\partial_{z_1}^m \Psi^\dagger(z_1)\partial_{z_2}^n \Psi(z_2): = \partial_{z_1}^m \partial_{z_2}^n ((w_1'(z_1))^{1/2}$$
$$\times (w_2'(z_2))^{1/2}:\Psi^\dagger(w_1)\Psi(w_2):)$$
$$- \partial_{z_1}^m \partial_{z_2}^n G(z_1, z_2). \qquad (24)$$

The propagator for fermionic holomorphic fields is given by

$$\langle \Psi^\dagger(z)\Psi(w)\rangle = \frac{\lambda}{z - w}. \qquad (25)$$

The value of λ is determined in such a way as to reproduce the transformation properties of the energy-momentum tensor and, in physical units, is proportional to $\hbar$. Eventually we will set $\lambda = \hbar$. We are interested in the transformation properties of fermionic currents when $w(z)$ is $w(z) = -e^{-\kappa z}$. Using this we have

$$G(z_1, z_2) = G(z_1 - z_2)$$
$$= -\lambda\left(\frac{\kappa/2}{\sinh(\frac{\kappa}{2}(z_1 - z_2))} - \frac{1}{z_1 - z_2}\right). \qquad (26)$$

Proceeding with our currents (20) we obtain

$$j_{z...z}^{(s)}(z) = -\frac{B(s)}{s}\sum_{k=1}^s (-1)^k\binom{s-1}{s-k}^2 \lim_{z_1 \to z_2} \partial_{z_1}^{s-k}\partial_{z_2}^{k-1}$$
$$\times ((w_1'(z_1))^{1/2}(w_2'(z_2))^{1/2}:\Psi^\dagger(w_1)\Psi(w_2):)$$
$$+ \langle X_s^F\rangle, \qquad (27)$$

where

$$\langle X_s^F\rangle \equiv -\frac{B(s)}{s}\sum_{k=1}^s (-1)^{k+1}\binom{s-1}{s-k}^2$$
$$\times \lim_{z_1 \to z_2} \partial_{z_1}^{s-k}\partial_{z_2}^{k-1}G(z_1, z_2). \qquad (28)$$

Now, using the familiar series

$$\frac{a}{\sinh(ax)} - \frac{1}{x} = -\sum_{p=1}^\infty \frac{a^{2p}(2^{2p-1}-1)B_{2p}}{p}\frac{x^{2p-1}}{(2p-1)!}$$

for $a = \kappa/2$, we obtain

$$\langle X_s^F\rangle = -\frac{B(s)}{s}\sum_{k=1}^s (-1)^{k+1}\binom{s-1}{s-k}^2(-1)^{k-1}$$
$$\times \lambda\frac{\kappa^s(1 - 2^{-(s-1)})B_s}{s}. \qquad (29)$$

Finally, using the value of the sum

$$\sum_{k=1}^s \binom{s-1}{s-k}^2 = 2^{s-1}\frac{(2s-3)!!}{(s-1)!},$$

we find

$$\langle X_s^F\rangle = -\lambda\frac{\kappa^s B_s}{s}(1 - 2^{1-s})(4q)^{s-2} = -\langle j_{z...z}^{(s)}\rangle_h, \qquad (30)$$

where $\langle\cdot\rangle_h$ denotes the value at the horizon. Notice that $\langle X_s^F\rangle = 0$ for an odd spin s. For $s > 1$ this is because $B_s = 0$ for odd $s > 1$. For $s = 1$ it is because of the other factor in (30).

V. HIGHER-SPIN COVARIANT CURRENTS

The holomorphic currents of the previous section refer to a background with a trivial Euclidean metric. In order to construct the corresponding covariant higher-spin currents from fermionic fields, first we recall some properties of fermions in two dimensions [15]. The equation of motion for a right-handed fermion with unit charge is given by

$$(\partial_u - iA_v + \tfrac{1}{4}\partial_v\varphi)\psi(u, v) = 0. \qquad (31)$$

In the Lorentz gauge, the gauge field can be written locally as $A_u = \partial_u\eta(u, v)$ and $A_v = -\partial_v\eta(u, v)$ where $\eta(u, v)$ is a scalar field. Since gravitational and gauge fields are not generally holomorphic, $\psi(u, v)$ is not holomorphic either. In order to construct holomorphic quantities from a fermionic field, we define a new field Ψ by

$$\Psi \equiv \exp(\tfrac{1}{4}\varphi(u, v) + i\eta(u, v))\psi(u, v) \qquad (32)$$

It is easy to show that the equation of motion implies $\partial_v\Psi = 0$ and hence Ψ is holomorphic. Similarly we can define $\Psi^\dagger$ as

$$\Psi^\dagger \equiv \exp(\tfrac{1}{4}\varphi(u, v) - i\eta(u, v))\psi^\dagger(u, v).$$

The equation of motion again guarantees that $\partial_v\Psi^\dagger = 0$, so that $\Psi^\dagger$ is also holomorphic. We will use Ψ and $\Psi^\dagger$ as the basic chiral fields to construct the $W_{1+\infty}$ algebra introduced in the previous section. To covariantize the expressions of the currents we reduce the problem to one dimension by considering only the u dependence and keeping v fixed. In one dimension a curved coordinate u in the presence of a background metric

$$g_{\mu\nu} = e^{\varphi(u,v)}\eta_{\mu\nu}$$

is easily related to the corresponding normal coordinate x by the equation $\partial_x = e^{-\varphi(u,v)}\partial_u$. We view u as $u(x)$, and by the above equation, we extract the correspondence between $j_{z...z}^{(s)}$ and $j_{u...u}^{(s)}$ by identifying u with the coordinate z of the previous section after Wick rotation. The expressions we get in this way are not yet components of the covariant currents. We have to remember the current conformal weights and introduce suitable factors in order to take them into account.

Under a holomorphic conformal transformation $u \to \tilde{u}$ the function $\varphi(u, v)$ and the field $\Psi(u)$ transform according to

$$\tilde{\varphi}(\tilde{u}, v) = \varphi(u, v) - \ln\left(\frac{d\tilde{u}}{du}\right), \qquad \tilde{\Psi}(\tilde{u}) = \left(\frac{d\tilde{u}}{du}\right)^{1/2}\Psi(u).$$

Therefore $e^{-\varphi/2}\Psi(u)$ [and analogously, $e^{-\varphi/2}\Psi^\dagger(u)$] transforms as a scalar with respect to a holomorphic coordinate transformation.

A remark is in order about the transformation property of the fermion field Ψ under (holomorphic) gauge transformations; in the Lorentz gauge there remains a residual holomorphic gauge symmetry,

$$\psi'(u, v) = e^{i\Lambda(u)}\psi(u, v), \qquad \eta'(u, v) = \eta(u, v) + \Lambda(u).$$

Under this transformation the field $\Psi(u)$ transforms as a field with twice the charge of ψ, i.e., $\Psi'(u) = e^{2i\Lambda(u)}\Psi(u)$.

As a consequence the covariant derivative of $\Psi(u)$ turns out to be

$$\nabla_u\Psi(u) = (\partial_u - \tfrac{1}{2}\partial_u\varphi - 2iA_u)\Psi(u),$$
$$\nabla_u\Psi^\dagger(u) = (\partial_u - \tfrac{1}{2}\partial_u\varphi + 2iA_u)\Psi^\dagger(u),$$

and for higher covariant derivatives we have

$$\nabla_u^{m+1}\Psi(u) = (\partial_u - (m+\tfrac{1}{2})\partial_u\varphi - 2iA_u)\nabla_u^m\Psi(u),$$
$$\nabla_u^{m+1}\Psi^\dagger(u) = (\partial_u - (m+\tfrac{1}{2})\partial_u\varphi + 2iA_u)\nabla_u^m\Psi^\dagger(u).$$

It can be shown that $e^{-(m+\frac{1}{2})\varphi}\nabla_u^m\Psi(u)$ and $e^{-(m+\frac{1}{2})\varphi}\nabla_u^m\Psi^\dagger(u)$ transform as scalars under holomorphic coordinate transformation, for every $m \in \mathbb{N}$.

After these preliminaries the covariant currents are constructed using the following bricks:

$$J_{u...u}^{(m,n)} = e^{(m+n+1)\varphi(u,v)}\lim_{\epsilon\to 0}\Bigg(e^{2i\int_{u_-}^{u_+}A_u(u',v)du'}e^{-(m+1/2)\varphi(u_+,v)}$$

$$\times \nabla_u^m\Psi^\dagger(u_+)e^{-(n+1/2)\varphi(u_+,v)}\nabla_u^n\Psi(u_-) - \frac{c_{m,n}^f}{\epsilon^{m+n+1}}\Bigg),$$

where we have used the abbreviations $u_+ \equiv u(x + \epsilon/2)$ and $u_- \equiv u(x - \epsilon/2)$. The numerical constants $c_{m,n}^f$, defined by

$$c_{m,n}^f = \lambda(-1)^m(m + n)!,$$

are determined in such a way that all singularities are canceled in the final expressions for $J^{(m,n)}$.

Finally, let us define the covariant currents corresponding to the $W_{1+\infty}$ fermionic currents:

$$J_{u...u}^{(s)} = -\frac{B(s)}{s}\sum_{k=1}^s(-1)^k\binom{s-1}{s-k}^2 J_{u...u}^{(s-k,k-1)}, \tag{33}$$

$$B(s) \equiv \frac{2^{s-3}s!}{(2s-3)!!}q^{s-2}.$$

The first few covariant $W_{1+\infty}$ fermionic currents can be written in pretty simple form, using the abbreviation $T \equiv \partial_u^2\varphi - \frac{1}{2}(\partial_u\varphi)^2$,

$$J_u^{(1)} = j_u^{(1)} + \frac{i\lambda}{2q}A_u, \tag{34}$$

$$J_{uu}^{(2)} = \left(2A_u^2 - \frac{T}{12}\right)\lambda - 2A_uJ_u^{(1)} + j_{uu}^{(2)}, \tag{35}$$

$$J_{uuu}^{(3)} = -4J_u^{(1)}A_u^2 - 4J_{uu}^{(2)}A_u + \left(\frac{8A_u^3}{3} - \frac{A_uT}{3}\right)\lambda$$
$$+ \frac{TJ_u^{(1)}}{6} + j_{uuu}^{(3)}, \tag{36}$$

$$J_{uuuu}^{(4)} = +\lambda\left(4A_u^4 - \frac{7TA_u^2}{5} - \frac{2}{5}(\nabla_u^2A_u)A_u + \frac{7T^2}{240}\right.$$
$$+ \frac{3}{5}(\nabla_uA_u)^2\right) - 8J_u^{(1)}A_u^3 - 12J_{uu}^{(2)}A_u^2$$
$$+ \left(\frac{1}{5}\nabla_u^2J_u^{(1)} + \frac{7TJ_u^{(1)}}{5} - 6J_{uuu}^{(3)}\right)A_u$$
$$- \frac{3}{5}(\nabla_uA_u)(\nabla_uJ_u^{(1)}) + \frac{1}{5}(\nabla_u^2A_u)J_u^{(1)} + \frac{7TJ_{uu}^{(2)}}{10}$$
$$+ j_{uuuu}^{(4)}. \tag{37}$$

The expression for the fifth order current can be found in Appendix B. For the other currents we have explored up to order 8, the expressions are so unwieldy that we have decided not to write them down explicitly.

Next we write down the covariant derivatives of the $W_{1+\infty}$ fermionic currents, $J^{(s)}$, defined above,

$$g^{uv}\nabla_vJ_u^{(1)} = -\lambda F_u{}^u, \tag{38}$$

$$g^{uv}\nabla_vJ_{uu}^{(2)} = \frac{1}{24}\lambda(\nabla_uR) + F_u{}^uJ_u^{(1)}, \tag{39}$$

$$g^{uv}\nabla_vJ_{uuu}^{(3)} = 2F_u{}^uJ_{uu}^{(2)} - \frac{1}{12}(\nabla_uR)J_u^{(1)}, \tag{40}$$

$$g^{uv}\nabla_vJ_{uuuu}^{(4)} = \frac{3}{10}(\nabla_uF_u{}^u)(\nabla_uJ_u^{(1)}) - \frac{1}{10}F_u{}^u(\nabla_u^2J_u^{(1)}) - \frac{1}{10}$$
$$\times (\nabla_u^2F_u{}^u)J_u^{(1)} - \frac{7}{20}(\nabla_uR)J_{uu}^{(2)} + 3F_u{}^uJ_{uuu}^{(3)}. \tag{41}$$

In the case of lowest spin current, $J^{(1)}$, (38) gives rise to the gauge anomaly

$$g^{\mu\nu}\nabla_\mu J_\nu^{(1)} = -\frac{\hbar}{2}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{42}$$

Apart from the gauge anomaly in the first current we are interested to check whether there are trace anomalies in the other currents. This is done as follows. After the right-hand side of the above equation is expressed in terms of covariant quantities, terms proportional to $\hbar$ (which is present only in λ) are identified as possible anomalies by proceeding in analogy to the energy-momentum tensor. One assumes that there is no anomaly in the conservation laws of covariant currents, that is, that the covariant derivatives of the higher-spin currents with the addition of suitable covariant terms [these terms are classical, i.e., not proportional to $\hbar$; see for instance the terms in (39) and (40)]

vanish. Since

$$\nabla \cdot J_{u...u} + \ldots = g^{uv}\nabla_v J_{u...u} + g^{uv}\nabla_u J_{vu...u} + \ldots = 0,$$

where dots denote the above mentioned classical covariant terms, one relates terms proportional to $\hbar$ in the u derivative of the trace ($vu\ldots u$ components) with the terms proportional to $\hbar$ in the v derivative of $u\ldots u$ components of the currents.

For the covariant energy-momentum tensor, $J^{(2)}$, we have $\mathrm{Tr}(J^{(2)}) = 2g^{vu}J^{(2)}_{vu} = -\frac{\hbar}{12}R$, which is the well-known trace anomaly. In the case of the $J^{(3)}$ current the terms that carry explicit factors of $\hbar$ cancel out in $g^{uv}\nabla_v J^{(3)}_{uuu}$, which implies absence of $\hbar$ in the trace, and consequently the absence of the trace anomaly. The same is true for $J^{(4)}$ and the higher currents.

A. Higher moments of the Hawking radiation

Now let us come to the description of the higher moments of the fermionic Hawking radiation. We will follow the pattern outlined in Sec. III and consider first the case in which the electromagnetic field is decoupled ($m = 0$).

In Sec. IV we evaluated $\langle j^{(s)}_{z...z}\rangle_h$. If we identify $j^{(s)}_{z...z}(z)$ via a Wick rotation with $j^{(s)}_{u...u}(u)$, we get the corresponding value at the horizon $\langle j^{(s)}_{u...u}\rangle_h$. We notice that since the problem we are considering is stationary and $j^{(s)}_{u...u}(u)$ is chiral, it follows that it is constant in t and r. Therefore $\langle j^{(s)}_{u...u}\rangle_h$ corresponds to its value at $r = \infty$. Since $j^{(s)}_{u...u}(u)$ and $J^{(s)}_{u...u}(u)$ asymptotically coincide, the asymptotic flux of these currents is

$$\langle J^{(s)r}_{\ t...t}\rangle = \langle J^{(s)}_{u...u}\rangle - \langle J^{(s)}_{v...v}\rangle = \langle j^{(s)}_{u...u}\rangle_h.$$

If we set $q = \frac{i}{4}$ and $\lambda = 1$ in conventional units and, as in [1,2], we multiply the currents by $-\frac{1}{2\pi}$ in order to properly normalize the (physical) energy-momentum tensor, we get

$$-\frac{1}{2\pi}\langle J^{(2n)r}_{\ t...t}\rangle = -(-1)^n \frac{\kappa^{2n}B_{2n}}{4\pi n}(1 - 2^{1-2n}), \quad (43)$$

while the odd currents give a vanishing value. These values correspond precisely to the fluxes of the Hawking thermal spectrum defined by (18) multiplied by two. This is so because our currents carry both particle and antiparticle contributions.

Next we wish to take into account the presence of the gauge field, which, in our case, vanishes at infinity but not at the horizon. This introduces a significant change in our method. In Sec. III the basic criterion was to regularity of $T^{(\mathrm{hol})}_{uu}$ at the horizon. Now the presence of the electromagnetic field interferes with the regularity of $T^{(\mathrm{hol})}_{uu}$ at the horizon. As a consequence we have to update our criterion.

Let us start with the first current (34). From now on we understand that the electromagnetic field A_u absorbs also the charge m, so that in the final results the replacement $A_t \to mA_t$ is understood. We easily get (remember that

$$J^{(1)}_{\tilde{u}} = j^{(1)}_{\tilde{u}} + \frac{i\lambda}{2q}A_{\tilde{u}} = \frac{1}{f_u}\left(j^{(1)}_u + \frac{i\lambda}{2q}A_u\right), \quad (44)$$

where f_u denotes the first derivative of $\tilde{u} = f(u)$ with respect to u. Now let us introduce the Kruskal coordinate $f(u) \equiv U = -e^{-\kappa u}$. It is evident that we have to require regularity at the horizon of $j^{(1)}_{\tilde{u}} + \frac{i\lambda}{2q}A_{\tilde{u}}$, not of $j^{(1)}_{\tilde{u}}$ alone. Therefore we get

$$\langle j^{(1)}_{\tilde{u}}\rangle_h + \frac{i\lambda}{2q}\langle A_{\tilde{u}}\rangle_h = 0, \quad (45)$$

where $\langle \cdot \rangle_h$ denotes the value at the horizon. Now $j^{(1)}_u(u)$ is constant in t and r. Therefore $\langle j^{(1)}_u\rangle_h = -\frac{i\lambda}{2q}\langle A_u\rangle_h$ corresponds to its value at $r = \infty$. Since $j^{(1)}_u(u)$ and $J^{(1)}_u(u)$ asymptotically coincide, because $A_u(u)$ asymptotically vanishes, we get

$$-\frac{1}{2\pi}\langle J^{(1)r}\rangle = -\frac{1}{2\pi}\langle J^{(1)}_u\rangle + \frac{1}{2\pi}\langle J^{(1)}_v\rangle$$
$$= \frac{i\lambda}{4\pi q}\langle A_u\rangle_h = \frac{1}{2\pi}A_t = \frac{\Omega}{2\pi}, \quad (46)$$

where $\langle \cdot \rangle$ represents the asymptotic value and we have assumed that there is no incoming flux $\langle J^{(1)}_v\rangle$ from infinity.

From this example we learn the obvious lesson. We have to assume that the currents $J^{(s)}_{U...U}$ are regular on the horizon in Kruskal coordinates $U = -e^{-\kappa u}$. Since these currents are covariant, we have

$$J^{(s)}_{U...U} = \frac{1}{(-\kappa U)^s}J^{(s)}_{u...u}(u).$$

It then follows that the currents $J^{(s)}_{u...u}$ and their $n - 1$ derivatives vanish. From (35)–(37), at the horizon we must get

$$j^{(2)}_{uu} = -\lambda\left(2A^2_u - \frac{T}{12}\right), \qquad j^{(3)}_{uuu} = -\lambda\left(\frac{8A^3_u}{3} - \frac{A_uT}{3}\right), \quad (47)$$

$$j^{(4)}_{uuuu} = -\lambda\left(4A^4_u - \frac{7TA^2_u}{5} - \frac{2}{5}(\nabla^2_u A_u)A_u\right.$$
$$\left. + \frac{7T^2}{240} + \frac{3}{5}(\nabla_u A_u)^2\right). \quad (48)$$

As already remarked, at infinity the background fields A_u and ϕ vanish so that

$$\langle J^{(s)}_{u...u}\rangle = \langle j^{(s)}_{u...u}\rangle_h. \quad (49)$$

Now, we evaluate the derivatives on right-hand side of (47) at the horizon. Setting $\lambda = \hbar = 1$ we get

$$\langle j^{(2)}_{uu}\rangle_h = \frac{\langle T\rangle_h}{12} - \frac{\langle A_t\rangle_h^2}{2},$$

$$\langle j^{(3)}_{uuu}\rangle_h = -\frac{1}{3}\langle A_t\rangle_h^3 - \frac{1}{6}\langle T\rangle_h\langle A_t\rangle_h,$$

$$\langle j^{(4)}_{uuuu}\rangle_h = -\frac{1}{4}\langle A_t\rangle_h^4 + \frac{1}{4}\langle T\rangle_h\langle A_t\rangle_h^2 - \frac{7}{240}\langle T\rangle_h^2.$$

Therefore at infinity we get

$$-\frac{1}{2\pi}\langle J_t^{(2)r}\rangle = \frac{\kappa^2}{48\pi} + \frac{\Omega^2}{4\pi},$$

$$-\frac{1}{2\pi}\langle J_{tt}^{(3)r}\rangle = \frac{\Omega^3}{6\pi} + \frac{\kappa^2\Omega}{24\pi}, \quad (50)$$

$$-\frac{1}{2\pi}\langle J_{ttt}^{(4)r}\rangle = \frac{7\kappa^4}{1920\pi} + \frac{\Omega^2\kappa^2}{16\pi} + \frac{\Omega^4}{8\pi},$$

where we have used $\langle f'(r_+)\rangle_h = 2\kappa$, $\langle T\rangle_h = -\frac{\kappa^2}{2}$, $A_t(r) = \frac{r}{r^2+a^2}$, $\Omega = A_t(r_+) = \langle A_t\rangle_h$. These results agree with formula (19) after the replacement $A_t \rightarrow mA_t$ [see the comment before Eq. (44)]. We checked the agreement up to spin 8 current $\langle J_{t..t}^{(8)r}\rangle$. The relevant expressions for $\langle j^{(i)}_{u...u}\rangle_h$ for $i = 1\ldots 8$ are given in Appendix B.

VI. HIGHER-SPIN CURRENTS AND TRACE ANOMALIES

Each of these higher-spin currents carries to infinity its own component of the Hawking radiation. Just in the same way as in the action the metric is a source for the energy-momentum tensor, these new (covariant) currents will have in the effective action suitable sources, with the appropriate indices and symmetries. In [1] they were represented by background fields $B^{(s)}_{\mu_1...\mu_s}$ (which will be eventually set to zero). So we have

$$J^{(s)}_{\mu_1...\mu_s} = \frac{1}{\sqrt{g}}\frac{\delta}{\delta B^{(s)\mu_1...\mu_s}}S. \quad (51)$$

We assume that all $J^{(s)}_{\mu_1...\mu_s}$ are maximally symmetric and classically traceless.

In addition to the series of $B^{(s)}$ fields, there must be other background fields with the same characteristics (i.e., maximally symmetric and asymptotically trivial). Their function is to explain the presence of the additional covariant terms in the conservation equations of the higher currents [to be specific, the terms at the right-hand side of (40) and (41), ...]. Let us call these additional fields $C^{(s)}$, $D^{(s)}$, ... As an example let us consider the conservation of $J^{(3)}$.

$$\nabla^\mu J^{(3)}_{\mu\nu\lambda} = 2F_\nu^\rho J^{(2)}_{\rho\lambda} - \tfrac{1}{6}(\nabla_\nu R)J^{(1)}_\lambda, \quad (52)$$

where symmetrization over the indices ν and λ is understood in the right-hand side. The left-hand side is due to assumed invariance of the effective action under

$$\delta_\xi B^{(3)}_{\mu_1\mu_2\mu_3} = \nabla_{\mu_1}\xi_{\mu_2\mu_3} + \text{cycl}, \quad (53)$$

where ξ is a symmetric traceless tensor and cycl denotes cyclic permutations of the indices. In order to explain the presence of the right-hand side terms, we assume that there exist, in the effective action, other background potentials $C^{(3)}$, coupled to the two terms in the right-hand side of (52), which transform like

$$\delta_\xi C^{(3)}_{\mu\nu} = \xi_{\mu\nu}, \quad (54)$$

while all the other fields in the game are invariant under ξ transformations. These fields must have transformation properties that guarantee the invariance of the terms they are involved in.

In an analogous way we can deal with the other conservation laws. We remark that the transformations of the $C^{(s)}$ potentials are intrinsically Abelian. Unfortunately we do not know how to derive the transformation (54) from first principles. But we can use consistency to conclude that these two equations represent the only possibility. For, although, in order to account for the $J^{(3)}$ conservation law, one can envisage possible (non-Abelian) transformations, one must check that these transformations form a Lie algebra. Such a condition strongly restricts the form of the transformations and, consequently, of the effective action.[1] One can indeed verify that the higher potentials transformation laws are so strongly restricted that it is generically impossible to avoid the conclusion that they must be Abelian (see also [2]). Under these circumstances (54) represents the generic case for higher-spin quantities. The presence of these additional background fields, which were not considered in [1,2], may complicate the anomaly analysis. However, to simplify it, one can remark that these potentials can increase the number of cocycles only if they explicitly appear in the cocycles themselves. Since eventually these potentials are set to zero, the corresponding cocycles vanish. As a consequence they cannot give rise to the anomalies we are interested in and their study is of academic interest. For this reason, for the sake of simplicity, we choose to dispense of it. Thus henceforth we will ignore the additional potentials.

This said we can now analyze the problem of the existence of trace anomalies in higher-spin currents with cohomological methods. With respect to [1] the analysis is complicated by the presence of the electromagnetic field. Of course the electromagnetic field gives rise to the gauge anomaly in the covariant derivative of the $J^{(1)}$ current [see

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[1]The presence of the terms in the right-hand side of (52) could be formally explained by different transformation laws of the other fields. In particular for the third order current such terms could be explained by

$$\delta_\tau g_{\mu\nu} \sim \tau_\mu^\lambda F_{\lambda\nu}, \qquad \delta_\tau A_\mu \sim \tau_\mu^\lambda \nabla_\lambda R.$$

But these are not good symmetry transformations, for they do not form an algebra. It is easy to see it for instance by promoting ξ to anticommuting parameters and verifying that such transformations are not nilpotent.

(38)]. The latter is induced by the gauge transformation $\delta_\lambda A_\mu = \partial_\mu \lambda$, and this is all we need to say about this anomaly.

With these premises, we want to show that the conclusion of [1] on the absence of trace anomalies in the higher-spin currents holds under the present conditions. Let us recall first the setting of [1] for this type of analysis [75–80]. Let us start from the analysis of $J^{(3)}$. Setting $B^{(3)}_{\mu\nu\lambda} = B_{\mu\nu\lambda}$ the Weyl transformation of the various fields involved is (see [1] for a comparison)

$$\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}, \qquad \delta_\sigma B_{\mu\nu\lambda} = x\sigma B_{\mu\nu\lambda}, \qquad (55)$$

$$\delta_\sigma A_\mu = 0, \qquad (56)$$

which induces the trace of the energy-momentum tensor, and

$$\delta_\tau g_{\mu\nu} = 0, \qquad \delta_\tau B_{\mu\nu\lambda} = \tau_\mu g_{\nu\lambda} + \text{cycl}, \qquad (57)$$

$$\delta_\tau A_\mu = 0, \qquad (58)$$

which induces the trace of $J^{(3)}$. Moreover, for consistency with (55) we must have

$$\delta_\sigma \tau_\mu = (x - 2)\sigma\tau_\mu, \qquad (59)$$

where x is an arbitrary number.

A comment on these transformations is in order. They are determined as follows: they must be expressed in terms of symmetry parameters and of the basic background fields $g_{\mu\nu}$ and A_μ and nothing else; they must form a Lie algebra, as was mentioned above; and they must leave unchanged the terms in the effective action, in particular, the terms involving the matter fields. The transformations are then dictated by the canonical dimensions of the various fields. The fields $B^{(s)}$ and $C^{(s)}$ have dimension $2 - s$ and $1 - s$, respectively.

We must now repeat the analysis we have done in [1]. We promote σ and τ_μ to anticommuting fields so that

$$\delta_\sigma^2 = 0, \qquad \delta_\tau^2 = 0, \qquad \delta_\sigma \delta_\tau + \delta_\tau \delta_\sigma = 0.$$

Integrated anomalies are defined by

$$\delta_\sigma \Gamma^{(1)} = \hbar \Delta_\sigma, \qquad \delta_\tau \Gamma^{(1)} = \hbar \Delta_\tau, \qquad (60)$$

where $\Gamma^{(1)}$ is the one-loop quantum action and Δ_σ, Δ_τ are local functionals linear in σ and τ, respectively. The unintegrated anomalies, i.e., the traces T^μ_μ and $J^{(3)\mu}{}_{\mu\lambda}$, are obtained by functionally differentiating with respect to σ and τ_λ, respectively.

By applying δ_σ, δ_τ to Eq. (60), we see that candidates for anomalies Δ_σ and Δ_τ must satisfy the consistency conditions

$$\delta_\sigma \Delta_\sigma = 0, \qquad \delta_\tau \Delta_\sigma + \delta_\sigma \Delta_\tau = 0, \qquad \delta_\tau \Delta_\tau = 0. \qquad (61)$$

Once we have determined these cocycles, we have to make sure that they are true anomalies, that is, that they are nontrivial. In other words there must not exist local counterterm C in the action such that

$$\Delta_\sigma = \delta_\sigma \int d^2x \sqrt{-g}\, \mathcal{C}, \qquad (62)$$

$$\Delta_\tau = \delta_\tau \int d^2x \sqrt{-g}\, \mathcal{C}. \qquad (63)$$

If such a C existed we could redefine the quantum action by subtracting these counterterms and get rid of the (trivial) anomalies.

Let us consider now the problem of the trace $J^{(3)\mu}{}_{\mu\lambda}$. We could repeat the complete analysis of [1], but there is a shortcut due to the simple form of the transformations (57). Suppose we find cocycle $\Delta_\tau^{(3)}$,

$$\Delta_\tau^{(3)} = \int d^2x \sqrt{-g}\, \tau^\mu I_\mu^{(3)}, \qquad (64)$$

where $I_\mu^{(3)}$ is a canonical dimension 3 tensor made of the metric, the gauge field, and their derivatives, such as $\nabla_\mu R$ or $\nabla_\nu F_\mu{}^\nu$, or even a non-gauge-invariant tensor such as $A_\mu R$. Then it is straightforward to write down a counterterm

$$\mathcal{C}^{(3)} \sim B^{\mu\lambda}_\lambda I_\mu^{(3)}, \qquad (65)$$

which cancels (64).[2]

As for the trace $J^{(4)\mu}{}_{\nu\lambda\rho}$ we can proceed in analogy to $J^{(3)\mu}{}_{\mu\lambda}$. Setting $B^{(4)}_{\mu\nu\lambda\rho} \equiv B_{\mu\nu\lambda\rho}$, the relevant Weyl transformations are as follows. The variation δ_τ acts only on $B_{\mu\nu\lambda\rho}$,

$$\delta_\tau B_{\mu\nu\lambda\rho} = g_{\mu\nu}\tau_{\lambda\rho} + \text{cycl}, \qquad (66)$$

and the other fields remain unchanged while the variations with respect to the ordinary Weyl parameter σ are

$$\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}, \qquad (67)$$

$$\delta_\sigma \tau_{\mu\nu} = (x - 2)\sigma\tau_{\mu\nu}, \qquad (68)$$

$$\delta_\sigma B_{\mu\nu\lambda\rho} = x\sigma B_{\mu\nu\lambda\rho}, \qquad (69)$$

where, again, x is an arbitrary number. Now we can repeat the previous argument. Let a cocycle have the form

$$\Delta_\tau^{(4)} = \int d^2x \sqrt{-g}\, \tau^{\mu\nu} I_{\mu\nu}^{(4)}, \qquad (70)$$

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[2]Of course the variation of (65) with respect to σ gives rise to a trivial Δ_σ cocycle, but this cocycle depends on the field B, which vanishes when we select the physical background. On the other hand, we have shown in [1], in an analogous case, that such cocycles can be consistently eliminated even without resorting to the vanishing of B.

where $I_\mu^{(4)}$ is a dimension 4 tensor made out of the metric, the gauge field, and their derivatives, such as $\nabla_\mu \nabla_\nu R$. The counterterm

$$\mathcal{C}^{(4)} \sim B_\lambda^{\mu\nu\lambda} I_{\mu\nu}^{(4)} \tag{71}$$

cancels (70).

It is not hard to generalize this conclusion to higher-spin currents. We believe these results together with those of [1,2] are evidence enough that anomalies may not arise in the higher-spin currents under any condition.

VII. CURRENT NORMALIZATION AND $W_{1+\infty}$ ALGEBRA

It is evident that our being able to describe the higher moments of the Hawking radiation is related to the transformation properties of the holomorphic higher-spin currents. Even in the case of the energy-momentum tensor, the Hawking flux is related to Weyl or Diff anomalies only in the sense that the latter determine the relation between the covariant and holomorphic part of the energy-momentum tensor (see our discussion in [2]). For higher-spin currents, as we have seen, there are no links with anomalies simply because anomalies cannot exist in the conservation laws of these currents. This much seems definitely clear. There are however other aspects of the problem which have remained implicit and are crucial in order to understand the central role of the $W_{1+\infty}$ algebra. In this section we would like to discuss these aspects.

Let us start from the remark that in formula (29) the summation over k does not affect the crucial term $\frac{\kappa^s}{s} \times (1 - 2^{-(s-1)}) B_s$ except for an overall multiplicative factor. This means that, had we used each one of the currents

$$j_{z\ldots z}^{(s,k)}(z) = \,:\!\partial_z^{s-k} \Psi^\dagger(z) \partial_z^{k-1} \Psi(z)\!: \tag{72}$$

instead of (20), we would have obtained (up to normalization) the same final result for the moments of the Hawking radiation.[3] This seems at first to deprive of any interest the role of the $W_{1+\infty}$ algebra, but the case is just the opposite. Using the currents $j_{z\ldots z}^{(s,k)}(z)$ we have two enormous disadvantages.

The first is that we do not have any means of normalizing these currents, thus rendering the results obtained by their means devoid of any predictive value. The $W_{1+\infty}$ algebra structure tells us how to normalize the currents in such a way as to get an algebra. There remain only two constants to be fixed, λ and q. The first is fixed in such a way as to get the right transformation laws (operator product expansion) of the energy-momentum tensor; the second is fixed by the U(1) algebra of $j^{(1)}$. Once these two constants are fixed the

normalization for all the higher-spin currents is uniquely determined and in agreement with the thermal spectrum of the Hawking radiation.

The second disadvantage of using currents such as $j_{z\ldots z}^{(s,k)}(z)$, which are not $W_{1+\infty}$, is the appearance of anomalies in their traces or in the conservation laws of their covariant version. This was shown in a very explicit way in [15]. As we have shown, these anomalies are cohomologically trivial and can be eliminated by suitable redefinitions or subtractions. As a result one ends up with the currents (20) and their $W_{1+\infty}$ algebra. In other words the $W_{1+\infty}$ algebra is the appropriate structure underlying the thermal spectrum of the Hawking radiation. This result seems to imply that the two-dimensional physics around the horizon is characterized by a symmetry larger than the Virasoro algebra, such as a W_∞ or $W_{1+\infty}$ algebra.

APPENDIX A: SPIN CONNECTION

With reference to Sec. II we list here the spin connection coefficients $\omega^a_{\ bc} = e_c^{\ \mu} \omega^a_{\ b\mu} = e_c^{\ \mu} e^a_{\ \nu} \nabla_\mu e_b^{\ \nu}$ for the Kerr metric (3) and the vierbein (4):

$$\omega^0_{\ 10} = \frac{(r - M)\Sigma(r,\theta) - r\Delta(r)}{\sqrt{\Delta(r)}\Sigma(r,\theta)^{3/2}},$$

$$\omega^0_{\ 20} = \omega^1_{\ 21} = -\frac{a^2 \cos\theta \sin\theta}{\Sigma(r,\theta)^{3/2}},$$

$$\omega^1_{\ 30} = \omega^0_{\ 31} = \omega^0_{\ 13} = -\frac{ar\sin\theta}{\Sigma(r,\theta)^{3/2}},$$

$$\omega^2_{\ 30} = -\omega^0_{\ 32} = \omega^0_{\ 23} = -\frac{a\cos\theta\sqrt{\Delta(r)}}{\Sigma(r,\theta)^{3/2}},$$

$$\omega^1_{\ 22} = \omega^1_{\ 33} = -\frac{r\sqrt{\Delta(r)}}{\Sigma(r,\theta)^{3/2}},$$

$$\omega^2_{\ 33} = -\frac{(2Mr + \Delta(r))\cos\theta}{\Sigma(r,\theta)^{3/2}\sin\theta}.$$

Apart from these coefficients and those related by using $\omega_{abc} = -\omega_{bac}$, where $\omega_{abc} = \eta_{ad}\omega^d_{\ bc}$, all other coefficients are zero. Christoffel symbols for the metric (3) can be found in Appendix D of [81].

---

[3]This, by the way, explains why the authors of [15,16] obtained the same predictions as in [1,2] for the higher moments of the Hawking radiation, using unnormalized currents and without invoking a W_∞ structure.

APPENDIX B: FIFTH ORDER CURRENT

In this Appendix we write down the explict expression for the fifth order current and the values of the holomorphic currents at the horizon, up to order eight.

The current $J^{(5)}_{uuuuu}$ is as follows:

$$
\begin{aligned}
J^{(5)}_{uuuuu} = \hbar\Bigg(&\frac{32A_u^5}{5} - \frac{104TA_u^3}{21} - \frac{16}{7}(\nabla_u^2 A_u)A_u^2 + \frac{27T^2A_u}{70} + \frac{24}{7}(\nabla_u A_u)^2 A_u + \frac{1}{35}(\nabla_u^2 T)A_u - \frac{1}{7}(\nabla_u A_u)(\nabla_u T) \\
&+ \frac{2}{21}T(\nabla_u^2 A_u)\Bigg) - 16J_u^{(1)}A_u^4 - 32J_{uu}^{(2)}A_u^3 + \frac{8}{7}(\nabla_u^2 J_u^{(1)})A_u^2 + \frac{52}{7}TJ_u^{(1)}A_u^2 - 24J_{uuu}^{(3)}A_u^2 - \frac{24}{7}(\nabla_u A_u)(\nabla_u J_u^{(1)})A_u + \frac{12}{35} \\
&\times (\nabla_u^2 J_{uu}^{(2)})A_u + \frac{16}{7}(\nabla_u^2 A_u)J_u^{(1)}A_u + \frac{52}{7}TJ_{uu}^{(2)}A_u - 8J_{uuuu}^{(4)}A_u + \frac{1}{14}(\nabla_u T)(\nabla_u J_u^{(1)}) - \frac{12}{7}(\nabla_u A_u)(\nabla_u J_{uu}^{(2)}) \\
&- \frac{1}{21}T(\nabla_u^2 J_u^{(1)}) - \frac{27T^2 J_u^{(1)}}{140} - \frac{12}{7}(\nabla_u A_u)^2 J_u^{(1)} - \frac{1}{70}(\nabla_u^2 T)J_u^{(1)} + \frac{8}{7}(\nabla_u^2 A_u)J_{uu}^{(2)} + \frac{13TJ_{uuu}^{(3)}}{7} + j^{(5)}_{uuuuu}.
\end{aligned}
$$

(B1)

For simplicity we omit the explicit expressions of $J^{(6)}_{u...u}$, $J^{(7)}_{u...u}$, $J^{(8)}_{u...u}$. Next we list the results for $j^{(i)}_{u...u}$ at the horizon, obtained using the condition that $J^{(i)}_{\tilde{u}...\tilde{u}}$, in Kruskal coordinates, be regular. We understand $\lambda = \hbar = 1$:

$$
\langle j_u^{(1)}\rangle_h = -\langle A_t\rangle_h, \qquad \langle j_{uu}^{(2)}\rangle_h = -\frac{1}{2}\langle A_t\rangle_h^2 + \frac{1}{12}\langle T\rangle_h, \qquad \langle j_{uuu}^{(3)}\rangle_h = -\frac{1}{3}\langle A_t\rangle_h^3 + \frac{1}{6}\langle T\rangle_h\langle A_t\rangle_h,
$$

$$
\langle j_{uuuu}^{(4)}\rangle_h = -\frac{1}{4}\langle A_t\rangle_h^4 + \frac{1}{4}\langle T\rangle_h\langle A_t\rangle_h^2 - \frac{7}{240}\langle T\rangle_h^2, \qquad \langle j_{uuuuu}^{(5)}\rangle_h = -\frac{1}{5}\langle A_t\rangle_h^5 + \frac{1}{3}\langle T\rangle_h\langle A_t\rangle_h^3 - \frac{7}{60}\langle T\rangle_h^2\langle A_t\rangle_h,
$$

$$
\langle j_{uuuuuu}^{(6)}\rangle_h = -\frac{1}{6}\langle A_t\rangle_h^6 + \frac{5}{12}\langle T\rangle_h\langle A_t\rangle_h^4 - \frac{7}{24}\langle T\rangle_h^2\langle A_t\rangle_h^2 + \frac{31}{1008}\langle T\rangle_h^3,
$$

$$
\langle j_{uuuuuuu}^{(7)}\rangle_h = -\frac{1}{7}\langle A_t\rangle_h^7 + \frac{1}{2}\langle T\rangle_h\langle A_t\rangle_h^5 - \frac{7}{12}\langle T\rangle_h^2\langle A_t\rangle_h^3 + \frac{31}{168}\langle T\rangle_h^3\langle A_t\rangle_h,
$$

$$
\langle j_{uuuuuuuu}^{(8)}\rangle_h = -\frac{1}{8}\langle A_t\rangle_h^8 + \frac{7}{12}\langle T\rangle_h\langle A_t\rangle_h^6 - \frac{49}{48}\langle T\rangle_h^2\langle A_t\rangle_h^4 + \frac{31}{48}\langle T\rangle_h^3\langle A_t\rangle_h^2 - \frac{127}{1920}\langle T\rangle_h^4.
$$

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