Quantum superposition as a resource for quantum communication

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Kvantna superpozicija kao resurs za kvantnu komunikaciju

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Master Thesis

Quantum superposition as a resource for quantum communication

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Zahvaljujem se svojoj obitelji na neizmjernoj ljubavi i potpori pruženoj svih ovih godina.

Many thanks to Borivoje Dakić for being such a wonderful supervisor.

Kvantna superpozicija kao resurs za kvantnu komunikaciju

Sažetak

Glavni cilj ovog rada je istraživanje prostorne superpozicije u kontekstu kvantne komunikacije. U klasičnoj fizici, u svakom vremenskom trenutku, informacija teće maksimalno u jednom smjeru, odnosno prenosioci informacija prenose poruku od pošiljatelja ka primatelju, ali ne i obratno. S druge strane, kvantna fizika otvara nove mogućnosti dozvoljavajući prostornu superpoziciju fizikalnih sustava što omogućava dvosmjerni prijenos informacija (od pošiljatelja ka primatelju i obratno). Naš cilj je doći do novih komunikacijskih procesa koje nije moguće implementirati u klasičnoj mehanici te stoga kvantificirati prostornu superpoziciju kao resurs za komunikaciju. Preciznije, cilj nam je klasificirati jednostavne komunikacijske protokole (npr. protokole koji uključuju izmjenu informacija između nekoliko igrača, s restrikcijom na resurse) putem analogona Bellovih nejednakosti. S druge strane, klasifikacija takvih nejednakosti pruža prirodnu operativnu metodu kojom se može pokazati da kvantni prenosioci informacija istinski predstavljaju resurs za ne-klasičnu komunikaciju (eksplicitnim narušenjem spomenutih nejednakosti).

Ključne riječi: prostorna superpozicija, kvantna komunikacija, Bellove nejednakosti, konveksna geometrija, ”device-independent” formalizam, teorije interferencije višeg reda

Quantum superposition as a resource for quantum communication

Abstract

The main objective of this thesis is to explore spatial superposition as a novel resource for quantum communication. In classical physics, at any instant of time, information flows in one direction only, i.e. an information carrier transmits the message from the sender to the receiver, but not vice versa. In contrast, quantum physics opens up the possibility of putting information carriers in spatial superposition which in turn enables simultaneous bi-directional transmission of information between the sender and the receiver. Our aim is explore this possibility to arrive at new communication tasks that cannot be accomplished within classical physics, thus quantifying the use of quantum superposition as a resource for communication. More precisely, we aim at classifying simple communication protocols (e.g. involving the exchange of information between several players, with restrictions on the available resources) by means of Bell-like inequalities, which set precise limits to classical communication. On the other hand, the classification of these inequalities provides a natural operational framework to show that quantum information carriers in spatial superposition are a genuine resource for non-classical communication (i.e. by explicitly violating the bounds imposed by Bell-like inequalities).

Keywords: spatial superposition, quantum communication, Bell's inequalities, convex geometry, device-independent formalism, higher order interference theories

Contents

1 Introduction

It has been already a 100 years since the advent of quantum mechanics and we are still unable to fully grasp its conceptual structure and how to interpret it. Its consequences often defy our most basic intuitions inherited from the classical world like determinism and locality, which forces us to adopt a radically different view of our experiences and of the world we are immersed in. This is exemplified in the plethora of interpretations of what quantum theory is telling us, ranging from the Copenhagen to the Bohmian and many worlds interpretations. Nevertheless, even without understanding its intrinsic meaning, quantum mechanics is what lies at the foundation of the most important discoveries in contemporary physics, including, among others, discoveries in particle, nuclear and condensed matter physics. On a more mundane note, today's technological reality also highly depends on quantum theory as one of its necessary ingredients. The immense success of quantum physics can thus distance us from questioning its foundations, since the latter may seem as unnecessary and redundant. However, this view has slowly been changing in the last thirty years which can be seen through the rise of quantum information theory. The latter provides a smooth dialogue between questions of seemingly purely foundational nature and questions regarding practical advantages and disadvantages posed by quantum theory. A paradigmatic example of this dialogue is given by the following question: is Nature compatible with a local hidden variable (LHV) model? The answer is negative, as Bell proved in his celebrated paper where he showed that a singlet state of two qubits violates bounds imposed by a LHV structure [1]. On the other hand, the latter lies at the heart of quantum cryptography [2] and of genuine randomness certification [3]. Another beautiful example of this phenomenon is the recent wave of operational reconstructions of quantum theory based on physically reasonable axioms, where various information-theoretic aspects (e.g. local tomography and the purification principle [4]) are seen as defining features that single out quantum physics from the vast sea of other theoretically conceivable theories [5–8]. In a similar fashion, in this work we ask what is the difference between the classical and the quantum world regarding communication tasks. What are the advantages and limitations that quantum theory poses with respect to the transmission of information with a finite amount of resources? Is it possible to communicate with arbitrarily many parties using a single quantum system (e.g. a single photon)? The answer to these and similar questions, besides having obvious practical consequences, might also offer insights into the differences between the classical and the quantum, and between the quantum and other counterfactual generalized theories.

We start by introducing the two-way communication protocol and by a brief revision of Bell's inequalities and of their related mathematical structure. We then proceed with the description of the scenario of interest and with the characterization of signaling correlations in a 'device-independent formalism', thereby providing various equalities and inequalities on a purely operational level. Finally, we analyze the possible underlying physical mechanisms, ranging through classical, quantum and supraquantum processes and prove the possibility of multi-way signaling with an arbitrary number of parties using a single particle in spatial superposition.

2 Preliminaries and motivation

2.1 Two-way communication with a single quantum particle

Recently it has been shown that it is possible to achieve two-way communication using a single quantum particle [9]. Let us briefly revise the protocol.

Imagine two parties, named Alice (A) and Bob (B), each of whom possesses one bit of information (i.e. a digit which can be either 0 or 1) which we will denote by x_A and x_B and suppose they want to communicate their information to each other. The parties are allowed to exchange their bits in a brief time window $t = D/c$, where D is the distance between A and B and *c* is the speed of light. If A possesses one photon as a resource for communication, then she can encode her input on the carrier and send it to B who can in principle perfectly retrieve A' s information; however, A can get to know nothing about B' s information. The analogous holds true for the converse situation.

On the other hand, quantum mechanics allows information carriers/particles to be in a superposition of spatial modes, which may allow bi-directional communication. Let Alice and Bob share a photon in spatial superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|A\rangle + |B\rangle \right), \tag{2.1}$$

where $|A\rangle$ and $|B\rangle$ represent states of the photon 'localized' respectively at A and B. Next, the parties encode their bits of information introducing local phases $e^{i\pi x_{A/B}}$. The state is thus transformed to

$$|\psi\rangle_{x_A, x_B} = \frac{1}{\sqrt{2}} \left(e^{i\pi x_A} |A\rangle + e^{i\pi x_B} |B\rangle \right). \tag{2.2}$$

In addition, we assume there is a 50:50 beam splitter (BS) in between the two parties which acts as a communication channel. The latter acts as a unitary device represented by the Hadamard gate in the $\{|A\rangle, |B\rangle\}$ basis [10]:

$$H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.3}$$

Immediately after encoding their inputs, Alice and Bob send their parts of the particle through the communication channel. The final state at $t = D/c$ is thus

$$|\psi\rangle_{x_A,x_B} = \pm |A\rangle \,, \quad \text{if} \quad s \equiv x_A \oplus x_B = 0 \tag{2.4}$$

$$|\psi\rangle_{x_A,x_B} = \pm |B\rangle \,, \quad \text{if} \quad s \equiv x_A \oplus x_B = 1. \tag{2.5}$$

Therefore, the two parties learn the parity s of the inputs by simply observing whether the particle is located at their respective locations or not. This then allows them to deterministically retrieve each others bits. For example, if Alice's bit is $x_A = 0$ and she observes the particle at her position, she can deduce that Bob's bit must have been $x_B = 0$, since the overall parity is $s = 0$; Bob simultaneously retrieves Alice's input by analogous reasoning.
Hence, quantum superposition indeed allows two-way communication using a single particle. Is it possible to go a step further and formalize this notion? Does quantum superposition enhance communication between arbitrarily many parties? In order to provide an answer to these and further questions, let us first briefly immerse ourselves in the world of Bell's nonlocality, since, as we will see later, its mathematical structure is reminiscent of the structure of our questions.

2.2 *Bell's nonlocality*

2.2.1 Black boxes and local hidden variables

Let us picture two agents named Alice and Bob, each of whom possesses a black box, as in Figure 2.1. The boxes are defined purely by their extrinsic behaviors in terms of inputs and outputs, i.e. we do not know anything about the content of the boxes. The two agents are free to input one bit of information into their respective boxes, which then produce outputs in the form of bits. Furthermore, the boxes may be non-deterministic, i.e. in each run they do not necessarily produce the same output for a given input. We describe this scenario via a set of conditional probability distributions, or *behavior*

$$\{P\,(ab|xy)\,; \quad \forall a,b,x,y \in \{0,1\}\}\,, \tag{2.6}$$

where *x* and *y* are respectively Alice's and Bob's inputs, while *a* and *b* are the outputs of their boxes.

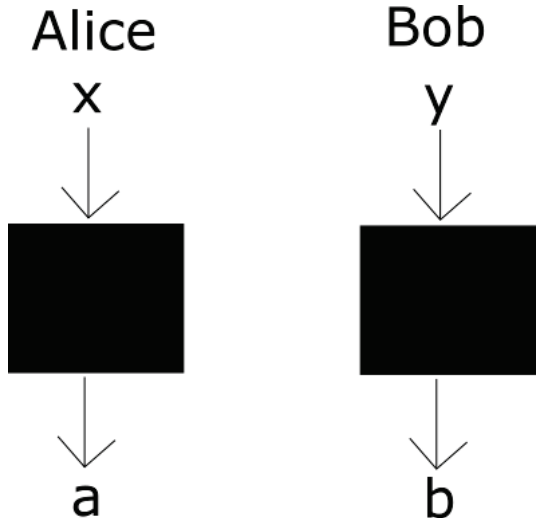


Figure 2.1: Alice and Bob freely choose input bits x and y and the unknown black boxes probabilistically produce output bits a and b.

A local hidden variable (LHV) model assumes that the probability distributions can be decomposed as

$$P\left(ab|xy\right) = \int_{\lambda} d\lambda q(\lambda) P\left(a|x,\lambda\right) P\left(b|y,\lambda\right), \tag{2.7}$$

which means that the outputs depend on the local inputs and on a common past cause denoted with λ, namely a hidden variable. Since these variables are unknown, we have taken an average over a non specified distribution $q(\lambda)$. Hidden variables are often called *shared randomness*, where 'randomness' refers to them being hidden/unspecified and 'shared' refers to them being common to both local distributions. The mechanism of how this common cause is produced remains fully general: it can be established via particles which interacted in the past, via a public announcement on the radio, or even by a past meeting of the two parties.

How can we prove whether a given set of probability distributions admits a LHV model? What Bell did in his work is that he found a necessary condition that a set of probability distributions has to satisfy in order to be describable by LHV-s [1]. Here we state a variation of Bell's result, namely the CHSH inequality [11]:

$$S \equiv E_{00} + E_{01} + E_{10} - E_{11} \leq 2, \quad E_{xy} \equiv P\left(a = b|xy\right) - P\left(a \neq b|xy\right). \tag{2.8}$$

Any set of distributions which violates the latter inequality cannot be modeled by LHV-s!

2.2.2 Violation of the inequality

The whole previous discussion is motivated by the question that was first addressed by Einstein, Podolsky and Rosen: can quantum mechanics be explained by a local deterministic model [12]? The answer is negative, and can be proven by constructing an explicit counterexample.

Assume Alice and Bob share a general quantum state ρ and are able to perform local measurements. The measurement settings are determined by inputs x and y, while the respective outputs are given by a and b. The conditional probabilities can then be expressed in terms of their underlying quantum mechanical mechanism as

$$P\left(ab|xy\right) = \mathrm{Tr}\left(\Pi_a^x \otimes \Pi_b^y \rho\right), \tag{2.9}$$

where $\Pi_{a/b}^{x/y}$ are local measurement operators (that may be more general than Von Neumann projectors, see for instance [13]) corresponding to outcomes a/b given the measurement settings x/y. Translating to the "black box language", the boxes are implemented by a shared quantum state, the inputs are represented as measurement settings and the outputs are given by measurement results.

Let us further specify the problem by assuming the shared state to be a pure singlet state of two spins $\frac{1}{2}$

$$\rho = \left|\psi_-\right\rangle\left\langle\psi_-\right|, \quad \left|\psi_-\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle\left|1\right\rangle - \left|1\right\rangle\left|0\right\rangle\right), \tag{2.10}$$

and the measurements to be local spin measurements in the plane orthogonal to the line connecting the two parties, under angles $x \in \left\{0, \frac{\pi}{4}\right\}$ and $y \in \left\{\frac{\pi}{8}, \frac{3\pi}{8}\right\}$ with respect to the initial spin direction. The measurement operators are thus

$$\Pi_a^x = \frac{1}{2}\left(\mathbb{1} + a\vec{n}_x \cdot \vec{\sigma}_A\right), \tag{2.11}$$

$$\Pi_b^y = \frac{1}{2}\left(\mathbb{1} + b\vec{n}_y \cdot \vec{\sigma}_B\right), \tag{2.12}$$

where $a, b \in \{1, -1\}$ and $\vec{n}_{x,y}$ are unit vectors in the directions of the spin measurements. The Bell expression (2.8) can now be computed analytically; one finds a clear violation

$$S_{QM} = 2\sqrt{2} > 2. \tag{2.13}$$

Thus, we explicitly constructed a quantum mechanical process whose statistics cannot be reproduced by a LHV model. Moreover, the inequality violation can be tested experimentally without addressing any underlying theoretical explanation, and indeed, all results up to date support the violation of the inequality [14–16] up to few loopholes [17].

Notice that in the preceding calculation we have not specified the spacetime coordinates of the two measurement events, which means that we can as well settle them at a spacelike distance. This then implies that, even though the events could not influence each other according to relativistic causality, the established correlations are stronger than the ones attributable to a common cause. This is a paradigmatic instance of what Einstein called ”spooky action at a distance”. However, relativistic causality is still retained on an operational level in form of the *no signaling principle* [22] which states that the marginal distributions depend only on their local inputs

$$\sum_b P\left(ab|xy\right) = \sum_b P\left(ab|xy'\right),$$
$$\sum_a P\left(ab|xy\right) = \sum_a P\left(ab|x'y\right),$$

thereby preventing the two parties to send superluminal signals using their boxes. More concretely, the statistics of Alice’s local measurement outcomes, which is represented by her marginal distribution, does not depend on Bob’s input, i.e. Bob cannot signal any message to Alice (the analogous reasoning holds for the converse statement). Quantum mechanical distributions (2.9) satisfy this constraint, due to the tensorial structure of composite systems

$$\sum_b P(ab|xy) = \sum_b \mathrm{Tr}\left(\Pi_a^x \otimes \Pi_b^y \rho\right) = \mathrm{Tr}\left(\Pi_a^x \otimes \left(\sum_b \Pi_b^y\right)\rho\right) = \mathrm{Tr}\left(\Pi_a^x \otimes \mathbb{1}\rho\right) = \sum_b P(ab|xy'),$$
$$\sum_a P(ab|xy) = \sum_a \mathrm{Tr}\left(\Pi_a^x \otimes \Pi_b^y \rho\right) = \mathrm{Tr}\left(\sum_a \Pi_a^x \otimes \Pi_b^y \rho\right) = \mathrm{Tr}\left(\mathbb{1} \otimes \Pi_b^y \rho\right) = \sum_a P(ab|x'y),$$

where we used the fact that measurement operators sum up to the identity, i.e. any measurement always produces an outcome. The main consequence of Bell's theorem is that, if one assumes physical quantities to possess objective preestablished values (which merely get uncovered in the measurement process), then one is forced to accept that these values influence each other in a nonlocal fashion, violating relativistic causality. A typical example of this feature is Bohmian mechanics where the role of hidden variables is played by particle trajectories which interact nonlocally [18].

2.2.3 Convex geometry

After reviewing the basics of Bell's inequalities, let us make a brief excursion into convex geometry [19], the purpose of which will be justified in the next section. For a start, we define what is a convex set.

Definition *Let $\mathbb{R}^D$ be a D-dimensional vector space over real numbers. A subset $C \subset \mathbb{R}^D$ is convex if*

$$\forall u, v \in C, \lambda \in [0,1]: \quad (1-\lambda)u + \lambda v \in C. \tag{2.14}$$

Translated into words, a set is convex if any convex mixture of any two of its elements still lies in the set. Notice that already at this step we have enough mathematical structure for a physical interpretation, since any convex mixture can be regarded as a probabilistic mixture. Thus, any set of elements which is closed under probabilistic mixing is convex.

Definition *A convex set C is named polyhedron if it is the set of solutions to a finite system of linear inequalities, and called polytope if it is a bounded polyhedron.*

Intuitively, polyhedra are convex sets without curved boundaries: e.g. a sphere is convex, but not a polyhedron, since it has curved boundaries and thus cannot be represented by a finite system of linear inequalities. A typical example of a polyhedron which is not a polytope is an infinite cone, while typical polytopes are triangles, squares, etc.

Theorem *(Minkowski-Weyl) Every polyhedron P has two equivalent representations:*
(a) (halfspace) H-representation, in terms of a finite set of linear inequalities

$$\vec{p} \in P \quad \textit{iff} \quad \forall i : \vec{h}_i \cdot \vec{p} \leq b_i,$$

where $\left\{\vec{h}_i\right\}$ is a finite set of vectors and $\{b_i\}$ is a finite set of real numbers.
(b) (vertex) V-representation, in terms of a finite number of points (vertices)

$$\vec{p} \in P \quad \textit{iff} \quad \exists \{q_k \geq 0\}, \sum_k q_k = 1, \quad \textit{s.t.} \quad \vec{p} = \sum_k q_k \vec{v}_k,$$

where $\{\vec{v}_k\}$ is a finite set of vectors, namely vertices.

A trivial example of this feature is a rectangular triangle lying in a two-dimensional plane labeled by coordinates (x, y). If its V representation is given by

$$V = \{(0,0), (0,1), (1,0)\},$$

its H representation is then

$$H = \{x \geq 0, y \geq 0, x + y \leq 1\}. \tag{2.15}$$

2.2.4 The local polytope

Let us return to the black box scenario of section 2.2.1 and regard the behavior

$$\{P(ab|xy); \forall a, b, x, y \in \{0,1\}\} \tag{2.16}$$

as a vector in a D-dimensional real vector space R^D where $D = 2^4 = 16$. Since the probabilities should be normalized, we must impose the following constraints

$$\sum_{a,b} P(ab|xy) = 1; \forall x, y \in \{0,1\}, \tag{2.17}$$

which represent 4 linear equalities and thus reduce the dimension to $D = 12$. Furthermore, assuming that the black boxes produce outputs at a spacelike distance, any physical process must satisfy the no-signaling principle

$$\sum_a P\left(ab|xy\right) = \sum_a P\left(ab|xy'\right); \forall x \in \{0,1\},$$
$$\sum_b P\left(ab|xy\right) = \sum_b P\left(ab|x'y\right); \forall y \in \{0,1\},$$

which reduces the dimensionality to $D = 8$.

What does the set of behaviors that can be modeled by LHV-s look like?

A crucial step required to provide an answer to this question was made by Fine in [20], where he showed that any local randomness can be incorporated in shared randomness, which means that it is sufficient to analyze deterministic hidden variable models, i.e. those in which λ and the local inputs completely determine the local outputs. Since the number of inputs and outputs is finite, the set of deterministic hidden variables is finite and the expression (2.7) can then be rewritten as

$$P\left(ab|xy\right) = \sum_{\lambda_i} q(\lambda_i)\delta_{a=f_{\lambda_i}(x)}\delta_{b=g_{\lambda_i}(y)}, \tag{2.18}$$

where f_{λ_i} and g_{λ_i} are functions mapping bits into bits depending on the value of the hidden variable λ_i. Any distribution arising from a LHV model is a convex combination of a finite number of points: the LHV set is thus a convex polytope, namely the 'local polytope', with vertices given by deterministic hidden variables.

The task of translating the V-representation of a polytope into its H-representation is generally known as the facet enumeration problem and is dealt with numerically. It can be shown [22] that the H-representation of the local polytope for the bipartite case with binary inputs and outputs can be fully constructed from a family of CHSH-like inequalities obtained from inequality (2.8) by relabeling parties, inputs and outputs and from trivial inequalities of the form

$$0 \leq P\left(ab|xy\right) \leq 1; \quad \forall a,b,x,y \in \{0,1\}. \tag{2.19}$$

The local polytope has thus been fully characterized, meaning that we found the set of necessary and sufficient conditions a behavior has to satisfy in order to be attributable to underlying LHV-s.

As we have seen earlier, some quantum distributions violate the CHSH inequality,

which means that the local polytope is strictly smaller than the set of quantum behaviors (i.e. behaviors that arise from performing local measurements on shared quantum states). There has been a lot of work invested in the characterization of the quantum set; one of the most notorious results is the *Tsirelson bound* [21], which states that the maximum possible violation of the CHSH inequality by quantum distributions is $2\sqrt{2}$ (achievable e.g. by the process provided in section 2.2.2).
On the other hand, Popescu and Rohrlich ”constructed” supraquantum boxes (the so called PR-boxes) which can achieve the logical bound of the CHSH inequality without violating the no-signaling conditions [23]. Therefore, the no-signaling principle does not single out quantum correlations. Which physical principle is missing then? Various ones have since been proposed, but a definite answer has not yet been constructed [24–27]. This whole line of thought is part of the recently emerging device-independent paradigm, in which the aim is to formulate physical processes operationally at the level of probabilities which can then be tested directly in an actual experiment. Quantum theory is then seen as just one of the many possible underlying theories: this perspective, besides offering a novel way towards the development of new physical theories, can also give insights into the nature of quantum theory itself by comparing it with various counterfactual theories. One thus works towards understanding what quantum theory *is* by shedding light on what quantum theory *is not*.

Inspired by the connections between local hidden variables, Bell’s inequalities and convex geometry, we are now ready to undertake the task of creating an operational framework for answering questions posed at the end of section 2.1. In the next chapter we will introduce a black box scenario which we find apt for analyzing the aforementioned questions and provide a purely *behavioristic* device independent description, i.e. without addressing any underlying physical process. Afterwards, we are going to ask ourselves what classes of behaviors are achievable by various classical, quantum or more general mechanisms.

3 Characterization of signaling correlations

3.1 Describing the scenario

As pictured in Figure 3.1, the scenario of interest consists of one party, whom we will refer to as Alice, and N equidistant surrounding parties each of whom possesses one piece of information $x_j \in I_k$, where I_k is an arbitrary set containing k elements (for $k = 2$, the x_j-s are simply bits). Alice is connected to each of the N parties with communication channels which enable a bidirectional transmission of information carriers. We assume that the N parties containing the local information pieces are not mutually connected by any channel, i.e. the information cannot flow in between different parties. This restriction can be understood as one forcing the pieces of information to be truly isolated/localized and removing the dependence on the geometry of the problem. Furthermore, Alice possesses a finite amount of resources available for communication purposes, e.g. a sheet of paper, a photon, etc.

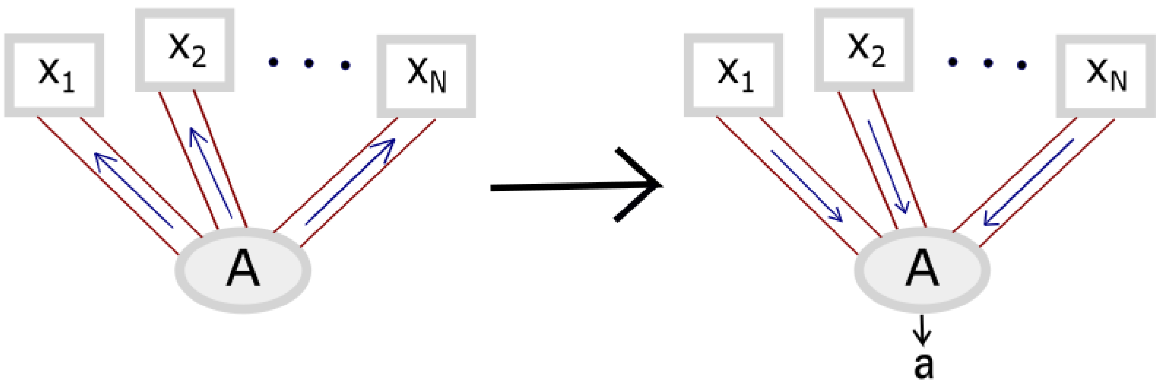


Figure 3.1: Alice is connected to each of the N surrounding parties via N communication channels. At $t = 0$ she sends her communication resources towards a subset of the N parties. At $t = 2\tau$, upon receiving back her signals, she decodes the message and produces a classical output a.

The process that we are going to analyze proceeds as follows: at time $t = 0$ Alice sends her resources to a subset of the N players; at $t = \tau$ the resources arrive at the destined parties who encode their inputs on the resources and send them back to Alice. Finally, at $t = 2\tau$, Alice receives and decodes the resources, thereby producing an output $a \in O_l$, where O_l is an arbitrary set containing l elements.

The system is thus mathematically fully characterized by the following set of conditional probabilities, or *behavior*

$$\{P\left(a|x_1, x_2, ..., x_N\right); \quad \forall a \in O_l, x_j \in I_k\}, \tag{3.1}$$

where $P\left(a|x_1, x_2, ..., x_N\right)$ indicates the probability that at time $t = 2\tau$ Alice otputs a, given that the N parties' pieces of information are $\{x_1, x_2, ..., x_N\}$. Next, we introduce the concept of *m-way signaling behaviors*.

Definition *A behavior* $\{P\left(a|x_1, x_2, ..., x_N\right), \quad \forall a, x_i\}$ *is said to be m-way signaling iff there exists a set of weights* $\{q_{j_1,j_2,...,j_m}, \forall j_1, ..., j_m\}$ *and a set of probability distributions* $\{P\left(a|x_{j_1}, x_{j_2}, ..., x_{j_m}\right), \forall j_1, ..., j_m\}$ *such that the following is satisfied:*

$$\begin{gathered} P\left(a|x_1, x_2, ..., x_N\right) = \sum_{j_1,j_2,...,j_m} q_{j_1,j_2,...,j_m} P\left(a|x_{j_1}, x_{j_2}, ..., x_{j_m}\right); \\ \sum_{j_1,j_2,...,j_m} q_{j_1,j_2,...,j_m} = 1; \\ q_{j_1,j_2,...,j_m} \geq 0, \quad \forall j_1, j_2, ..., j_m, \end{gathered}$$

where the domain of the indices $\{j_1, j_2, ..., j_m\}$ *ranges over all* $\binom{N}{m}$ *subsets of the* N *surrounding parties.*

The intuition behind the latter definition is the following: if the system exhibits m-way signaling, it means that its behavior can be modeled by Alice choosing to communicate with parties $\{j_1, j_2, ..., j_m\}$ with probability $q_{j_1,j_2,...,j_m}$.
For example, for $N = 3$, a two-way signaling distribution can be decomposed as

$$\begin{gathered} P\left(a|x_1, x_2, x_3\right) = q_{12} P\left(a|x_1, x_2\right) + q_{13} P\left(a|x_1, x_3\right) + q_{23} P\left(a|x_2, x_3\right); \\ \sum_{ij} q_{ij} = 1; \quad q_{ij} \geq 0, \forall i < j, \end{gathered} \tag{3.2}$$

where q_{ij} denotes the probability of Alice communicating with parties who possess inputs x_i and x_j.

3.2 Mathematical structure

The set of conditional probabilities (3.1) can be regarded as components of a vector in a D-dimensional real vector space $\mathbb{R}^D$, where $D = lk^N$. Since the probabilities must be normalized, we are interested only in a subset of the total vector space, i.e. we require

$$\sum_{a=0}^{L-1} P\left(a|x_1, x_2, ..., x_N\right) = 1, \quad \forall x_1, x_2, ..., x_N, \tag{3.3}$$

which constitute k^N linear constraints. We can thus reduce the dimension D to $D = (l-1)k^N$ by simply ignoring one component of each distribution, say the component $a = l - 1$.

Of course, not all vectors in $\mathbb{R}^D$ will be valid behaviors: we must still require all components to be non negative and to respect the normalization conditions. The set of the valid ones form a D-dimensional polytope which we will denote with L_D and which we will refer to as the *logical polytope*, since it contains all logically admissible behaviors.

The set of all k-way signaling behaviors forms a subset of L_D which we will denote with S_m. Intuitively, S_m should be closed under convex mixtures, since the latter can be interpreted as Alice choosing probabilistically among two different m-way signaling behaviors, which must again exhibit m-way signaling.

This intuition can be justified mathematically by first noting that a general conditional probability can be written as a convex sum of deterministic distributions

$$P(a|b) = \sum_f \mu_f \delta_{a,f(b)}, \tag{3.4}$$

where the sum runs over all functions *f* from $\mathcal{I}$ (input) to $\mathcal{O}$ (output), the latter being alphabets to which *b* and *a* pertain respectively. A general m-way signaling correlation can then be expressed as

$$P\left(a|x_1, x_2, ..., x_N\right) = \sum_{j_1, j_2, ..., j_m} q_{j_1, j_2, ..., j_m} P\left(a|x_{j_1}, x_{j_2}, ..., x_{j_m}\right)$$

$$= \sum_{j_1, j_2, ..., j_m} q_{j_1, j_2, ..., j_m} \sum_f \mu_f \delta_{a, f(x_{j_1}, ..., x_{j_m})} = \sum_{f, j_1, ..., j_m} \lambda_{f, j_1, ..., j_m} \delta_{a, f(x_{j_1}, ..., x_{j_m})},$$

where we defined a new set of weights

$$\lambda_{f,j_1,\dots,j_m} \equiv \mu_f q_{j_1,j_2,\dots,j_m}; \quad \lambda_{f,j_1,\dots,j_m} \geq 0, \sum_{f,j_1,\dots,j_m} \lambda_{f,j_1,\dots,j_m} = 1. \tag{3.5}$$

Therefore, every element of S_m can be written as a convex combination of a finite number of deterministic distributions, and is thus a polytope.
Now we are finally ready to embark on the mission of characterizing S_m in terms of its V and H representations. In what follows, we will assume for simplicity binary inputs and outputs, while the generalization will be given later.

3.3 *Two parties*

For a start, we are going to analyze the bipartite case $N = 2$ where the input and output alphabets are fixed to

$$I_2 = O_2 = \{0, 1\}.$$

We are dealing with a real D-dimensional vector space $\mathbb{R}^D$, where $D = (l-1)k^N = (2-1)2^2 = 4$. Let us choose a basis $\{\vec{e}_{ij}\}$ such that all vectors in the logical polytope can be decomposed as

$$\vec{P} = \sum_{x_1,x_2=0}^{1} P(0|x_1,x_2)\vec{e}_{x_1,x_2}, \tag{3.6}$$

where we chose to eliminate all $a = 1$ components via the normalization constraints. Since all components pertain to different probability distributions, the only constraint imposed on them is to lie in the interval $[0, 1]$, which implies that the logical polytope L_4 is a four-dimensional hypercube.
Our goal now is to characterize the set of all one-way signaling correlations S_1, i.e. we need to find its vertices and its facet inequalities.
According to the definition given in section 3.1, $\forall P \in S_1, \exists q \in [0, 1]$, such that

$$\begin{aligned}\vec{P} = \sum_{x_1,x_2=0}^{1} P(0|x_1,x_2)\vec{e}_{x_1,x_2} &= \sum_{x_1,x_2=0}^{1} \left[qP_1(0|x_1) + (1-q)P_2(0|x_2)\right]\vec{e}_{x_1,x_2} \\ &= q\vec{P}_1 + (1-q)\vec{P}_2,\end{aligned}$$

where

$$\vec{P}_1 = \sum_{x_1,x_2=0}^{1} P_1(0|x_1)\vec{e}_{x_1,x_2} \quad \text{and} \quad \vec{P}_2 = \sum_{x_1,x_2=0}^{1} P_2(0|x_2)\vec{e}_{x_1,x_2} \tag{3.7}$$

are respectively vectors that represent communication with exclusively the first or the second party and which form two new polytopes which we will call $S_1^{(1)}$ and $S_1^{(2)}$. Since S_1 is a convex hull of $S_1^{(1)}$ and $S_1^{(2)}$, the following holds for their sets of vertices:

$$V(S_1) = S_1^{(1)} \cup S_1^{(2)}. \tag{3.8}$$

Finding the sets $S_1^{(1,2)}$ amounts to finding all deterministic distributions compatible with the form (3.7). Taking the union of these two sets, we obtain the set of vertices of all one-way signaling distributions (written in columns in the previously defined basis)

$$V(S_1) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \tag{3.9}$$

The next step is translating the vertex representation into its equivalent facet representation, i.e. solve the facet enumeration problem. As mentioned in section 2.2.4, this problem is usually dealt with numerically, but here we will use a sleight of hand. Let us first define a unitary transformation $U \equiv H_2 \otimes H_2$, where H_2 is the standard Hadamard matrix

$$U \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \tag{3.10}$$

and apply it to our vertices in order to obtain

$$U\left(V(S_1)\right) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}. \tag{3.11}$$

We notice that the last component of each vertex vanishes, i.e. $\forall \vec{v} \in V(S_1)$

$$v_{00} - v_{01} - v_{10} + v_{11} = \sum_{x_1, x_2 = 0}^{1} (-)^{x_1 + x_2} v_{x_1, x_2} = 0, \tag{3.12}$$

where $v_{ij} \equiv \vec{e}_{ij} \cdot \vec{v}$.

Since all one-way signaling distributions are a convex combination of the given vertices, it follows that the same constraint holds for all one-way signaling distributions, i.e.

$$\forall \vec{P} \in S_1 : \quad \sum_{x_1, x_2 = 0}^{1} (-)^{x_1 + x_2} P\left(0 | x_1, x_2\right) = 0. \tag{3.13}$$

The latter is a necessary condition that a set of distributions has to satisfy in order to be one-way signaling. It can be rewritten as

$$P\left(0 | x_1 \oplus x_2 = 0\right) = P\left(0 | x_1 \oplus x_2 = 1\right), \tag{3.14}$$

which means that using only one-way communication, Alice cannot get to know anything about the parity of the inputs of the two surrounding parties. In other words, if the parties played a game where Alice is supposed to guess the parity of the two parties' inputs, the winning probability using one-way communication would not increase with respect to a random guess.

Let us now return to the set of rotated vertices (3.11) and apply to them a translation by

$$\vec{T} = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.15}$$

to obtain

$$T\left[U\left(V(S_1)\right)\right]=\left\{\begin{bmatrix}-1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\-1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}0\\-1\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}, \tag{3.16}$$

where we omitted the last component of each vertex, i.e. we projected the vertices on the hyperplane (3.13).

In this form, the structure of our polytope is manifest: it turns out to be a three-dimensional octahedron. Now, we can intuitively write down the facet inequalities, while taking into account the unitary transformation and translation that we have performed. There are 8 inequalities and they can all be summarized in the following form:

$$|P\left(0|00\right)+P\left(0|01\right)+P\left(0|10\right)+P\left(0|11\right)-2|+|P\left(0|00\right)-P\left(0|01\right)+P\left(0|10\right)-P\left(0|11\right)|$$
$$+|P\left(0|00\right)+P\left(0|01\right)-P\left(0|10\right)-P\left(0|11\right)|\leq 2.$$

We managed to fully characterize the set of one-way signaling distributions in the bipartite case: we figured out that it is a three-dimensional octahedron lying in the intersection of the *logical hypercube* and the hyperplane (3.13).

3.4 Three parties

Now we are going to generalize the previous analysis to the tripartite case. We start by eliminating the $a=1$ components of each distribution via the normalization conditions. A general vector $\vec{P}\in\mathbb{R}^8$ that describes the system is now

$$\vec{P}=\sum_{x_1,x_2,x_3=0}^{1}P(0|x_1,x_2,x_3)\vec{e}_{x_1,x_2,x_3}, \tag{3.17}$$

and the logical polytope L_8 is an 8-dimensional hypercube.

In order to characterize the set S_1 we proceed analogously to the bipartite case. First, we notice that any $\vec{P}\in S_1$ can be written as

$$\vec{P}=q_1\vec{P}_1+q_2\vec{P}_2+q_3\vec{P}_3;\quad \sum q_i=1; q_i\geq 0, \tag{3.18}$$

where $\vec{P}_j$ is a vector that represents communication exclusively with the j-th party

$$\vec{P}_j = \sum_{x_1,x_2,x_3=0}^{1} P_j(0|x_j)\vec{e}_{x_1,x_2,x_3}. \tag{3.19}$$

The set of vertices of S_1 is thus

$$V(S_1) = V(S_1^{(1)}) \cup V(S_1^{(2)}) \cup V(S_1^{(3)}), \tag{3.20}$$

and can be obtained analogously to the bipartite case:

$$S_1 = \left\{ \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 1\\1\\1\\1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\1\\1\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\1\\0\\0\\1\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\1\\0\\0\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\0\\1\\0\\1\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\1\\0\\1\\0\\1 \end{bmatrix}, \begin{bmatrix} 1\\1\\1\\1\\1\\1\\1\\1 \end{bmatrix} \right\}. \tag{3.21}$$

We proceed by applying the unitary transformation $U = H_2 \otimes H_2 \otimes H_2$ to the obtained vertices:

$$U(V(S_1)) = \left\{ \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 1\\0\\0\\0\\1\\0\\0\\0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 1\\0\\0\\0\\-1\\0\\0\\0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 1\\0\\1\\0\\0\\0\\0\\0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 1\\0\\-1\\0\\0\\0\\0\\0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 1\\1\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 1\\-1\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 2\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix} \right\}, \tag{3.22}$$

and we notice that some components are null for all vertices, which implies that all one-way signaling distributions necessarily satisfy the following equalities

$$\sum_{x_1,x_2,x_3=0}^{1} (-)^{x_1+x_2}\, P\left(0|x_1,x_2,x_3\right) = 0, \tag{3.23}$$

$$\sum_{x_1,x_2,x_3=0}^{1} (-)^{x_1+x_3}\, P\left(0|x_1,x_2,x_3\right) = 0, \tag{3.24}$$

$$\sum_{x_1,x_2,x_3=0}^{1} (-)^{x_2+x_3}\, P\left(0|x_1,x_2,x_3\right) = 0, \tag{3.25}$$

$$\sum_{x_1,x_2,x_3=0}^{1} (-)^{x_1+x_2+x_3}\, P\left(0|x_1,x_2,x_3\right) = 0. \tag{3.26}$$

The latter can be cast as

$$\begin{aligned}
&P\left(0|x_1 \oplus x_2 = 0\right) = P\left(0|x_1 \oplus x_2 = 1\right),\\
&P\left(0|x_1 \oplus x_3 = 0\right) = P\left(0|x_1 \oplus x_3 = 1\right),\\
&P\left(0|x_2 \oplus x_3 = 0\right) = P\left(0|x_2 \oplus x_3 = 1\right),\\
&P\left(0|x_1 \oplus x_2 \oplus x_3 = 0\right) = P\left(0|x_1 \oplus x_2 \oplus x_3 = 1\right).
\end{aligned}$$

Thus, limiting herself to purely one-way communication, Alice cannot make better than guess nor the total parity of the inputs, nor any ”bipartite” parity (i.e. parity of any bipartite subset of the three parties).
Next, we perform a translation by $\vec{T} = -\vec{e}_{000}$:

$$T\left[U\left(V\left(S_1\right)\right)\right] = \sqrt{2}\left\{\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}, \tag{3.27}$$

where we omitted the null components. The polytope is a 4-dimensional octahedron; its H-representation is:

$$|\sum_{x_1,x_2,x_3=0}^{1} P\left(0|x_1,x_2,x_3\right)-4|+|\sum_{x_1,x_2,x_3=0}^{1}(-)^{x_1}P\left(0|x_1,x_2,x_3\right)|$$
$$+|\sum_{x_1,x_2,x_3=0}^{1}(-)^{x_2}P\left(0|x_1,x_2,x_3\right)|+|\sum_{x_1,x_2,x_3=0}^{1}(-)^{x_3}P\left(0|x_1,x_2,x_3\right)|\leq 4.$$

Therefore, the set of one-way signaling correlations S_1 for $N=3$ is a four-dimensional octahedron lying in the intersection of the four hyperplanes (3.23) and the logical hypercube.

3.5 N parties

We proceed by generalizing the previous results to characterizing the S_1 set in the N-partite case, while still focusing on binary inputs and outputs. The dimension of the probability space is $D=2^N$ and the logical polytope is a D-dimensional hypercube (after eliminating the $a=1$ component of each distribution using normalization conditions). In order to make the notation more compact, we will reverse the order of the inputs and label them as $\{x_0,x_1,...x_{N-1}\}$, i.e.

$$P\left(a|x_1,x_2,...,x_N\right)\rightarrow P\left(a|x_{N-1}x_{N-2}...x_1x_0\right). \tag{3.28}$$

A general vector $\vec{P}\in L_D$ can be then written as

$$\vec{P}=\sum_{i=0}^{2^N-1}P_i\vec{e}_i,\quad P_j=P(0|\vec{x}^{(j)})=P(0|x_{N-1}^{(j)}x_{N-2}^{(j)}...x_1^{(j)}x_0^{(j)}), \tag{3.29}$$

such that $\vec{x}^{(j)}=(x_{N-1}^{(j)}x_{N-2}^{(j)}...x_1^{(j)}x_0^{(j)})$ is the number *j* written in binary form.

Any one-way signaling distribution can be decomposed in terms of distributions which represent communication with a specific party, i.e. $\forall \vec{P} \in S_1$

$$\vec{P} = \sum_{s=0}^{N-1} q_s \vec{P}_s; \quad \sum_s q_s = 1, \quad q_s \geq 1,$$

$$\vec{P}_s = \sum_i P(0|x_s^{(i)})\vec{e}_i.$$

As in the bipartite and tripartite case, we know that two vertices are achievable with no communication at all, i.e. by Alice deterministically outputting 0 or 1 independently of the N inputs. We label these two vertices by $\vec{u}$ and $\vec{\omega}$:

$$\vec{u} = \vec{0}, \qquad \vec{\omega} = \sum_i \vec{e}_i. \tag{3.30}$$

The other $2N$ vertices refer to one-way communication with the N parties (for each party j, Alice can either output x_j or $x_j \oplus 1$, since other combinations are already included in $\vec{u}$ and $\vec{\omega}$):

$$\vec{v}^{(rs)} = \sum_{i=0}^{2^N-1} \delta_{x_r^{(i)},s} \vec{e}_i, \qquad r \in [0, N-1], \quad s \in \{0, 1\}, \tag{3.31}$$

where the index *r* refers to the player involved in the communication, while *s* indicates which of the two functions Alice is using in the process of producing her result. Next, we apply the transformation $U = H_2^{\otimes N}$, the components of which can be written in the following concise form

$$(U)_{ij} = \frac{1}{2^{N/2}}(-)^{i \cdot j} \tag{3.32}$$

where $i \cdot j = \sum_s x_s^{(i)} x_s^{(j)}$ is the bitwise dot product of numbers i and j. After some algebra, we obtain the rotated vertices (the components are written in the standard basis $\{\vec{e}_i\}$):

$$\begin{aligned}
(U\vec{u})_i &= 0, \\
(U\vec{\omega})_i &= 2^{N/2}\delta_{i,0}, \\
(U\vec{v}^{(rs)})_i &= 2^{N/2-1}\delta_{i,0} + (-1)^l 2^{N/2-1}\delta_{i,2^r}.
\end{aligned}$$

Translating by $\vec{T} = 2^{N/2-1}\vec{e}_0$ we obtain:

$$\begin{aligned}
(TU\vec{u})_i &= -2^{N/2-1}\delta_{i,0}, \\
(TU\vec{\omega})_i &= 2^{N/2-1}\delta_{i,0}, \\
(TU\vec{v}^{(rs)})_i &= (-1)^l 2^{N/2-1}\delta_{i,2^r}.
\end{aligned}$$

Therefore, $(2^N - N + 1)$ components are equal to 0; the corresponding hyperplane equations can be summed up as

$$\sum_{x_0,\ldots x_{N-1}=0}^{1} (-1)^{\sum_j \sigma_j x_j} P\left(0|x_{N-1},\ldots,x_0\right) = 0; \quad \forall \sigma_i \in \{0,1\}\,, \quad \text{s.t.} \quad \sum_j \sigma_j \geq 2. \tag{3.33}$$

The latter implies that using one-way communication, Alice cannot do better than guessing the parity of any subset of at least two parties.

The form of the S_1 polytope is once again manifestly an octahedron of dimension $N+1$ and its 2^{N+1} facet inequalities are

$$|\sum_{\vec{x}} P\left(0|x_{N-1},\ldots,x_0\right) - 2^{N-1}| + \sum_{r=0}^{N-1} |\sum_{\vec{x}} (-1)^{x_r} P\left(0|x_{N-1},\ldots,x_0\right)| \leq 2^{N-1}. \tag{3.34}$$

3.6 Necessary and sufficient conditions

In the previous section we fully characterized the set of one-way signaling correlations S_1 for binary inputs and outputs in terms of equalities and inequalities. Let us write them once again for clarity: a behavior $\vec{P}$ is one-way signaling if and only if it satisfies the following constraints

$$\sum_{x_0,\ldots x_{N-1}=0}^{1} (-1)^{\sum_j \sigma_j x_j} P\left(0|x_{N-1},\ldots,x_0\right) = 0; \quad \forall \sigma_i \in \{0,1\}\,, \quad \text{s.t.} \quad \sum_j \sigma_j \geq 2, \tag{3.35}$$

$$|\sum_{\vec{x}} P\left(0|x_{N-1},\ldots,x_0\right) - 2^{N-1}| + \sum_{r=0}^{N-1} |\sum_{\vec{x}} (-1)^{x_r} P\left(0|x_{N-1},\ldots,x_0\right)| \leq 2^{N-1}. \tag{3.36}$$

The $(2^N - N + 1)$ linear equalities (3.35) define an $(N+1)$-dimensional subspace which we will denote with B_1. The inequalities (3.36), when projected on B_1, describe an octahedron.

What is the qualitative difference between a behavior that lies outside of B_1 and a behavior that lies in B_1 but violates bounds imposed by the octahedron? Considering this question we examined the bipartite and tripartite case and found out, surprisingly, that if a distribution lies in B_1, then it necessarily also lies in the octahedron. Could this be the case for any N?

Any vector $\vec{P}$ lying in B_1 can be written as

$$\vec{P} = v_0 \vec{f}_0 + \sum_{r=0}^{N-1} P_2^r \vec{f}_{2^r}, \tag{3.37}$$

where the basis $\left\{\vec{f}_j\right\}$ is constituted of *rows/columns* of the matrix $U = H_2^{\otimes N}$ written in the $\{\vec{e}_j\}$ basis. The components missing in (3.37) are exactly those invoked in the equalities that define B_1.

Since $\{\vec{e}_j\}$ is the basis in which our vectors' components are interpreted as probabilities, in order for a behavior $\vec{P}$ to be part of the logical hypercube, it must satisfy

$$0 \leq \vec{e}_i \cdot \vec{P} \leq 1, \quad i = 0, 1, ..., 2^{N-1}. \tag{3.38}$$

Furthermore, the relation between the two bases is, as already stated, the matrix U:

$$\begin{aligned} \vec{e}_i \cdot \vec{f}_0 &= (U)_{i0} = \frac{1}{2^{N/2}} \\ \vec{e}_i \cdot \vec{f}_{2^r} &= (U)_{i2^r} = \frac{1}{2^{N/2}} (-1)^{x_r^{(i)}}. \end{aligned}$$

Thus, any vector $\vec{P}$ which lies in the intersection of B_1 and the logical hypercube necessarily satisfies the following inequalities

$$0 \leq \vec{f}_0 \cdot \vec{P} + \sum_{k=0}^{N-1} (-1)^{x_k^{(i)}} \vec{f}_{2^k} \cdot \vec{P} \leq 2^{N/2-1}, \quad i = 0, 1, ..., 2^{N-1}. \tag{3.39}$$

On the other hand, any one-way signaling behavior satisfies (3.36) which can be rewritten as

$$|\vec{f}_0 \cdot \vec{P} - 2^{N/2-1}| + \sum_{k=0}^{N-1} |\vec{f}_{2^k} \cdot \vec{P}| \leq 2^{N/2-1}. \tag{3.40}$$

The absolute values in the latter equation introduce alternating minus signs analogously to (3.39); indeed, depending on the sign in front of the first term, half of them

are equal to the *0 inequalities* and the other half to the $2^{N/2-1}$ *inequalities*. Thus we proved that intersecting the logical hypercube with hyperplanes (3.13) one obtains exactly the octahedron (3.36).
Now we state the latter result in form of a theorem.

Theorem *Let us fix the scenario to binary inputs, binary outputs and an arbitrary number of parties N. The complete set of necessary and sufficient conditions a behavior $\vec{P}$ needs to satisfy in order to be one-way signaling is given by the following set of hyperplane equalities*

$$\sum_{x_0,...x_{N-1}=0}^{1} (-1)^{\sum_j \sigma_j x_j} P\left(0|x_{N-1}...x_0\right) = 0; \quad \sigma_i \in \{0,1\}\,, \quad \textit{s.t.} \quad \sum_j \sigma_j \geq 2. \tag{3.41}$$

3.7 General scenario

Up to this point we have been focused solely on describing the S_m set for binary inputs and outputs. Let us now characterize general m-way signaling behaviors for k inputs and l outputs, where k and l are arbitrary natural numbers.
Behaviors are now vectors in a $D = (l-1)k^N$ dimensional vector space; we define the basis $\{\vec{e}_{aj}\}$ such that an arbitrary behavior in the logical polytope can be written as

$$\vec{P} = \sum_{a=0}^{l-2} \sum_{j=0}^{k^N-1} P\left(a|x_{n-1}^{(j)}, x_{n-2}^{(j)}, ..., x_1^{(j)}, x_0^{(j)}\right) \vec{e}_{aj}, \tag{3.42}$$

where $\vec{x}^{(j)} = (x_{N-1}^{(j)}, x_{N-2}^{(j)}, ..., x_1^{(j)}, x_0^{(j)})$ is the number *j* written in basis k. The complete set of vertices is then given by

$$\vec{v}_{f,p_1,p_2,...,p_m} = \sum_{a,j} \delta_{a,f(k^{p_1}\cdot j, k^{p_2}\cdot j,...,k^{p_m}\cdot j)} \vec{e}_{aj}, \tag{3.43}$$

for all $p_1, ..., p_m$ m-players subsets and for all functions f: $I_k^{\otimes m} \to O_l$. The products $k^{p_r} \cdot j$ are k-wise dot products and are equal to the r-th component of j written in basis k. The set of vertices (3.43) is overcomplete, since not all $\binom{N}{m}\, l^{k^m}$ combinations produce different behaviors.
Inspired by the previous cases, we rotate the given vertices into a form which will manifestly exhibit hyperplane equalities: the desired unitary transformation is given

by

$$U \equiv \bigoplus_{l-1} F_k^{\otimes N}, \tag{3.44}$$

where F_k is a Fourier matrix of order k, whose components are given by

$$(F_k)_{ps} = \frac{1}{\sqrt{k}} (\omega_k)^{p \cdot s}, \quad \omega_k \equiv e^{i2\pi/k}, \quad p \cdot s \equiv \sum_i x_i^{(p)} x_i^{(s)}. \tag{3.45}$$

Notice that U reduces to the right form for the binary cases since F_2 is a Hadamard matrix.
Applying U to the set of vertices (3.43) and singling out the zero components, we get the following equalities that any m-way signaling behavior necessarily satisfies:

$$\sum_{x_0, x_1, \ldots, x_{N-1}} (\omega_k)^{\sum_i \sigma_i x_i} P\left(a | x_{N-1} \ldots x_0\right) = 0; \quad a = 0, 1, \ldots, l-2, \quad \sigma_i = 0, 1, \ldots, k-1, \tag{3.46}$$

where $\vec{\sigma}$ has at least m non-zero components.

In the characterization of the S_1 set for binary inputs and outputs we were able to recognize the polytope intuitively, by simply looking at the vertices in the rotated (and translated) coordinate system. However, in the general case this procedure is hopeless, since the components in the rotated basis are complex and the structure cannot be visualized. Hence, we are not able to provide the H-representation in the fully general scenario and we will need to analyze specific cases numerically.
However, an easier way out may arise from the following question: is it possible that the set of equalities (3.46) is not only a necessary, but also a sufficient condition for a behavior to be m-way signaling, as it was the case for one-way signaling distributions with binary inputs and outputs? Can the S_m polytope be constructed purely by intersecting the hyperplanes (3.46) with the logical polytope? The answer is *no*, as can be seen from the following counterexample. Consider the $N = 3$ case with binary inputs and outputs and focus on two-way signaling behaviors which constitute the set S_2. According to the previous discussion:

$$\forall \vec{P} \in S_2 : \quad \sum_{x_1, x_2, x_3} (-1)^{x_1 + x_2 + x_3} P(0 | x_1, x_2, x_3) = 0. \tag{3.47}$$

Now, if the latter equality was a sufficient condition for $\vec{P}$ to be two-way signaling, all deterministic behaviors lying in the hyperplane (3.47) would be two-way signaling. However, a simple counterexample is the following behavior

$$\vec{V} = \vec{e}_{011} + \vec{e}_{100}, \tag{3.48}$$

which clearly satisfies (3.47) but cannot be generated by two-way communication. Consequently, a complete characterization of the signaling polytopes requires the construction of a full set of facet inequalities, which we are not able to provide in the general case. Nonetheless,we can still perform numerical calculations on some particular cases.

3.8 *Example of the H-representation of S_2*

In this section we will provide the full characterization of the two-way signaling polytope S_2 in the tripartite case with binary inputs and outputs.
The behavior space is $\mathbb{R}^8$ and the vertices are given by two-way signaling deterministic distributions

$$\vec{v}^{f}_{p_1,p_2} = \sum_{x_1,x_2,x_3} \delta_{0,f(x_{p_1},x_{p_2})} \vec{e}_{x_1,x_2,x_3}, \tag{3.49}$$

with $p_{1,2}$ being the parties chosen for communication and *f* being the function used by Alice to produce the output. There are altogether 38 non equivalent vertices, which can be translated to the complete set of facet inequalities using the programming package *cdd* [28]. The resulting H-representation is given by 96 inequalities, 16 of which are of the trivial form

$$0 \leq P\left(0|x_1, x_2, x_3\right) \leq 1, \quad \forall x_1, x_2, x_3 \in \{0, 1\}\,. \tag{3.50}$$

The remaining 80 inequalities can be cast into equivalence classes with respect to the symmetries of the scenario: permutations of parties, relabeling of inputs for each party and relabeling of Alice's outputs, resulting with solely three non equivalent inequalities:

$$P(0|100) + P(0|010) + P(0|001) - P(0|000) \leq 2, \tag{3.51}$$

$$P(0|011) + P(0|100) - P(0|101) - P(0|110) - P(0|111) \leq 1, \tag{3.52}$$

$$P(0|100) + P(0|001) - P(0|101) - P(0|110) \leq 1. \tag{3.53}$$

The latter inequalities and the equality (3.47) thus provide a full characterization of S_2 for $N = 3$. We now also see explicitly that the vertex (3.48) violates the second inequality by achieving the logical bound of 2.

3.9 *An inequality in the general scenario*

Next, we provide a straightforward generalization of inequality (3.51) to an arbitrary number of parties and arbitrary cardinality of input and output sets. Indeed, the following inequality

$$B \equiv -P(1|0,0,...,0) + P(1|1,0,...,0) + P(1|0,1,...,0) + ... + P(1|0,0,...,1) \leq N - 1 \tag{3.54}$$

is satisfied by any $(N-1)$-way signaling distribution.

To see that this is the case, notice that any $(N-1)$-way signaling behavior can be expressed as a convex sum of processes which leave out one party from the communication. If the i-th party is left out, then

$$P(a|0,0,...,x_i = 0,...,0) = P(a|0,0,...,x_i = 1,...,0), \tag{3.55}$$

so the first negative term in B cancels at least one of the positive terms and leaves the maximum achievable value equal to $N-1$.

Notice that in the previous observations we have not assumed anything about the input and output spaces: indeed, inequality (3.54) holds in full generality, i.e. for an arbitrary number of parties and cardinalities of input and output sets. Several other $(N-1)$-way signaling inequalities can be constructed from (3.54) by applying symmetry operations.

Fortunately, despite the impossibility of finding the full set of necessary and sufficient conditions in the general case, we managed to find one particular inequality for arbitrary N by generalizing an inequality which we obtained numerically.

4 Classical, quantum and beyond

Up to this point, there has been no reference whatsoever to what lies beneath the analyzed behaviors. In this chapter, we will borrow the results from the device independent formulation from the previous section and see what classes of behaviors are achievable by various underlying physical processes.

4.1 Classical processes

Classical mechanics is a theory that describes the world around us as a conglomerate of particles with definite positions and momenta, a trivial consequence of which is that at any given instant, each particle has a well defined trajectory putting a strong limit on its signaling capacity. Thus, one classical particle allows at most one-way signaling, or more generally, P classical particles allow at most P-way signaling.

4.2 Quantum processes

Juxtaposed to classical mechanics, quantum theory allows particles to be in coherent superpositions of trajectories which opens up possibilities for achieving higher levels of signaling. Let us first put the result from chapter 2.1 on a more formal footing.

4.2.1 Two-way communication with a single quantum particle revisited

The system consists of Alice who possesses a single photon and of two surrounding parties who can implement local phase shifts depending on their binary inputs x_1 and x_2. At $t = 0$, Alice prepares her photon in a superposition of trajectories directed respectively towards the two parties:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|1\rangle + |2\rangle \right). \tag{4.56}$$

At t= τ, the two parties receive their parts of the photon and implement local phase shifts depending on their inputs

$$|\psi\rangle_{x_1,x_2} = \frac{1}{\sqrt{2}}\left((-1)^{x_1}|1\rangle + (-1)^{x_2}|2\rangle\right), \tag{4.57}$$

and immediately send the resource back to Alice. Finally, at t=2τ, Alice performs a measurement on the incoming wave packets in the Hadamard basis $\{|+\rangle, |-\rangle\}$, defined as

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle \pm |2\rangle\right). \tag{4.58}$$

The outcome statistics is given by

$$P(0|x_1,x_2) = \frac{1}{4}\left(1+(-1)^{x_1+x_2}\right)^2 = \begin{cases} 1 & \text{if } x_1 \oplus x_2 = 0 \\ 0 & \text{if } x_1 \oplus x_2 = 1. \end{cases} \tag{4.59}$$

Alice can guess the inputs' parity with perfect efficiency, thereby maximally violating the one-way signaling equality:

$$\sum_{x_1,x_2} (-1)^{x_1+x_2} P(0|x_1,x_2) = 2 \neq 0. \tag{4.60}$$

This result formally quantifies the statement given in section 2.1, namely the possibility of achieving two-way communication using a single quantum particle.

4.2.2 A peculiar constraint

We now proceed by tackling the quantum scenario in full generality by considering an arbitrary number of parties N and arbitrary cardinalities of inputs' and outputs' sets, denoted respectively with k and l. In what follows, we will assume the quantum states to be pure: all the results are trivially extendable to mixed states because of the linearity of quantum operations and measurements. Also, without loss of generality, we will assume the N parties can implement only unitary operations (one might also analyze generic completely positive maps, but this would not introduce anything qualitatively new into the discussion).

For a start, let us fix the resources to a single quantum particle. The particle is completely unspecified and can have arbitrarily many internal degrees of freedom of unspecified dimensionality. At the initial moment, Alice prepares an arbitrary pure state

$$|\psi\rangle = \sum_{n=0}^{N-1} c_n |\phi_n\rangle \otimes |n\rangle , \tag{4.61}$$

where $\{|n\rangle\}$ is the basis of defined trajectories directed towards each of the N parties, while $\{|\phi\rangle ...\}$ spans the state space of the internal degrees of freedom (e.g. spin orientations). At $t = \tau$ the parties receive the resource and perform local unitary transformations

$$|\psi\rangle_{\vec{x}} = \sum_{n=0}^{N-1} c_n [U_n(x_n) |\phi_n\rangle] \otimes |n\rangle , \tag{4.62}$$

where $\{U_n(x_n)\}$ are local unitary transformations that depend on the inputs and act on the internal state of the particle. Immediately after encoding, the parties send their resources back to Alice who performs a general measurement producing the final output

$$P(a|\vec{x}) = \mathrm{Tr}\,[\Pi_a \rho_{\vec{x}}] , \tag{4.63}$$

where $\{\Pi_a, \forall a = 0, ..., l-1\}$ is a generic POVM (positive operator valued measure) with l elements corresponding to l outcomes and $\rho_{\vec{x}}$ is the density operator corresponding to the state (4.62)

$$\rho_{\vec{x}} = \sum_{n,m=0}^{N-1} c_n c_m^* U_n(x_n) |\phi_n\rangle \langle\phi_m| U_m^\dagger(x_m) \otimes |n\rangle \langle m| . \tag{4.64}$$

Let us now insert the behavior (4.63) into the hyperplane equalities (3.46)

$$\sum_{\vec{x}} (\omega_k)^{\sum_i \sigma_i x_i} P(a|\vec{x}) = \sum_{n,m=0}^{N-1} c_n c_m^* |n\rangle \langle m| \otimes \sum_{\vec{x}} (\omega_k)^{\sum_i \sigma_i x_i} U_n(x_n) |\phi_n\rangle \langle\phi_m| U_m^\dagger(x_m)$$
$$= \sum_{n,m=0}^{N-1} c_n c_m^* |n\rangle \langle m| \otimes \left[\sum_{\vec{x}\setminus\{x_n,x_m\}} (\omega_k)^{\sum_i \sigma_i x_i} \right] (\omega_k)^{\sigma_n x_n + \sigma_m x_m} U_n(x_n) |\phi_n\rangle \langle\phi_m| U_m^\dagger(x_m).$$

The expression

$$\left[\sum_{\vec{x}\setminus\{x_n,x_m\}} (\omega_k)^{\sum_i \sigma_i x_i} \right]$$

is equal to 0 if $\sigma_i \neq 0$ for at least one element in the sum, since

$$\sum_{x=0}^{k-1} (\omega_k)^{\sigma x} = 0, \quad \sigma \in N.$$

If $\vec{\sigma}$ that defines the hyperplane has at least three non-zero components, then the latter condition is satisfied $\forall n, m$ which implies that the hyperplane equality is satisfied. Thus, any behavior arising from communication using one quantum particle lies in the subset defined by the following equalities

$$\sum_{x_0,x_1,...,x_{N-1}} (\omega_k)^{\sum_i \sigma_i x_i} P\left(a|x_{N-1}...x_0\right) = 0; \quad a = 0,1,...,l-2, \quad \sigma_i = 0,1,...,k-1, \tag{4.65}$$

for all $\vec{\sigma}$ with at least three non-zero components.

This constrains one-particle-quantum distributions to a very small subset of possible behaviors; specifically the one which contains all two-way signaling distributions. However, as we have seen in section 3.7, despite this constraint, distributions lying in the plane may still exhibit higher orders of signaling.

Now we are going to extend the previous constraint to an arbitrary number P of non-identical particles. The particles are again completely generic and can have any internal structure. Adopting the previously used notation, the initial quantum state is

$$|\psi_0\rangle = \sum_{n^{(0)},n^{(1)},...,n^{(P-1)}=0}^{N-1} c_{n^{(0)},n^{(1)},...,n^{(P-1)}} \left|\Phi_{n^{(0)},...,n^{(P-1)}}\right\rangle \bigotimes_{p=0}^{P-1} \left|n^{(p)}\right\rangle, \tag{4.66}$$

where $\left|\Phi_{n^{(0)},...,n^{(P-1)}}\right\rangle$ are potentially entangled states of the internal degrees of freedom of the P particles.

The density operator after encoding at time t=τ is then

$$\rho_{\vec{x}} = \sum_{\substack{n^{(0)},...,n^{(P-1)}=0 \\ m^{(0)},...,m^{(P-1)}=0}}^{N-1} c_{\vec{n}} c^*_{\vec{m}} \bigotimes_{p=0}^{P-1} U_{n^{(p)}}\left(x_{n^{(p)}}\right) |\Phi_{\vec{n}}\rangle \langle\Phi_{\vec{m}}| U^\dagger_{m^{(p)}}\left(x_{m^{(p)}}\right) \otimes \left|n^{(p)}\right\rangle \left\langle m^{(p)}\right|, \tag{4.67}$$

where $\vec{n}$ and $\vec{m}$ stand for $n^{(0)}, ..., n^{(P-1)}$ and $m^{(0)}, ..., m^{(P-1)}$. Analogously to the single particle case, the structure of the density matrix implies that the following hyperplane equalities

$$\sum_{x_0,x_1,\ldots,x_m} (\omega_k)^{\sum_i \sigma_i x_i} P\left(a|x_{N-1},...,x_0\right) = 0; \quad a = 0,1,...,l-2, \quad \sigma_i = 0,1,...,k-1 \tag{4.68}$$

are satisfied for all $\vec{\sigma}$ with at least $2P+1$ non-zero components.

4.3 Higher order interference theories

What is it in the nature of quantum theory that imposes the constraints derived in the preceding chapter? What would a theory that violates these constraints look like?
In order to provide an answer to these questions, let us focus on the single particle case and notice that in the derivation of the constraint, the fully specific mathematical form of quantum processes was not necessary: it was sufficient to use the linearity of probabilities with respect to quantum states

$$\sum_{\vec{x}} \omega^{\vec{\sigma}\cdot\vec{x}} \operatorname{Tr}\left[\Pi_r \rho_{\vec{x}}\right] = \operatorname{Tr}\left[\Pi_r \sum_{\vec{x}} \omega^{\vec{\sigma}\cdot\vec{x}} \rho_{\vec{x}}\right], \tag{4.69}$$

the linearity of the unitary encoding of local inputs

$$\sum_{\vec{x}} \omega^{\vec{\sigma}\cdot\vec{x}} \rho_{\vec{x}} = \sum_{\vec{x}} \omega^{\vec{\sigma}\cdot\vec{x}} \sum_{n,m} \rho_{n,m}\left(\vec{x}\right) |n\rangle \langle m| = \sum_{n,m} \sum_{\vec{x}} \omega^{\vec{\sigma}\cdot\vec{x}} \rho_{n,m}\left(\vec{x}\right) |n\rangle \langle m|, \tag{4.70}$$

and the fact that each matrix element $\rho_{n,m}$ depends on at most two inputs, namely *n* and *m*:

$$\rho_{n,m}\left(\vec{x}\right) = \rho_{n,m}\left(x_n, x_m\right). \tag{4.71}$$

A quantum-like theory that violates the first two conditions without violating some reasonable physical principle (e.g. the no-signaling constraint) is almost inconceivable.
On the other hand, the third property is a direct consequence of the specific form of the density matrix which is a (1,1) tensor and thus contains couplings of at most two coefficients of the wave function. Furthermore, since the form of the density matrix is determined by the Born rule on pure states, this property can be seen as a direct consequence of probabilities being obtainable by ”squaring” the wave function.
We will now briefly describe a class of generalized probabilistic theories called higher

order interference theories [29–31]. The idea is to generalize the double slit experiment, which represents a clear demarcation between classical and quantum theory, to multi-slit experiments, which might be used to demarcate quantum theory from other generalized probabilistic theories.

In a double slit experiment, a particle is sent on a plate pierced by two parallel slits. After passing through the plate, the particle can be detected on a spot on the wall. Each of the slits may be open (which we indicate with 1) or closed (which we indicate with 0). The figure of merit is the interference term

$$I_2 = P_{11} - P_{01} - P_{10}, \tag{4.72}$$

where P_{ij} denotes the probability of detecting the particle on a point on the screen given that one slit is in state i and the other in state j. Classical mechanics predicts I_2 to be 0. However, quantum theory allows the particle to be in a spatial superposition and to reinterfere with itself on the screen, which generates $I_2 > 0$.

Now, if we add one more slit, and define a third order interference term as

$$I_3 = P_{111} - P_{110} - P_{101} - P_{011} + P_{100} + P_{010} + P_{001}, \tag{4.73}$$

both quantum and classical theory predict I_3 to be zero. A theory which predicts a non-zero I_3 is said to be part of the class of third-order-interference theories. Analogously, one can define *n-th order interference theories* as those which, in an n-slit experiment, predict a non-zero n-th order term defined as

$$I_n = (-1)^n \sum_{x_1,\dots,x_n=0}^{1} (-1)^{\sum_i x_i} P_{x_1 \dots x_n}. \tag{4.74}$$

Since what separates classical and quantum theory are the terms that couple different 'branches' of the wave function, which arise because of the probability being obtained by squaring the total wave function, a natural way to come up with higher order interference theories is to define quantum-like theories with generalized Born rules or generalized density matrices. One example of such a theory is the theory of *density cubes* [32] which postulates that a state of a system is described by a rank-three tensor, namely a density cube

$$\rho = \sum_{e_1,e_2,e_3} \rho_{e_1,e_2,e_3} \vec{e_1} \otimes \vec{e_2} \otimes \vec{e_3}. \tag{4.75}$$

Since density cubes couple three coefficients of the wave function, when applied to our scenario, they can couple up to three localized inputs. Hence, behaviors arising from communication using one particle obeying the density cube theory, satisfy

$$\sum_{x_0,x_1,...,x_{N-1}} (\omega_k)^{\sum_i \sigma_i x_i} P\left(a|x_{N-1}...x_0\right) = 0; \quad a = 0,1,...,l-2, \quad \sigma_i = 0,1,...,k-1, \tag{4.76}$$

for all $\vec{\sigma}$ with at least *four* non-zero components. The set of behaviors arising from density cube processes is thus strictly larger than the quantum one and spans a higher dimensional subspace of the probability space.

Tentatively, we can generalize the latter example by introducing a class of n-th order interference theories which to each state of a physical system ascribe a rank-n tensor

$$\rho = \sum_{e_1,...,e_n} \rho_{e_1,...,e_n} \vec{e_1} \otimes \vec{e_2} \otimes ... \otimes \vec{e_n}. \tag{4.77}$$

Such a theory predicts the possibility of coupling up to n localized inputs using a single particle, or up to *np* localized inputs using *p* particles, constraining its behaviors to lie in the following hyperplanes

$$\sum_{x_0,x_1,...,x_{N-1}} (\omega_k)^{\sum_i \sigma_i x_i} P\left(a|x_{N-1}...x_0\right) = 0; \quad a = 0,1,...,l-2, \quad \sigma_i = 0,1,...,k-1, \tag{4.78}$$

for all $\vec{\sigma}$ with at least $np+1$ non-zero components. However, the possibility of constructing such a theory without contradicting some natural physical principles is far from obvious.

The peculiar constraints which arose in the previous section are thus a direct manifestation of quantum mechanics being a second-order interference theory. We see that this feature highly reduces the dimensionality of quantum distributions in the space of all logically conceivable distributions. However, we stress once again that these constraints do not necessarily limit the amount of signaling the theory can produce, as has been seen in section 3.7: indeed, in the next section we question the possibility of violating the general inequality presented in section 3.9 using a single quantum particle in spatial superposition and thereby certifying arbitrarily high levels of signaling exhibited by quantum processes with limited resources.

4.4 N-way communication with a single quantum particle

Getting back to quantum theory, we ask ourselves whether it is possible to achieve N-way signaling for arbitrary N using a single quantum particle, despite constraints derived in the previous sections. We will examine this question by focusing on the inequality derived in section 3.9:

$$B \equiv -P(1|0,0,...,0) + P(1|1,0,...,0) + P(1|0,1,...,0) + ... + P(1|0,0,...,1) \leq N-1. \tag{4.79}$$

Any behavior that violates the latter constraint cannot be simulated by $(N-1)$-way communication and thus genuinely exhibits N-way signaling.

The communication process we are going to analyze is analogous to the bipartite case described in section 4.2.1. Again, Alice's resources consist of one particle (without internal degrees of freedom, or simply neglecting them) which she sends in a homogeneous spatial superposition to the N surrounding parties; each of the latter applies a local phase shift depending on their input and sends their part of the particle back to Alice; finally, Alice performs a generic measurement thereby producing a classical outcome.

The initial wave function is:

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_n |n\rangle \tag{4.80}$$

where $\{|n\rangle\}$ is the basis of spatial modes corresponding to the N parties.

After encoding, the wave function is transformed to

$$|\psi\rangle_{\vec{x}} = \frac{1}{\sqrt{N}} \sum_n e^{[i(\phi_n x_n + b_n]} |n\rangle\,, \tag{4.81}$$

where $\{(\phi_1, b_1)\,, ..., (\phi_n, b_n)\}$ are generic numbers which fully specify the encoding procedure.

The final measurement process is described by a general POVM with l elements

$\{\Pi_0, \Pi_1, ..., \Pi_{l-1}\}$.

Let us denote the quantum state that arises via encoding when $\{x_1 = 0, x_2 = 0, ..., x_N = 0\}$ with ρ_0, and the one that arises from encoding when $\{x_1 = 0, x_2 = 0, ..., x_i = 1, ..., x_N = 0\}$ with ρ_i. Then, if we introduce the following averaged state

$$\rho_1 = \frac{1}{N} \sum_i \rho_i, \tag{4.82}$$

the conditional probabilities can be connected to the quantum states using the Born rule

$$\begin{aligned} &P(1|0,0,...,0) = \mathrm{Tr}(\Pi_1 \rho_0), \\ &P(1|1,0,...,0) + P(1|0,1,...,0) + ... + P(1|0,0,...,1) = N\,\mathrm{Tr}(\Pi_1 \rho_1). \end{aligned}$$

Expression (4.79) can then be rewritten as

$$B = -1 + (N+1)\left[\frac{1}{N+1}\,\mathrm{Tr}\big(\bar{\Pi}_1 \rho_0\big) + \frac{N}{N+1}\,\mathrm{Tr}(\Pi_1 \rho_1)\right] \equiv -1 + (N+1) P_W, \tag{4.83}$$

where we introduced the coarse-grained POVM $\left\{\Pi_1, \bar{\Pi}_1 \equiv \sum_{k\neq 1} \Pi_k\right\}$.

The expression $P_W = \left[\frac{1}{N+1}\,\mathrm{Tr}\big(\bar{\Pi}_1 \rho_0\big) + \frac{N}{N+1}\,\mathrm{Tr}(\Pi_1 \rho_1)\right]$ is the probability of successfully distinguishing the quantum states ρ_0 and ρ_1 given their respective prior probabilities $p_0 = \frac{1}{N+1}$ and $p_1 = \frac{N}{N+1}$. It is known [33] that this probability is bounded by

$$\max_{\Pi} P_W = \frac{1}{2}(1 + ||p_1 \rho_1 - p_0 \rho_0||_1), \tag{4.84}$$

where $||..||_1$ denotes the trace norm.

The maximum achievable value of expression (4.79) for a given encoding scheme is then given by

$$\max_{\Pi} B = -1 + \frac{N+1}{2}(1 + ||p_1 \rho_1 - p_0 \rho_0||_1) \equiv N - 1 + \delta, \tag{4.85}$$

where $\delta = \frac{1}{2} - \frac{N}{2} + \frac{N+1}{2}||p_1 \rho_1 - p_0 \rho_0||_1$ is the maximum violation.

According to the described encoding procedure, the density operators of interest are

$$\rho_0 = \frac{1}{N} \sum_{n,m} |n\rangle \langle m|\, e^{i(b_n - b_m)}, \tag{4.86}$$

$$\rho_1 = \frac{1}{N^2}\sum_{n,m} |n\rangle\langle m| e^{i(b_n-b_m)} \sum_k e^{i(\phi_n\delta_{n,k}-\phi_m\delta_{m,k})}$$
$$= \frac{1}{N^2}\sum_{n,m} |n\rangle\langle m| e^{i(b_n-b_m)} \left[N + (1-\delta_{n,m})(e^{i\phi_n} + e^{-i\phi_m} - 2)\right].$$

Since our formula already includes the optimization over measurement operators, we are free to set all the offsets b_j to 0 without loss of generality.

The goal is to calculate the trace norm of the following operator

$$p_1\rho_1 - p_0\rho_0 = \frac{1}{N(N+1)}\sum_{n,m} |n\rangle\langle m| \left\{(N-3) + e^{i\phi_n} + e^{-i\phi_m} + 2\delta_{n,m} - \delta_{n,m}\left(e^{i\phi_n} + e^{-i\phi_m}\right)\right\}. \tag{4.87}$$

The trace norm of an operator M can be expressed succinctly as

$$||M||_1 = \sum_i |\lambda_i|, \tag{4.88}$$

where λ_i are the eigenvalues of the given operator. The problem has thus been reduced to an eigenvalue problem.

Let us further specify the encoded phases by setting half of them equal to an arbitrary phase ϕ and the other half to $-\phi$; more specifically, if $N = 2K$ we set K of them to ϕ and K of them to $-\phi$, while, if $N = 2K+1$, we set $K+1$ of them to ϕ and K of them to $-\phi$. The operator (4.87) is then equal to

$$p_1\rho_1 - p_0\rho_0 = \frac{1}{N+1}\left\{\frac{2}{N}(1-\cos(\phi))\mathbb{1} + (N-3)|\psi_0\rangle\langle\psi_0| + |\phi_0\rangle\langle\psi_0| + |\psi_0\rangle\langle\phi_0|\right\}, \tag{4.89}$$

where we introduced an auxiliary phase vector

$$|\phi\rangle \equiv \frac{1}{\sqrt{N}}\sum_n e^{i\phi_n}|n\rangle. \tag{4.90}$$

Now we define the operator M as

$$M \equiv (N-3)|\psi_0\rangle\langle\psi_0| + |\phi_0\rangle\langle\psi_0| + |\psi_0\rangle\langle\phi_0|, \tag{4.91}$$

and diagonalize it in the two-dimensional subspace on which it has nonvanishing support. Two orthogonal vectors in the given subspace are

$$\begin{aligned} |0\rangle &\equiv |\psi_0\rangle \\ |1\rangle &\equiv \frac{1}{\sqrt{1-|\langle\psi_0|\phi\rangle|^2}}\left(|\phi\rangle - \langle\psi_0|\phi\rangle\,|\psi_0\rangle\right). \end{aligned}$$

We are going to treat the even and odd N cases separately. For even $N = 2K$, the following holds:

$$\begin{aligned} |\psi_0\rangle &= |0\rangle\,, \\ |\phi\rangle &= \cos(\phi)\,|0\rangle + \sin(\phi)\,|1\rangle\,, \end{aligned}$$

where we used

$$\langle\psi_0|\phi\rangle = \frac{1}{N}\frac{N}{2}(e^{i\phi} + e^{-i\phi}) = \cos(\phi). \tag{4.92}$$

Substituting the latter into M we obtain:

$$M = (N - 3 + 2\cos(\phi))\,|0\rangle\langle 0| + \sin(\phi)\,|0\rangle\langle 1| + \sin(\phi)\,|1\rangle\langle 0|\,. \tag{4.93}$$

The eigenvalues are

$$\begin{aligned} \lambda_\pm &= \frac{1}{2}\left[A \pm \sqrt{A^2 + (2\sin(\phi))^2}\right], \\ A &\equiv N - 3 + 2\cos(\phi). \end{aligned}$$

Now we have to return to the full operator (4.89); since the identity matrix is diagonal in any basis, the eigenvalues trivially follow: two of them are equal to

$$\begin{aligned} \mu_+ &= \frac{1}{N+1}\left(\frac{2}{N}(1-\cos\phi) + \lambda_+\right), \\ \mu_- &= \frac{1}{N+1}\left(\frac{2}{N}(1-\cos\phi) + \lambda_-\right), \end{aligned}$$

and $(N-2)$ of them which correspond to eigenvectors orthogonal to our two-dimensional subspace are equal to

$$\mu_i = \frac{1}{N+1}\frac{2}{N}(1-\cos\phi).$$

The trace norm is then

$$||p_1\rho_1 - p_0\rho_0||_1 = \sum_j |\mu_j| = \frac{1}{N+1}((N-2)|\frac{2}{N}(1-\cos(\phi))| + |\frac{2}{N}(1-\cos(\phi)) + \lambda_+|$$
$$+ |\frac{2}{N}(1-\cos(\phi)) + \lambda_-|\,).$$

If $|\frac{2}{N}(1-\cos(\phi))| > |\lambda_-|$ then δ turns out to be 0 and independent of ϕ, so there is no violation of the $(N-1)$-way signaling bound.
On the other hand, if $|\frac{2}{N}(1-\cos(\phi))| < |\lambda_-|$:

$$\delta = \frac{3}{2} - \frac{N}{2} - \frac{2}{N} + \frac{2}{N}\cos(\phi) - \cos(\phi) + \frac{1}{2}\sqrt{A^2 + (2\sin(\phi))^2}. \tag{4.94}$$

Inserting the assumed inequality in the previous expression we get

$$\delta > \frac{3}{2} - \frac{N}{2} - \cos(\phi) + \frac{1}{2}\left[A - \sqrt{A^2 + (2\sin(\phi))^2}\right] + \frac{1}{2}\sqrt{A^2 + (2\sin(\phi))^2} = 0, \tag{4.95}$$

which means that the inequality is violated. Now it only remains to be shown that for any N there exists ϕ such that $|\frac{2}{N}(1-\cos(\phi))| < |\lambda_-|$ is satisfied.
Rearranging and squaring the inequality we obtain

$$\left[\frac{4}{N}(1-\cos(\phi)) + A\right]^2 < A^2 + (2\sin(\phi))^2. \tag{4.96}$$

A few trigonometric manipulations lead to

$$\frac{8}{N^2}\sin^2\left(\frac{\phi}{2}\right)\left\{(N-2)^2\cos(\phi) - N(N-6) - 4\right\} > 0, \tag{4.97}$$

which is satisfied if

$$\cos(\phi) > \frac{N(N-6)+4}{(N-2)^2}. \tag{4.98}$$

This means that for any even N it is possible to find ϕ such that our communication scheme achieves N-way signaling using one particle. In particular, ϕ has to be chosen such that $\frac{N(N-6)+4}{(N-2)^2} < \cos(\phi) < 1$.
The previous analysis holds only for even N; for odd $N = 2K+1$ we get

$$\langle\psi_0|\phi\rangle = \frac{1}{N}\left((N-1)\cos(\phi) + e^{i\phi}\right) = \cos(\phi) + \frac{i}{N}\sin(\phi), \tag{4.99}$$

and

$$M = (N - 3 + 2\cos(\phi)) |0\rangle \langle 0| + \sin(\phi)\sqrt{1 - \frac{1}{N^2}} |0\rangle \langle 1| + \sin(\phi)\sqrt{1 - \frac{1}{N^2}} |1\rangle \langle 0| , \tag{4.100}$$

which is equivalent to the even N case up to the factor $\sqrt{1 - \frac{1}{N^2}}$ in the off-diagonal elements. Following a procedure analogous to the even case, one obtains a clear violation $\delta > 0$ if $|\frac{2}{N}(1 - \cos(\phi))| < |\lambda_-|$, where λ_- is the negative eigenvalue of the operator M. The assumed inequality can be cast in a simpler form using trigonometric relations and can be shown to be equivalent to the condition

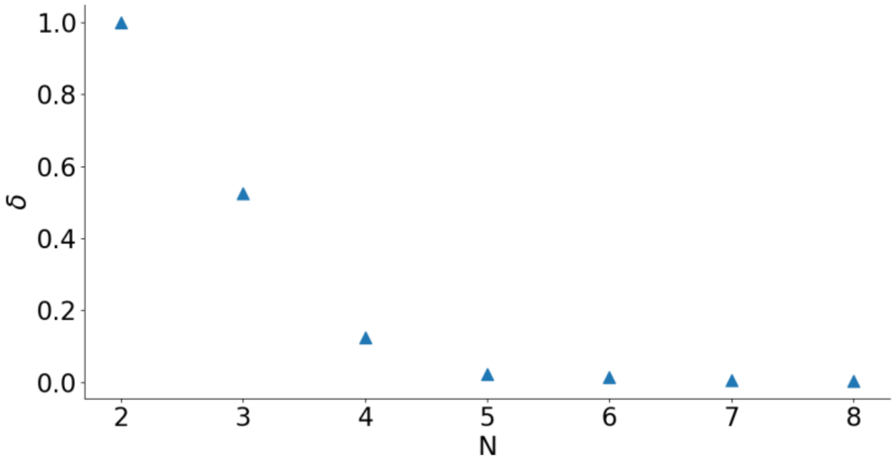
$$\cos(\phi) > \frac{N(N-6)+5}{N^2 - 2N + 3}. \tag{4.101}$$

Therefore, we showed the possibility of signaling with an arbitrary number of parties using a single quantum particle.

We also computed numerically the amount of violation of our inequality for various N: the results are shown in Figure 4.2. As expected, the violation decreases drastically with increasing N; thus, our proof should be seen more as proof of principle (i.e. proof of the theoretical possibility of arbitrarily high levels of signaling) than as a source for practical applications, unless one deals with a fairly low number of parties.

As mentioned in section 4.1, p particles with defined trajectories can achieve at most p-way signaling. Consequently, in order to violate our inequality for a given N, Alice requires at least N classical particles. In this sense, spatial coherence drastically reduces the amount of resources needed for achieving the given task.

Another interesting remark about our scheme is the fact that the quantum particle used by Alice is completely unspecified and no internal degrees of freedom are used, since all the information is encoded in local phases. The particle could thus be a photon of arbitrarily small energy and the local encodings may be implemented by simple phase shifters (e.g. crystals). It still remains open whether internal degrees of freedom could increase the performance: one may expect enhancement coming from entangling internal and spatial degrees of freedom since this might increase the ”correlation” between spatial modes.



N	B_{CL}	B_{QM}
2	1	2.000
3	2	2.524
4	3	3.125
5	4	4.024
6	5	5.014
7	6	6.005
8	7	7.004

Figure 4.2: **Violation of the inequality**. The left graph represents the quantum violation of inequality (3.36) as function of the number of information locations N. The table on the right compares the value of the inequality achievable using classical particles (B_{CL}) and the bound achievable with quantum particles in spatial superposition (B_{QM}).

5 Summary and outlook

5.1 Summary

In this work we touched upon questions regarding the advantages offered by spatial coherence in tasks regarding transfer of information. The line of thought started with a revision of the two-way communication protocol, which shows the possibility of achieving two-way signaling using a single particle in spatial superposition. However, in order to generalize this result, we needed to reformulate the scenario to make it more apt for a formal mathematical analysis. Inspired by the device-independent paradigm which is usually used in the context of Bell's inequalities, we introduced the concept of *m-way signaling behaviors* on a purely probabilistic level, without addressing any underlying physical mechanism. Since the set of m-way signaling behaviors, when embedded in the vector space of probabilities, can be regarded as a polytope, the natural mathematical language we used was convex geometry. Thus, we characterized various polytopes in terms of hyperplane equalities and facet inequalities, thereby providing an operational way to verify the amount of signaling present in the system.

We proceeded by analyzing possible underlying physical processes which may give rise to various levels of signaling. Almost by definition, p classical particles (or objects with defined trajectories) can give rise up to p-way signaling distributions. On the other hand, a single object in spatial superposition can be used to achieve arbitrarily high levels of signaling. The latter was shown by proving the violation of a generic inequality which demarcates $(N-1)$-way signaling behaviors from N-way signaling behaviors. The quantum object had a completely unspecified internal structure, since all the information was encoded in local phases.

While analyzing behaviors arising from quantum processes, we stumbled upon various peculiar constraints, which highly limit the dimensionality of the set of quantum behaviors in the total probability space. The origin of these constraints was traced down to the specific form of the density matrix, i.e. it being a second rank tensor, the latter being a consequence of the quantum mechanical Born rule. This motivated us to analyze a class of generalized probabilistic theories called *higher order interference theories*, which are usually defined with respect to the order of interference they exhibit in multi-slit experiments. We have shown that, if Nature was governed by such theories, the space of possible behaviors would be much less constrained and would have a higher dimensionality in the total space of logically possible behaviors.

5.2 Outlook

In this section, we mention possible further research that could be done along the lines presented in this work.

Causal structures. The characterization of signaling correlations presented in section 3 can be regarded as a purely mathematical classification of conditional probability distributions. In light of this point, a possible application of the latter might be the theory of causal modeling [34]. Briefly speaking, a causal model aims to explain cause-effect relations present in a physical or more general system (e.g. biological, economical, sociological, etc.). Mathematically, a causal structure is a set of classical random variables $\{X_i\}$ and a set of ordered pairs $\{(X_i, X_j)\}$ which indicate that the variable X_i is a cause of X_j. This set of relations can be graphically framed in terms of a directed acyclic diagram (DAG), where the nodes represent the random variables and the directed edges represent causal influences (see Figure 5.1). The latter can be furnished with a set of conditional probabilities $\{P(X_i|Pa(X_i))\}$ for each node, where $Pa(X_i)$ denotes the parents of X_i, i.e. all the random variables that have a direct causal influence on X_i. A causal structure and its pertaining conditional probability distributions define a *causal model*.

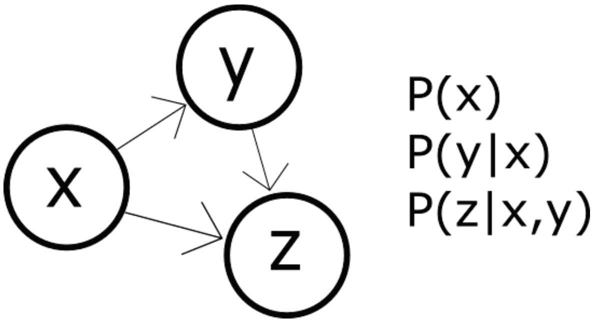


Figure 5.1: Example of a causal model. The causal structure consists of three random variables and is fully specified by its pertaining conditional probability distributions.

Crucially, each node is characterized by a set of conditional probability distributions, which is mathematically identical to the behaviors analyzed in section 3. Therefore, our mathematical results regarding the characterization of signaling correlations can be applied to the classification of causal models. Concretely, suppose we are given a causal model (that someone inferred from experimental data using a causal inference algorithm) and we want to know whether the causal structure can be decomposed as a convex combination of simpler causal structures. The simplest example of such a problem is shown in Figure 5.2. Our results provide sets of necessary and sufficient conditions in terms of various equalities and inequalities for such a decomposition to be possible. The interpretation and the significance of this classification should be further investigated. For instance, one might regard it as verifying whether the random variable pertaining to a node is a genuine function of all of its parents in *each run* of the process, i.e. whether the information is flowing through all the connections in the diagram at once.

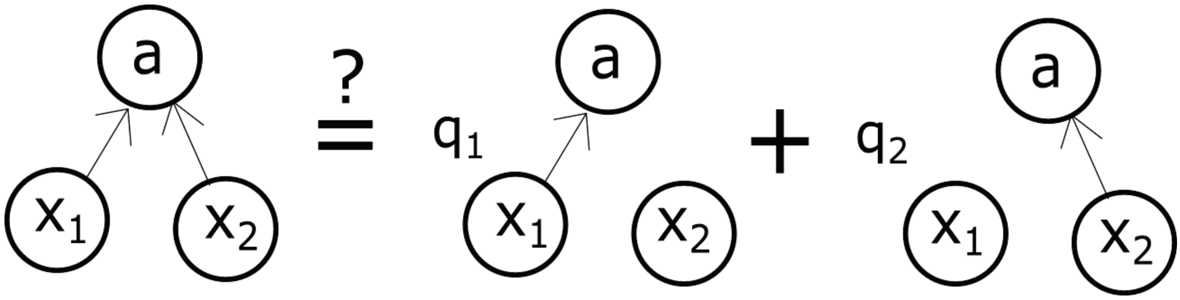


Figure 5.2: Can the causal structure on the LHS be decomposed in a convex combination of simpler causal structures as on the RHS? The answer to this question is provided by the set of equalities and inequalities derived in section 3.

Higher-order interference theories. The connection found between the amounts of signaling with finite resources and higher-order interference theories requires further investigation. An interesting step would be to regard our communication scheme as a generalized version of the multi-slit experiment: the N slits which can be either open or closed would then be replaced by the N parties which implement arbitrary local operations, while the detection of the particle on the screen would be replaced by an arbitrary measurement performed by Alice. Higher-order interference theories are defined with respect to single systems/particles; our generalization might offer a way to define the concept for an arbitrary number of systems.

Acquisition of information globally encoded in space. The mere possibility of achieving arbitrarily high levels of signaling using a single object is a very theoretical statement. In practice, one wants to efficiently transmit useful information. In this sense, one can interpret our results as showing an enhancement of the transmission of information globally encoded in space. More concretely, take the $N = 2$ example: there, the useful information can be identified as the overall parity of the pieces of information held by the two parties that surround Alice. Alice can then retrieve this information using a single particle in superposition twice faster than if she had at disposal only one classical particle. For higher N, the practical applicability is less manifest, since the violation of the inequality drastically decreases with the number of parties. Future work might thus consist in analyzing which multipartite information-theoretic tasks could give rise to serious practical advantages. We suspect such an analysis would be strongly related to generic quantum algorithmic speed-ups.

Communication using single photons. Furthermore, this work offers a novel view on communication using single-photon states, namely Fock states. In experiments, one usually deals with coherent light pulses produced by lasers of limited power. It would thus be interesting to compare the power consumption using a single-photon scheme versus one using coherent light in the context of signaling with several parties. One may find that some useful communication tasks are more efficiently implementable using Fock states (of course, the efficiency of converting coherent pulses into single photons and other potential losses need to be taken into account).

6 Prošireni sažetak

6.1 Uvod

Prošlo je već 100 godina od zasnivanja kvantne mehanike, no i dalje ne razumijemo u potpunosti njenu konceptualnu strukturu te kako ju interpretirati. Njene se posljedice često kose s najosnovnijim konceptima naslijeđenima iz svakodnevnog života kao što su determinizam i lokalnost, što nas navodi na drastičnu alteraciju pogleda na naše iskustvo i na svijet u kojem živimo. Posljedica toga je mnoštvo drastično različitih interpretacija kvantne teorije, od kopenhagenške i bohmske interpretacije do teorije mnoštva svijetova. No, čak i bez potpunog razumijevanja njenog značenja, kvantna mehanika leži u srcu gotovo svih grana moderne fizike, od teorije čestica do teorije čvrstog stanja. Čak je i velik dio današnjeg tehnološkog svijeta zasnovan na kvantnoj teoriji. Taj značajan uspjeh i mnogostrukost primjena nas može udaljiti od propitkivanja temelja i značenja kvantne teorije, budući da prividno sve funkcionira sasvim u redu i bez toga. No, u proteklih trideset godina, situacija se počela drastično mijenjati nastankom kvantne teorije informacija. Potonja pruža čvrstu vezu između naizgled potpuno teoretskih pitanja i pitanja vezanih za praktične prednosti i mane koje nam pruža kvantna fizika. Savršen primjer tog dijaloga je naredno pitanje: mogu li se procesi u Prirodi modelirati lokalnim skrivenim varijablama? Odgovor je negativan, kao što je Bell to pokazao na primjeru prepletenih spinova [1]. S druge strane, potonji rezultat je ključan u kvantnoj kriptografiji [2] i u verifikaciji intrinzične slučajnosti [3]. Drugi primjer ovog fenomena je suvremeni val operativnih rekonstrukcija kvantne teorije baziranima na fizikalno razumnim aksiomima, gdje razni aspekti teorije procesiranja informacija služe kao svojstva koja izdvajaju kvantnu fiziku iz mora drugih generaliziranih teorija informacija [5–8].

U istom duhu, u ovom radu se pitamo koja je razlika između klasičnog i kvantnog svijeta što se tiče komunikacijskih protokola. Koje prednosti i mane pruža kvantna teorija u kontekstu izmjene informacija ograničenim resursima? Je li moguće komunicirati s proizvoljno mnogo igrača koristeći samo jedan kvantni sustav (npr. jedan foton)? Odgovori na ova pitanja, osim što imaju očitu praktičnu primjenu u komunikacijskim tehnologijama, mogu dati uvid u razlike između klasične i kvantne teorije, te između kvantne teorije i hipotetskih generaliziranih teorija.

6.2 *Preliminarni koraci i motivacija*

6.2.1 Dvosmjerna komunikacija jednom kvantnom česticom

Nedavno je pokazana mogućnost dvosmjerne komunikacije jednom kvantnom česticom u prostornoj superpoziciji [9]. Sada ćemo ukratko opisati taj protokol.
Zamislimo dva igrača, koji se zovu Alice (A) i Bob (B) te koji posjeduju svatko po jedan bit informacija (odnosno broj 0 ili 1) koje ćemo označavati s x_A i x_B. Cilj igre je da Alice sazna Bob-ov bit te da Bob sazna Alice-in. Igrači smiju izmijeniti bitove u vremenskom intervalu $t = D/c$, gdje je D udaljenost između A i B te c brzina svjetlosti. Ukoliko A posjeduje samo jedan foton kao resurs za komunikaciju, onda ona može kodirati svoj input na prenosioc informacija i poslati ga Bobu koji u principu može savršeno iščitati Alice-in bit; no, Alice ne može saznati ništa o Bobovoj informaciji. S druge strane, kvantna mehanika omogućava prostornu superpoziciju prenosioca informacija što, kao što ćemo vidjeti, omogućava dvosmjernu komunikaciju.
Alice i Bob sada posjeduju jedan foton u prostornoj superpoziciji

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|A\rangle + |B\rangle\right), \tag{6.1}$$

gdje $|A\rangle$ i $|B\rangle$ predstavljaju stanja fotona 'lokaliziranog' kod Alice i kod Boba. Igrači kodiraju svoje bitove lokalnim fazama $e^{i\pi x_{A/B}}$. Kvantno stanje nakon kodiranja je dakle

$$|\psi\rangle_{x_A,x_B} = \frac{1}{\sqrt{2}} \left(e^{i\pi x_A} |A\rangle + e^{i\pi x_B} |B\rangle\right). \tag{6.2}$$

Nadalje, između A i B nalazi se 50:50 beam splitter koji implementira unitarnu transformaciju

$$H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{6.3}$$

Odmah nakon kodiranja, Alice i Bob šalju svoje dijelove fotona jedno prema drugome; konačno kvantno stanje nakon prolaska kroz beam splitter je

$$|\psi\rangle_{x_A,x_B} = \pm |A\rangle, \quad \text{ako} \quad s \equiv x_A \oplus x_B = 0 \tag{6.4}$$

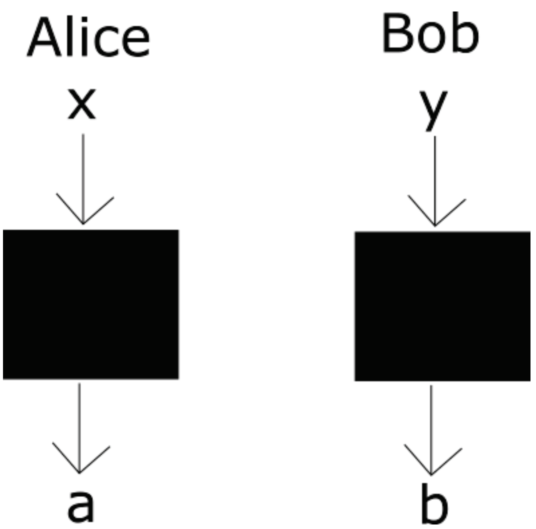
$$|\psi\rangle_{x_A,x_B} = \pm |B\rangle, \quad \text{ako} \quad s \equiv x_A \oplus x_B = 1. \tag{6.5}$$

Dakle, oba igrača mogu pogledati nalazi se foton na njihovoj lokaciji te posljedično saznati paritet njihovih bitova. To im onda omogućava determinističnu izmjenu informacija. Npr., ukoliko je Alice-in bit $x_A = 0$ te ako Alice nađe foton na svojoj lokaciji, to znači da je Bobov bit isto nužno 0, kako bi ukupan paritet bio također 0; analogno, Bob može istovremeno iščitati Alice-in bit.
Je li moguće generalizirati i formalizirati prethodni rezultat na više igrača? Kao što ćemo vidjeti, struktura ovog problema je veoma slična matematičkoj strukturi Bellove nelokalnosti.

6.2.2 Bellova nelokalnost

U ovom poglavlju je ukratko opisan 'device-independent' formalizam te Bellova nelokalnost. Zamislimo situaciju kao na slici 6.3 gdje Alice i Bob ubacuju bitove x i y u nepoznate crne kutije koje potom generiraju outpute a i b. Kutije nisu nužno determističke te su stoga opisane narednim skupom distribucija vjerojatnosti, odnosno 'ponašanjem'

$$\{P\left(ab|xy\right); \quad \forall a, b, x, y \in \{0, 1\}\}. \tag{6.6}$$



Slika 6.3: Alice i Bob odabiru input bitove x i y, a nepoznate crne kutije probabilistički generiraju output bitove a i b.

Modeli lokalnih skrivenih varijabli su matematički modeli kojima se pokušava objasniti dano ponašanje putem naredne dekompozicije

$$P\left(ab|xy\right) = \int_\lambda d\lambda q(\lambda) P\left(a|x, \lambda\right) P\left(b|y, \lambda\right), \tag{6.7}$$

gdje su skrivene varijable λ distribuirane nepoznatom distribucijom vjerojatnosti $q(\lambda)$. Nužan uvjet koji mora biti zadovoljen kako bi ponašanje bilo kompatibilno s modelom lokalnih skrivenih varijabli dan je narednom generalizacijom originalne Bellove nejednakosti, tzv. CHSH nejednakošću [11]

$$S \equiv E_{00} + E_{01} + E_{10} - E_{11} \leq 2, \quad E_{xy} \equiv P\left(a = b|xy\right) - P\left(a \neq b|xy\right). \tag{6.8}$$

Narušenje potonje nejednakosti moguće je ostvariti lokalnim mjerenjima na singlet stanju dvaju spinova. U tom slučaju, crne kutije su implementirane putem zajedničkog kvantnog stanja, inputi su postavke mjernih uređaja, a outputi su rezultati mjerenja. Eksperimentalno je pokazano da se nejednakost narušava neovisno o prostorvremenskoj udaljenosti između mjerenja [14–16]. To povlači falsifikaciju svih modela lokalnih skrivenih varijabli (do na eksperimentalne nedoumice [17]).
Nadalje, možemo shvatiti ponašanje (6.6) kao vektor u 16-dimenzionalnom realnom vektorskom prostoru te identificirati kakav podskup čine distribucije kompatibilne s lokalnim skrivenim varijablama. Jednostavnosti radi, možemo eliminirati neke komponente vektora putem normalizacijskih uvjeta

$$\sum_{a,b} P\left(ab|xy\right) = 1; \forall x, y \in \{0,1\}, \tag{6.9}$$

te putem 'no-signaling' uvjeta (koji osiguravaju nemogućnost slanja informacija brže od svjetlosti [22])

$$\sum_{a} P\left(ab|xy\right) = \sum_{a} P\left(ab|xy'\right); \forall x \in \{0,1\},$$
$$\sum_{b} P\left(ab|xy\right) = \sum_{b} P\left(ab|x'y\right); \forall y \in \{0,1\}.$$

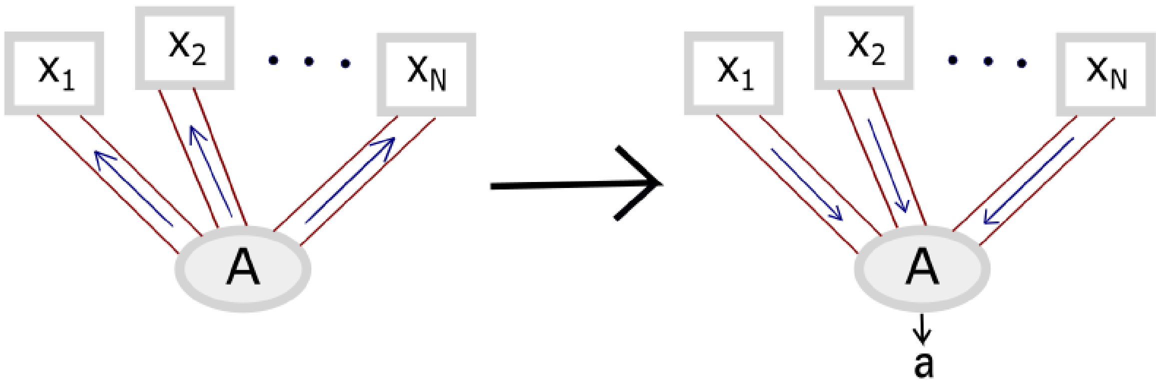
Navedeni uvjeti čine sveukupno 8 linearnih jednadžbi što reducira ukupan prostor na 8 dimenzija. U literaturi [20] je pokazano da skup svih distribucija koje je moguće simulirati lokalnim skrivenim varijablama čini politop, odnosno konveksnu strukturu koju je moguće opisati konačnim brojem verteksa ili konačnim brojem linearnih nejednakosti. Jedna takva nejednakost je upravo CHSH nejednakost (6.8).
U ovom poglavlju ključan je uvid taj da je koncept skrivenih varijabli moguće definirati i u potpunosti analizirati bez ikakvog spomena fizikalnih procesa, odnosno op-

erativno, na razini distribucija vjerojatnosti. Dobivene nejednakosti se također mogu direktno provjeriti eksperimentom bez ikakvog spomena kvantne mehanike. Takav pristup se zove 'device-independent formalizam'.

6.3 *Karakterizacija signalizirajućih korelacija*

6.3.1 Opis scenarija i matematička struktura

U ovom poglavlju uvodimo 'device-independent' scenarij pogodan za formalizaciju i generalizaciju dvosmjerne komunikacije jednom kvantnom česticom.
Zamislimo Alice i N okolnih igrača kao na slici 6.4. Svaki od igrača posjeduje neku informaciju $x_j \in I_k$ gdje I_k predstavlja proizvoljan skup s k elemenata. Svaki od N igrača povezan je s Alice putem N komunikacijskih kanala. Nadalje, Alice posjeduje ograničenu količinu resursa koje može koristiti za komunikaciju, npr. komadić papira, foton ili slično.



Slika 6.4: Alice je povezana s okolnim igračima putem N komunikacijskih kanala. U trenutku $t = 0$ Alice šalje svoje prenosioce informacija nekom podskupu N igrača, dok u trenutku $t = 2\tau$, generira ouput a ovisno o povratnim signalima.

Zanima nas naredni proces: u trenutku $t = 0$ Alice šalje svoje resurse nekom podskupu N igrača; u trenutku $t = \tau$ resursi stižu do igrača koji kodiraju svoje inpute na resurse te ih šalju nazad prema Alice. Konačno, u $t = 2\tau$, Alice prima i dekodira signale, pritom generirajući output $a \in O_l$, gdje O_l predstavlja proizvoljan skup s l elemenata.
Proces je matematički karakteriziran narednim skupom distribucija vjerojatnosti, ili 'ponašanjem'

$$\{P\left(a|x_1, x_2, ..., x_N\right); \quad \forall a \in O_l, x_j \in I_k\}, \tag{6.10}$$

gdje $P(a|x_1, x_2, ..., x_N)$ označava vjerojatnost da Alice u trenutku $t = 2\tau$ generira output a, uz uvjet da okolni igrači posjeduju inpute $\{x_1, x_2, ..., x_N\}$. Nadalje, uvodimo koncept *m-way signaling* ponašanja.

Kažemo da je ponašanje $\{P(a|x_1, x_2, ..., x_N), \quad \forall a, x_i\}$ *m-way signaling* ako i samo ako postoje težine $\{q_{j_1,j_2,...,j_m}, \forall j_1, ..., j_m\}$ i skup distribucija $\{P(a|x_{j_1}, x_{j_2}, ..., x_{j_m}), \forall j_1, ..., j_m\}$ tako da su zadovoljene naredne relacije:

$$
\begin{aligned}
P(a|x_1, x_2, ..., x_N) = \sum_{j_1,j_2,...,j_m} q_{j_1,j_2,...,j_m} P(a|x_{j_1}, x_{j_2}, ..., x_{j_m});\\
\sum_{j_1,j_2,...,j_m} q_{j_1,j_2,...,j_m} = 1;\\
q_{j_1,j_2,...,j_m} \geq 0, \quad \forall j_1, j_2, ..., j_m,
\end{aligned}
$$

gdje se domena indeksa $\{j_1, j_2, ..., j_m\}$ prostire po svih $\binom{N}{m}$ podskupova N okolnih igrača.

Ukoliko je proces opisan m-way signaling ponašanjem, to znači da se dano ponašanje može simulirati komunikacijom s igračima $\{j_1, j_2, ..., j_m\}$ vjerojatnošću $q_{j_1,j_2,...,j_m}$.

Npr., za tri okolna igrača, 2-way signaling distribucija se može izraziti kao

$$
\begin{aligned}
P(a|x_1, x_2, x_3) = q_{12} P(a|x_1, x_2) + q_{13} P(a|x_1, x_3) + q_{23} P(a|x_2, x_3);\\
\sum_{ij} q_{ij} = 1; \quad q_{ij} \geq 0, \forall i < j,
\end{aligned}
\tag{6.11}
$$

gdje je q_{ij} vjerojatnost da Alice komunicira s igračima koji posjeduju informacije x_i i x_j.

Nadalje, distribucije (6.10) mogu se shvatiti kao komponente vektora u D-dimenzionalnom realnom vektorskom prostoru, gdje je $D = lk^N$. Možemo eliminirati k^N komponenti putem normalizacijskih uvjeta

$$
\sum_{a=0}^{L-1} P(a|x_1, x_2, ..., x_N) = 1, \quad \forall x_1, x_2, ..., x_N. \tag{6.12}
$$

Skup svih distribucija koje zadovoljavaju normalizaciju te pozitivnost vjerojatnosti nazivamo 'logičkim politopom' te ga označavamo s L_D. Prostor svih m-way signaling distribucija označavamo sa S_m. Može se pokazati da potonji skup čini politop, što znači da ga je moguće u potpunosti karakterizirati konačnim brojem verteksa, ili alternativno, konačnim brojem linearnih nejednakosti.

6.3.2 Razne jednakosti i nejednakosti

Za početak, fokusirajmo se isključivo na binarne inpute i outpute te na karakterizaciju 1-way signaling ponašanja za proizvoljan N. U glavnom tekstu je pokazano da se S_1 politop u tom slučaju može u potpunosti okarakterizirati putem raznih *jednakosti*, te da su nejednakosti redundantne. Geometrijski, pokazano je da je S_1 politop identičan presjeku logičkog politopa i hiperravnina definiranih narednim jednakostima

$$\sum_{x_0,...x_{N-1}=0}^{1} (-1)^{\sum_j \sigma_j x_j} P\left(0|x_{N-1}...x_0\right) = 0; \quad \sigma_i \in \{0,1\}\,, \quad \sum_j \sigma_j \geq 2. \tag{6.13}$$

Dakle, ukoliko su inputi i outputi binarni, potpun skup nužnih i dovoljnih uvjeta da neko ponašanje bude 1-way signaling dan je gornjim jednakostima.

Karakterizacija S_m skupa za nebinarne inpute i outpute te proizvoljan N je podosta kompliciraniji. U tom slučaju, uspjeli smo naći skup nužnih, ali ne i dovoljnih uvjeta koje dano ponašanje mora zadovoljavati kako bi bilo m-way signaling. Potonje je ponovno dano u obliku hiperravninskih jednakosti

$$\sum_{x_0,x_1,...,x_{N-1}} (e^{i2\pi/k})^{\sum_i \sigma_i x_i} P\left(a|x_{N-1}...x_0\right) = 0; \quad a = 0,1,...,l-2, \quad \sigma_i = 0,1,...,k-1, \tag{6.14}$$

gdje vektor $\vec{\sigma}$ sadrži minimalno m pozitivnih komponenti.

Nužni uvjeti se mogu suplementirati dovoljnim uvjetima u vidu numerički dobivenih nejednakosti za pojedine slučajeve. U glavnom tekstu je u potpunosti analiziran skup S_2 za slučaj $\{N = 3; k = l = 2\}$ te su dobivene razne nejednakosti.

Jedna od tih nejednakosti je nadalje intuitivno generalizirana na scenarij s proizvoljno mnogo igrača te s proizvoljnom kardinalnošću alfabeta inputa i outputa: svaki $(N-1)$-way signaling proces zadovoljava narednu relaciju

$$B \equiv -P(1|0,0,...,0) + P(1|1,0,...,0) + P(1|0,1,...,0) + ... + P(1|0,0,...,1) \leq N-1. \tag{6.15}$$

Narušenje prethodne nejednakosti implicira N-way signaling ponašanje za proizvoljan N.

6.4 *Klasični, kvantni i suprakvantni procesi*

Budući da klasični objekti imaju definirane trajektorije, pomoću jedne klasične čestice moguće je postići najviše 1-way signaling, a pomoću P čestica P-way signaling. S druge strane, kvantna mehanika omogućava prostornu superpoziciju prenosioca informacija.

6.4.1 Ponovni pogled na dvosmjernu komunikaciju

Kao što smo vidjeli u poglavlju 6.2.1, jednom kvantnom česticom moguće je postići dvosmjernu komunikaciju. Za početak, prevedimo potonji rezultat na naš formalizam. Alice posjeduje jedan foton te je okružena dvama igračima koji posjeduju bitove x_1 i x_2. Igrači mogu kodirati svoje bitove lokalnim fazama $(-1)^{x_{1,2}}$. Alice šalje svoj foton u homogenoj superpoziciji dvama igračima; igrači kodiraju svoje lokalne inpute te šalju foton nazad prema Alice. Konačno, Alice prima foton te vrši projektivno mjerenje u bazi

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle \pm |2\rangle\right), \tag{6.16}$$

gdje su $|1,2\rangle$ stanja određenih trajektorija prema prvom i drugom igraču. Statistika rezultata Alice-inih mjerenja je dana sa

$$P(0|x_1,x_2) = \frac{1}{4}\left(1 + (-1)^{x_1+x_2}\right)^2 = \begin{cases} 1, & x_1 \oplus x_2 = 0 \\ 0, & x_1 \oplus x_2 = 1. \end{cases} \tag{6.17}$$

Nadalje, ubacivanjem dobivenog ponašanja u jednakost (6.13) za $N = 2$,

$$\sum_{x_1,x_2} (-1)^{x_1+x_2} P(0|x_1,x_2) = 2 \neq 0, \tag{6.18}$$

vidimo da Alice maksimalno narušava 1-way signaling jednakost. Stoga, pokazali smo da Alice može generirati 2-way signaling ponašanje jednim fotonom.

6.4.2 Generalni kvantni procesi

Nastavljamo analizu kvantnih procesa u generalnom scenariju s proizvoljnim brojem igrača i proizvoljnom kardinalnošću skupova inputa i outputa. Za početak, pretpostavimo da Alice posjeduje jednu česticu s nedefiniranim unutarnjim stupnjevima slobode te da igrači implementiraju svoje informacije lokalnim unitarnim transformacijama. Alice priprema arbitrarno čisto stanje

$$|\psi\rangle = \sum_{n=0}^{N-1} c_n |\phi_n\rangle \otimes |n\rangle\,, \tag{6.19}$$

gdje je $\{|n\rangle\}$ baza definiranih trajektorija prema okolnim igračima, dok $\{|\phi\rangle ...\}$ označava stanja unutarnjih stupnjeva slobode (npr. orijentacije spina). U trenutku $t = \tau$ igrači primaju resurse i kodiraju svoje informacije putem lokalnih unitarnih transformacija

$$|\psi\rangle_{\vec{x}} = \sum_{n=0}^{N-1} c_n \left[U_n(x_n) |\phi_n\rangle\right] \otimes |n\rangle\,. \tag{6.20}$$

Odmah nakon kodiranja, igrači šalju resurse nazad prema Alice koja vrši generično kvantno mjerenje

$$P(a|\vec{x}) = \mathrm{Tr}\left[\Pi_a \rho_{\vec{x}}\right], \tag{6.21}$$

gdje $\{\Pi_a, \forall a = 0, ..., l-1\}$ predstavlja operatore mjerenja s l elemenata, a $\rho_{\vec{x}}$ označava matricu gustoće pridruženu stanju (6.20):

$$\rho_{\vec{x}} = \sum_{n,m=0}^{N-1} c_n c_m^* U_n(x_n) |\phi_n\rangle \langle\phi_m| U_m^\dagger(x_m) \otimes |n\rangle \langle m|\,. \tag{6.22}$$

Nadalje, može se pokazati da sve kvantne distribucije (6.21) leže u hiperravninama definiranima narednim jednakostima

$$\sum_{x_0,x_1,...,x_{N-1}} (e^{i2\pi/k})^{\sum_i \sigma_i x_i} P\left(a|x_{N-1}...x_0\right) = 0; \quad a = 0,1,...,l-2, \quad \sigma_i = 0,1,...,k-1, \tag{6.23}$$

za sve vektore $\vec{\sigma}$ s minimalno 3 pozitivnih komponenti. Posljedično, jednočestične kvantne distribucije su ograničene na veoma mali podskup logički dozvoljenih ponašanja. Analogan rezultat se dobiva generalizacijom na više čestica; naime, sve P-čestične kvantne distribucije zadovoljavaju naredne jednakosti

$$\sum_{x_0,x_1,...,x_m} (e^{i2\pi/k})^{\sum_i \sigma_i x_i} P\left(a|x_{N-1},...,x_0\right) = 0; \quad a = 0,1,...,l-2, \quad \sigma_i = 0,1,...,k-1, \tag{6.24}$$

za sve $\vec{\sigma}$ s minimalno $(2P+1)$ pozitivnih komponenti.

6.4.3 Teorije interferencije višeg reda

Prethodno dobivena ograničenja su posljedica kvantnomehaničkog Bornovog pravila koje implicira da je matrica gustoće tenzor ranga dva. Raznim modifikacijama Bornovog pravila mogu se konstruirati teorije interferencije više reda [29]. Primjer teorije interferencije trećeg reda je 'density cube' teorija [32], koja postulira da se svakom fizikalnom sustavu pridružuje tenzor ranga 3:

$$\rho = \sum_{e_1,e_2,e_3} \rho_{e_1,e_2,e_3} \vec{e_1} \otimes \vec{e_2} \otimes \vec{e_3}. \tag{6.25}$$

Hipotetski jednočestični procesi koji zadovoljavaju 'density cube' teoriju leže u narednim hiperravninama

$$\sum_{x_0,x_1,...,x_{N-1}} (e^{i2\pi/k})^{\sum_i \sigma_i x_i} P\left(a|x_{N-1}...x_0\right) = 0; \quad a = 0,1,...,l-2, \quad \sigma_i = 0,1,...,k-1, \tag{6.26}$$

za sve $\vec{\sigma}$ s minimalno 4 pozitivnih komponenti. Zaključujemo da je razlog iznimno izražene supresije kvantnih distribucija u ukupnom prostoru mogućih ponašanja taj što je kvantna fizika teorija interferencije drugog reda.

6.4.4 N-way signaling jednom kvantnom česticom

Vratimo se sada na pitanje mogućnosti komunikacije s proizvoljno mnogo igrača koristeći jednu kvantnu česticu. Može li se postići N-way signaling za proizvoljan N jednom česticom u prostornoj superpoziciji? Na to pitanje možemo odgovoriti analizom nejednakosti (6.15) u kontekstu kvantnih procesa.

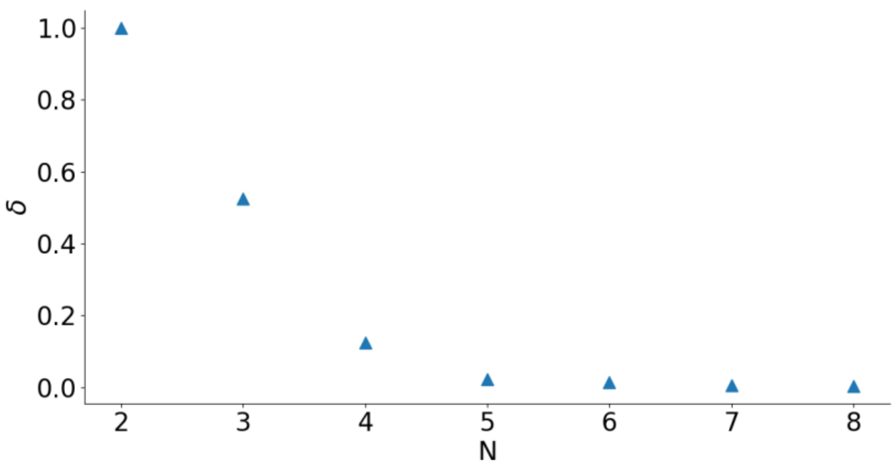
Pretpostavimo da Alice posjeduje jednu česticu bez unutarnje strukture koju može poslati u homogenoj superpoziciji okolnim igračima. Početno kvantno stanje je

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_n |n\rangle \, . \tag{6.27}$$

Igrači kodiraju svoju informaciju lokalnim fazama:

$$|\psi\rangle_{\vec{x}} = \frac{1}{\sqrt{N}} \sum_n e^{[i(\phi_n x_n + b_n]} |n\rangle \, , \tag{6.28}$$

gdje su $\{(\phi_1, b_1), ..., (\phi_n, b_n)\}$ generični brojevi koji u potpunosti specificiraju kodiranje. Konačno mjerenje opisano je generičnim operatorima mjerenja $\{\Pi_0, \Pi_1, ..., \Pi_{l-1}\}$. U glavnom tekstu pokazano je da je uvijek moguće naći specifično kodiranje i mjerenje koje narušava nejednakost (6.15) za proizvoljni N. Narušenje nejednakosti drastično opada s N kao što se vidi na slici 6.5.



N	B_{CL}	B_{QM}
2	1	2.000
3	2	2.524
4	3	3.125
5	4	4.024
6	5	5.014
7	6	6.005
8	7	7.004

Slika 6.5: Lijevi graf predstavlja kvantno narušenje nejednakosti (6.15) kao funkciju broja okolnih igrača N. Tablica uspoređuje vrijednost lijeve strane nejednakosti dobivene klasičnim česticama (B_{CL}) te jednom kvantnom česticom u superpoziciji (B_{QM}).

Dakle, pokazali smo da Alice može postići N-way signaling jednom česticom u prostornoj superpoziciji. S druge strane, potrebno je minimalno N čestica s definiranim trajektorijama kako bi se narušila nejednakost; stoga, prostorna koherencija drastično smanjuje resurse potrebne za komunikaciju s više igrača. Zanimljivo svojstvo našeg protokola je to što nikakva unutarnja struktura čestice nije korištena, budući da je sva informacija zapisana u lokalnim fazama.

6.5 Zaključak

U ovom radu istražili smo ulogu prostorne superpozicije u kontekstu transfera informacija. Priču smo započeli opisom dvosmjerne komunikacije jednom kvantnom česticom te smo se zapitali kako generalizirati i kvantificirati potonji rezultat na više igrača. To nas je navelo na formulaciju scenarija u kojem Alice komunicira s okolnim igračima ograničenom količinom resursa. Nadalje, uveli smo koncept m-way signaling ponašanja u 'device-independent' formalizmu, odnosno na razini distribucija vjerojatnosti, bez spomena fizikalnih procesa. Budući da je skup svih m-way signaling distribucija politop, prirodan jezik koji se prožimao kroz analizu je konveksna geometrija. Dakle, opisali smo razne m-way signaling politope nejednakostima i hiperravninskim jednakostima.
Nadalje, analizirali smo skup ponašanja dobivenih kvantnomehaničkim procesima te vidjeli da ga Bornovo pravilo značajno sužava na veoma mali podskup prostora svih logički dozvoljenih distribucija. Usputno, prokomentirali smo i teorije interferencije višeg reda te pokazali da takve teorije leže u podskupovima znatno većim od onog kvantnog. Konačno, unatoč spomenutim ograničenjima, pokazali smo mogućnost postizanja N-way signaling ponašanja jednom kvantnom česticom u prostornoj superpoziciji, za proizvoljan N. Dokaz se bazira na narušenju linearne nejednakosti koja razdvaja $(N-1)$-way signaling i N-way signaling ponašanja.
Neka od potencijalnih budućih istraživanja baziranih na ovom radu su: klasifikacija kauzalnih modela [34], dublja razumijevanje veze između m-way signaling ponašanja i teorija interferencije višeg reda, prikupljanje informacija globalno kodiranih u prostoru te praktična prednost Fockovih stanja svjetlosti nad koherentnim stanjima u specifičnim komunikacijskim protokolima.

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