UNIVERSITY OF ZAGREB
FACULTY OF SCIENCE
DEPARTMENT OF PHYSICS

Ivana Ćesić

Newton-Cartan Gravity

Master Thesis

Zagreb, 2019

SVEUČILIŠTE U ZAGREBU
PRIRODOSLOVNO-MATEMATIČKI FAKULTET
FIZIČKI ODSJEK

Ivana Ćesić

Newton-Cartanova gravitacija

Diplomski rad

Zagreb, 2019.

UNIVERSITY OF ZAGREB
FACULTY OF SCIENCE
DEPARTMENT OF PHYSICS

INTEGRATED UNDERGRADUATE AND GRADUATE UNIVERSITY PROGRAMME IN PHYSICS

Ivana Ćesić

Master Thesis

Newton-Cartan Gravity

Advisor: dr. sc. Athanasios Chatzistavrakidis

Co-Advisor: dr. sc. Maro Cvitan, Associate Professor

Master Thesis grade: ______________________

Committee: 1. ________________________

2. ________________________

3. ________________________

Master Thesis defence date: ________________

Zagreb, 2019

Newton-Cartan gravitacija

Sažetak

Geometrijskim reformulacijama nerelativističke gravitacije, poput Newton-Cartanove, narasla je popularnost zbog njihovih primjena u holografiji i fizici čvrstog stanja. Ovaj diplomski rad bavi se konstrukcijom Newton-Cartanove gravitacije baždarenjem Bargmannove algebre, koja je centralno proširena Galileieva algebra. Da bismo to postigli, prvo se prisjećamo vielbein formalizma opće teorije relativnosti te kako se baždarenjem Poincaréove algebre dobije navedena formulacije van ljuske. Ista procedura koristi se za nerelativistički slučaj te daje pravila transformacije svih baždarnih polja i pripadnih zakrivljenosti.

Ključne riječi: gravitacija, Newton-Cartan, tenzori, baždarenje, Poincaré algebra, Bargmann algebra

Newton-Cartan Gravity

Abstract

Geometrical reformulations of non-relativistic gravity, such as Newton-Cartan, have seen a surge in popularity due to their applications in holography and condensed matter physics. In this thesis, we discuss the construction of Newton-Cartan gravity through a gauging procedure of the Bargmann algebra, which is the central extension of the Galilei algebra. In order to achieve this goal, we recall the vielbein formalism of general relativity and how a gauging of the Poincare algebra results in its off shell formulation. The procedure is then generalized in the non-relativistic case and it yields the transformation rules for all fields and the corresponding curvatures.

Keywords: gravity, Newton-Cartan, tensors, gauging, Poincaré algebra, Bargmann algebra

Sadržaj

1 Introduction

Gravity is omnipresent and the most common force we experience, and as such has perplexed even the earliest scientists. Aristotle, a Greek philosopher, attempted to explain the effects of gravity by claiming that objects were drawn from one place to another because they inherently belong there. He also believed that the speed at which objects fall is proportional to their weight, which was proven false by Galileo Galilei's famous Pisa experiment that showed that the gravitational acceleration is the same for all objects. The first man to actually give a mathematical description of gravity was Sir Isaac Newton, who hypothesized the inverse-square law of universal gravitation. Together with his laws of motion, it formed a system of gravity and motion that would remain uncontested for over two centuries.

A major step to our understanding of gravity was made when Albert Einstein formulated the general theory of relativity, reconciling Newton's laws of gravity and special relativity. With it, he provided a unified and coordinate independent description of gravity as a geometric property of spacetime. Spacetime is no longer a static arena on which physics takes place, but a dynamical background whose dynamics depend on its content.

The question arises whether there exists a coordinate independent description of Newton's gravity. This problem was solved by the French mathematician Cartan, and the theory is consequently named Newton-Cartan gravity and it is the primary subject of this thesis. However, the motivation to study this geometrical reformulation of non-relativistic gravity is not purely academic, as interest in non-relativistic gravity is rekindled in recent years due to its useful applications. One of them is holography or gauge/gravity duality that allows us to establish a connection between a theory with gravity and a gauge theory without gravity in one dimension less. The most famous example of this is AdS/CFT correspondence [1] which allows a study of strongly coupled theories in terms of its weakly coupled dual and has also found applications in condensed matter physics. Since the physical systems under study may often be non-relativistic, a non-relativistic theory of gravity, such as Newton-Cartan or an extension thereof, is used as the gravity dual [2] [3] [4] [5] [6] [7]. Newton-Cartan gravity is also used in studying thermal transport in a resistive medium. It has been argued [8] that a varying gravitational field produces energy flows and temperature fluctuations just like a varying external electric potential produces electric currents and density variations. In case of working with a non-relativistic system, one would couple matter to non-relativistic gravity, and Newton-Cartan gravity has been proven to be a good formulation for this use. It has been used to construct effective field theories for the quantum Hall effect [9] [10] and chiral superfluids [11].

To work our way towards Newton-Cartan gravity, we will first go over the basics of differential geometry in the second chapter. In the third chapter we introduce the vielbein formalism and show how to obtain Einstein's gravity by gauging the Poincaré algebra.

2 Gravity and differential geometry

2.1 From Newtonian Gravity to General Relativity

Newton's law of gravitation states that every object in the universe attracts every other object in the universe with a force proportional to the mass of each object and inversely proportional to the square of the distance between them. The mathematical expression for universal gravitation in vector form is

$$\mathbf{F}_{21} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^2}\hat{\mathbf{r}}_{12}, \tag{2.1}$$

where $\mathbf{F}_{21}$ is the attracting gravitational force applied on object 2 and exerted by object 1, m_1 and m_2 are the objects' masses, $|\mathbf{r}_{12}| = |\mathbf{r}_2 - \mathbf{r}_1|$ is the distance between them, $\hat{\mathbf{r}}_{12} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$ is the unit vector from object 1 to 2, and $G = 6.674\times10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is the gravitational constant.

The gravitational field

$$\mathbf{g}(\mathbf{r}) = -G\frac{m_1}{|\mathbf{r}|^2}\hat{\mathbf{r}} \tag{2.2}$$

is a conservative field so it can be written as a gradient of a scalar potential

$$\mathbf{g} = -\nabla\phi. \tag{2.3}$$

Substituting this into differential form of Gauss' law for gravity

$$\nabla\cdot\mathbf{g} = -4\pi G\rho, \tag{2.4}$$

where ρ is mass density, gives Poisson's equation for gravity,

$$\nabla^2\phi = 4\pi G\rho. \tag{2.5}$$

Newton's laws, however, do not hold in all frames of reference, as they are stated only for inertial frames. Galilean transformation is used in Newtonian mechanics for the transformation between inertial frames which differ by constant relative velocity. On the other hand, Einstein's theory of special relativity, which also postulates the equivalence of all inertial frames, also postulates that the speed of light in free space is invariant. Because of that, the transformation between inertial frames in special relativity is the Lorentz transformation.

The guiding principle for generalizing special relativity to include gravity was the equivalence principle. Let's define a gravitational test particle as a test particle which experiences a gravitational field but does not itself alter the field or contribute to the field. The strong form of the principle of equivalence states that the motion of a gravitational test particle in a gravitational field is independent of its mass and composition. This is the embodiment of the empirical result of the Pisa experiments into a principle. The weak form of the principle of equivalence states that the gravitational field is coupled to everything.

The principle of special relativity states that all inertial observers are equivalent. Arguing that all observers, inertial or not, should be capable of discovering the laws of physics, Einstein proposed the principle that all observers are equivalent. This is known as the principle of general relativity.

If any observer can discover the laws of physics, it means that any coordinate system can be used. This is different from special relativity, where the metric is flat and a canonical or preferred coordinate system exists, specifically Minkowski coordinates. In a curved spacetime, i.e. a manifold with a non-flat metric, there is no canonical coordinate system. However, in some situations, there are preferred coordinate systems in a way that they yield simpler descriptions of what's going on. When a problem possesses a symmetry, it is best to adapt the coordinate system to the underlying symmetry. But the theory should be invariant under a coordinate transformation, which leads to the principle of general covariance which states that the equations of physics should have tensorial form.

2.2 Tensors

A tensor is an object defined on a geometric entry called a differential manifold, which is a manifold that is locally similar enough to a linear space to allow one to do calculus. We will simply take an n-dimensional manifold M to be a set of points such that each point possesses a set of n coordinates $(x^1, x^2, \ldots, x^n)$. A coordinate system that covers only a portion of a manifold is called a coordinate patch, and a point in a manifold can be covered by many different coordinate patches. Consider the passive change of coordinates $x^a \to x'^a$ given by n equations

$$x'^a = f^a(x^1, x^2, \ldots, x^n) \; (a = 1, 2, \ldots, n), \tag{2.6}$$

where the f's are single-valued continuous differentiable functions, at certain ranges of their arguments. The equation can be written more simply as

$$x'^a = x'^a(x), \tag{2.7}$$

where $x'^a(x)$ denote the n functions $f^a(x)$.

Differentiating the above equation with respect to each of the coordinates produces the $n \times n$ transformation matrix of coefficients:

$$\left[\frac{\partial x'^a}{\partial x^b}\right] = \begin{bmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \cdots & \frac{\partial x'^1}{\partial x^n} \\ \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \cdots & \frac{\partial x'^2}{\partial x^n} \\ \vdots & & & \\ \frac{\partial x'^n}{\partial x^1} & \frac{\partial x'^n}{\partial x^2} & \cdots & \frac{\partial x'^n}{\partial x^n} \end{bmatrix}. \tag{2.8}$$

The determinant of this matrix is called the Jacobian of the transformation:

$$J' = \left|\frac{\partial x'^a}{\partial x^b}\right|. \tag{2.9}$$

The total differential is

$$\mathrm{d}x'^a = \frac{\partial x'^a}{\partial x^b}\mathrm{d}x^b. \tag{2.10}$$

A contravariant vector or a contravariant tensor of rank 1 is a set of quantities, written X^a in the x^a-coordinate system, associated with a point P, which transforms under a change of coordinates according to

$$X'^a = \frac{\partial x'^a}{\partial x^b}X^b, \tag{2.11}$$

where the transformation matrix is evaluated at P.

A contravariant tensor of rank 2 is a set n^2 quantities associated with a point P, denoted by X^{ab} in the x^a-coordinate system, which transforms according to

$$X'^{ab} = \frac{\partial x'^a}{\partial x^c}\frac{\partial x'^b}{\partial x^d}X^{cd}. \tag{2.12}$$

The definitions of third- and higher-order contravariant tensors are analogous.

Tensor of zero rank is called a scalar or a scalar invariant, and it transforms according to

$$\phi' = \phi \tag{2.13}$$

at P.

A covariant vector or covariant tensor of rank 1 is a set of quantities, written X_a in the x^a-coordinate system, associated with a point P, which transforms under a change of coordinates according to

$$X'_a = \frac{\partial x^b}{\partial x'^a} X_b. \tag{2.14}$$

Again, covariant tensors of higher ranks are defined similarly.

Mixed tensors are tensors that are neither strictly covariant nor strictly contravariant. For example, a mixed tensor of rank 2, with one contravariant rank and two covariant ranks, transforms according to

$$X'^a_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} X^d_{ef}. \tag{2.15}$$

A mixed tensor with contravariant rank p and covariant rank q is said to have type or valence (p, q).

The importance of tensors in mathematical physics lies in the fact that tensor equations that hold in one coordinate system, necessarily hold in all coordinate systems.

A tensor field defined over some region of the manifold is an association of a tensor of the same valence to every point of the region

$$P \to T^{a\cdots}_{b\cdots}(P). \tag{2.16}$$

The transformation of a contravariant vector field is given by

$$X'^a(x') = \left[\frac{\partial x'^a}{\partial x^b}\right] X^b(x) \tag{2.17}$$

at each point P in the region.

We introduce notation

$$\partial_a \equiv \frac{\partial}{\partial x^a} \tag{2.18}$$

and define X as the operator

$$X = X^a \partial_a \tag{2.19}$$

so that

$$Xf = (X^a \partial_a)f = X^a(\partial_a f) \tag{2.20}$$

for any real-valued function f. It can be shown that operating on f by X will be the same irrespective of the coordinate system

$$X'^a \partial'_a = X^a \partial_a. \tag{2.21}$$

Partial differentiation of tensors is not tensorial because it does not transform like a tensor:

$$\partial'_c X'^a = \frac{\partial x'^a}{\partial x^b} \frac{\partial x^d}{\partial x'^c} \partial_d X^b + \frac{\partial^2 x'^a}{\partial x^b \partial x^d} \frac{\partial x^d}{\partial x'^c} X^b. \tag{2.22}$$

The core reason for this behavior is that the differentiation involves comparing a quantity evaluated at two neighbouring points P and Q, in our case

$$\lim_{\delta u \to 0} \frac{[X^a]_P - [X^a]_Q}{\delta u} \tag{2.23}$$

for some parameter δu. When transforming the tensors X^a_P and X^a_Q to another coordinate system, each of them are evaluated at different points, from which it should be clear that $X^a_P - X^a_Q$ is not a tensor.

2.2.1 Lie derivative

We define a congruence of curves as a set of curves such that only one curve goes through each point in the manifold. Given any one curve of the congruence,

$$x^a = x^a(u), \tag{2.24}$$

the tangent vector field $\mathrm{d}x^a/\mathrm{d}u$ along the curve can be defined.

Conversely, a congruence of curves can be obtained from a non-zero vector field $X^a(x)$ defined on the manifold by solving the ordinary differential equations

$$\frac{\mathrm{d}x^a}{\mathrm{d}u} = X^a(x(u)). \tag{2.25}$$

Solution is guaranteed by the existence and uniqueness theorem for ordinary differential equations.

Suppose that X^a has been given and a local congruence of curves obtained from it. The idea behind the Lie derivative of some tensor $T^{a\cdots}_{b\cdots}(x)$ is to use the congruence of curves to drag the tensor at some point P, $T^{a\cdots}_{b\cdots}(P)$, along the curve passing through P to some neighbouring point Q, and compare this tensor with the tensor already there, $T^{a\cdots}_{b\cdots}(Q)$. The derivative will be defined by subtracting the two tensors at Q in the limiting process as Q tends to P.

The Lie derivative of a general tensor field $T^{a\cdots}_{b\cdots}$ is given by [19]:

$$\mathrm{L}_X T^{a\cdots}_{b\cdots} = X^c \partial_c T^{a\cdots}_{b\cdots} - T^{c\cdots}_{b\cdots} \partial_c X^a - \cdots + T^{a\cdots}_{c\cdots} \partial_b X^c + \cdots \,. \tag{2.26}$$

Some important properties of Lie differentiation are:

- It is linear.
- It is Leibniz:
$$\mathrm{L}_X (Y^a Z_{bc}) = Y^a (\mathrm{L}_X Z_{bc}) + (\mathrm{L}_X Y^a) Z_{bc} \tag{2.27}$$
- It is type preserving. The Lie derivative of a tensor of type (p,q) is again a tensor of type (p,q).
- It commutes with contraction, for example
$$\delta^b_a \mathrm{L}_X T^a_b = \mathrm{L}_X T^a_a \tag{2.28}$$
- The Lie derivative of a scalar field ϕ is given by
$$\mathrm{L}_X \phi = X\phi = X^a \partial_a \phi \tag{2.29}$$
- The Lie derivative of a contravariant vector field Y^a is given by
$$\mathrm{L}_X Y^a = X^b \partial_b Y^a - Y^b \partial_b X^a \tag{2.30}$$

- The Lie derivative of a covariant vector field Y_a is given by

$$\mathrm{L}_X Y_a = X^b \partial_b Y_a + Y_b \partial_a X^b \tag{2.31}$$

2.2.2 Covariant differentiation

Covariant derivative of a contravariant vector field $X^a(x)$ is defined by the limiting process

$$\nabla_c X^a = \lim_{\delta x^c \to 0} \frac{1}{\delta x^c} \{X^a(x + \delta x) - [X^a(x) + \bar{\delta} X^a(x)]\} \tag{2.32}$$

where $\bar{\delta} X^a(x)$ denotes a difference vector between $X^a(x)$ at point P and a vector at point Q, with coordinates $x^a + \delta x^a$, that is in some general sense 'parallel' to $X^a(x)$ at P. It is natural to require that $\bar{\delta} X^a(x)$ vanishes whenever $X^a(x)$ or δx^a does, which means that there exist multiplicative factors Γ^a_{bc} where

$$\bar{\delta} X^a(x) = -\Gamma^a_{bc}(x) X^b(x) \delta x^c, \tag{2.33}$$

which, along with Taylor's theorem

$$X^a(x + \delta x) = X^a(x) + \delta x^b \partial_b X^a, \tag{2.34}$$

gives

$$\nabla_c X^a = \partial_c X^a + \Gamma^a_{bc} X^b. \tag{2.35}$$

If we require that $\nabla_c X^a$ is a tensor of type (1, 1), then Γ^a_{bc} must transform according to

$$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \Gamma^d_{ef} - \frac{\partial x^d}{\partial x^b} \frac{\partial x^e}{\partial x^c} \frac{\partial^2 x^a}{\partial x^d \partial x^e}. \tag{2.36}$$

The transformation law shows that Γ^a_{bc} is not a tensor because of the presence of the second term. Any quantity that transforms according to 2.36 is called an affine connection, or simply connection or affinity. A manifold with a continuous connection prescribed on it is called an affine manifold.

The covariant derivative of a scalar field is defined to be the same as its ordinary derivative

$$\nabla_c \phi = \partial_c \phi. \tag{2.37}$$

The covariant derivative of a covariant vector is given by

$$\nabla_c X_a = \partial_x X_a - \Gamma^b_{ac} X_b. \tag{2.38}$$

The name covariant derivative originates from the fact that the derivative of a tensor of type $(p,\ q)$ has one extra covariant rank, and is of type $(p,\ q+1)$. The covariant derivative of a general tensor is

$$\nabla_c T^{a\cdots}_{b\cdots} = \partial_c T^{a\cdots}_{b\cdots} + \Gamma^a_{dc} T^{d\cdots}_{b\cdots} + \cdots - \Gamma^d_{bc} T^{a\cdots}_{d\cdots} - \cdots\ . \tag{2.39}$$

It is obvious from the transformation law that the sum of two connections is not a connection or a tensor. However, in the difference of two connections the inhomogeneous term cancels out and the result is a tensor of valence (1, 2). The anti-symmetric part of Γ^a_{bc},

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} \tag{2.40}$$

is a tensor called the torsion tensor. The connection is symmetric if the torsion tensor vanishes.

2.2.3 Affine geodesics

We introduce notation

$$\nabla_X T^{a\cdots}_{b\cdots} = X^c \nabla_c T^{a\cdots}_{b\cdots} \tag{2.41}$$

for any tensor $T^{a\cdots}_{b\cdots}$. Then we define the absolute derivative of a tensor along a curve C of the local congruence determined by the vector X, by

$$\frac{\mathrm{D}}{\mathrm{D}u}\{T^{a\cdots}_{b\cdots}\} = \nabla_X T^{a\cdots}_{b\cdots}. \tag{2.42}$$

The tensor $T^{a\cdots}_{b\cdots}$ is parallely propagated or transported along curve C if

$$\frac{\mathrm{D}}{\mathrm{D}u}\{T^{a\cdots}_{b\cdots}\} = 0. \tag{2.43}$$

Affine geodesic is defined as a privileged curve along which the tangent vector is propagated parallel to itself:

$$\nabla_X X^a = \lambda X^a, \tag{2.44}$$

or equivalently

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}u^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}u}\frac{\mathrm{d}x^c}{\mathrm{d}u} = \lambda\frac{\mathrm{d}x^a}{\mathrm{d}u} \tag{2.45}$$

If λ vanishes, the tangent vector is transported into itself, and the geodesic equation is reduced to

$$\nabla_X X^a = 0. \tag{2.46}$$

2.2.4 The Riemann tensor

Covariant differentiation is not in general commutative. For any tensor $T^{a\cdots}_{b\cdots}$, we define its commutator as

$$\nabla_c\nabla_d T^{a\cdots}_{b\cdots} - \nabla_d\nabla_c T^{a\cdots}_{b\cdots}. \tag{2.47}$$

In case of a vector X^a and its covariant derivative

$$\nabla_c X^a = \partial_c X^a + \Gamma^a_{bc}X^b, \tag{2.48}$$

we obtain the result

$$\nabla_c\nabla_d X^a - \nabla_d\nabla_c X^a = R^a{}_{bcd}X^b + (\Gamma^e_{cd} - \Gamma^e_{dc})\nabla_e X^a, \tag{2.49}$$

where $R^a{}_{bcd}$ is defined by

$$R^a{}_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \tag{2.50}$$

If the connection is symmetric (torsion-free), the last term in 2.49 vanishes and we obtain

$$2\left(\nabla_c\nabla_d X^a - \nabla_d\nabla_c X^a\right) = \nabla_{[c}\nabla_{d]}X^a = \frac{1}{2}R^a{}_{bcd}X^b. \tag{2.51}$$

Because the left hand side of 2.62 is a tensor, it follows that $R^a{}_{bcd}$ is a tensor of type (1, 3). This tensor is called the Riemann tensor. For a symmetric connection, the commutator of any tensor can be expressed in terms of the tensor itself and the Riemann tensor, and consequently, the vanishing of the Riemann tensor is a necessary and sufficient condition for the vanishing of the commutator of any tensor.

2.2.5 Geodesic coordinates

At any point P in a manifold, a geodesic coordinate system can be introduced in which

$$[\Gamma^a_{bc}]_P = 0. \tag{2.52}$$

This can easily be proven by choosing P to be at the origin of coordinates $x^a = 0$ and considering a transformation to a new coordinate system

$$x^a \to x'^a = x^a + \frac{1}{2} Q^a_{bc} x^b x^c, \tag{2.53}$$

where $Q^a_{bc} = Q^a_{cb}$ are constants that can be chosen to be

$$Q^a_{bc} = [\Gamma^a_{bc}]_P \tag{2.54}$$

in order to obtain the result.

It can be shown that a coordinate system can be obtained in which the connection vanishes along a curve, but not in general over the whole manifold. If there exists a coordinate system in which the connection vanishes everywhere, the manifold is called affine flat or simply flat.

2.2.6 Affine flatness

Parallely transporting a vector in a general affine manifold from one point to another along two different curves yields two different vectors. However, if the transported vector is independent of the path taken, the connection is called integrable.

2.2.7 The metric

Any symmetric covariant tensor field of rank 2, for example $g_{ab}(x)$, defines a metric, and a manifold endowed with a metric is called a Riemann manifold. The line element is given by

$$\mathrm{d}s^2 = g_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b, \tag{2.55}$$

and the square of the length or norm of a contravariant vector X^a is defined by

$$X^2 = g_{ab}(x)X^aX^b. \tag{2.56}$$

The metric is positive definite or negative definite if for all vectors X, $X^2 > 0$ or $X^2 < 0$, respectively. It is indefinite otherwise.

Vectors X^a and Y^a are orthogonal if

$$g_{ab}X^aY^b = 0. \tag{2.57}$$

If the metric is indefinite, then there exist vectors which are orthogonal to themselves called null vectors:

$$g_{ab}X^aX^b = 0. \tag{2.58}$$

If the metric is non-singular ($g \neq 0$), the inverse of g_{ab}, g^{ab}, is given by

$$g_{ab}g^{bc} = \delta^c_a. \tag{2.59}$$

The metric g_{ab} and the contravariant metric g^{ab} can be used to lower and raise tensorial indices by defining

$$T^{\cdots\ \cdots}_{\cdots a \cdots} = g_{ab}T^{\cdots b \cdots}_{\cdots\ \cdots} \tag{2.60}$$

and

$$T^{\cdots a \cdots}_{\cdots\ \cdots} = g^{ab}T^{\cdots\ \cdots}_{\cdots b \cdots} \tag{2.61}$$

Since indices can be freely raised and lowered, the order in which the contravariant and covariant indices are written is important. In general, $X_a{}^b$ is different from $X^b{}_a$.

2.2.8 Metric flatness

At any point P, g_{ab} is a symmetric matrix of real numbers, so there exists a transformation which reduces the matrix to a diagonal form whose terms are either -1 or +1. The prevalence of plus signs over negative signs is called the signature of the metric. The signature is invariant under the assumption that that the metric is continuous over the manifold and non-singular. If there exists a coordinate system in which the metric is reduced to this diagonal form everywhere, then the metric is called flat. The necessary and sufficient condition for a metric to be flat is that its Riemann tensor vanishes.

2.2.9 The curvature tensor

The curvature tensor or Riemann tensor is given by

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \tag{2.62}$$

and Γ^a_{bc} is the Levi-Civita connection, given by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}). \tag{2.63}$$

From the definition it is easy to deduct that $R^a{}_{bcd}$ is anti-symmetric on its last pair of indices

$$R^a{}_{bcd} = -R^a{}_{bdc}, \tag{2.64}$$

which, together with the fact that the connection is symmetric, leads to identity

$$R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} \equiv 0. \tag{2.65}$$

By using geodesic coordinates, it can be easily shown that the Riemann tensor with lowered indices is symmetric under interchange of the first and last pair of indices

$$R_{abcd} = R_{cdab}, \tag{2.66}$$

which, after combining with (2.64), gives that the lowered tensor is anti-symmetric on its first pair of indices, too:

$$R_{abcd} = -R_{bacd}. \tag{2.67}$$

It can also be shown that the curvature tensor satisfies so-called Bianchi identities:

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec}. \tag{2.68}$$

The Ricci tensor is given by contraction

$$R_{ab} = R^c_{acb}, \tag{2.69}$$

and a second contraction gives the curvature scalar or Ricci scalar

$$R = g^{ab} R_{ab}. \tag{2.70}$$

These are used to define the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \tag{2.71}$$

which satisfies the contracted Bianchi identities

$$\nabla_b G_a^{\ b} = 0. \tag{2.72}$$

Einstein's equation that relate local spacetime curvature with the local energy and momentum within the spacetime is given by

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}, \tag{2.73}$$

where Λ is the cosmological constant which today we know is positive and has value of $1.1056 \times 10^{-52}\mathrm{m}^{-2}$, $\kappa = 8\pi G c^{-4}$ is the Einstein constant and T_{ab} is the stress-energy tensor. It is acquired by minimizing the Hilbert action

$$S = \frac{1}{2\kappa} \int R \sqrt{-\det g_{ab}} \mathrm{d}^4 x. \tag{2.74}$$

3 The vielbein and gauging of the Poincaré algebra

3.1 Vielbein formulation

Consider first a classical case of a particle in a constant gravitational field

$$m_I \frac{\mathrm{d}^2 \mathbf{r}}{\mathrm{d}t^2} = m_G \mathbf{g}, \tag{3.1}$$

where m_I is the inertial mass of the particle, $\vec{r}(t)$ its position vector, m_G its gravitational mass and $\vec{g}$ the acceleration due to the external constant gravitational field. Assuming that the inertial and gravitational masses are the same ($m_I = m_G = m$), we get

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\mathbf{r}(t)-\frac{1}{2}\mathbf{g}t^2\right]=0, \tag{3.2}$$

which means that when viewed from a freely falling frame

$$\mathbf{r}(t)\to\mathbf{r}'(t)=\mathbf{r}(t)-\frac{1}{2}\mathbf{g}t^2 \tag{3.3}$$

the particle experiences no gravity.

Gravitational fields are generally not constant. To account for this, Einstein postulated that at each point in spacetime gravitational fields can be transformed away by choosing an appropriate set of coordinates. Hence to include gravity, one should take any local quantity and identify the coordinates with the "freely falling" coordinates. The interaction with gravity will be generated when expressed in arbitrary coordinates. For clarity, Latin tensor indices $m, n, p, q, \cdots$ will be used in the freely falling frame and Greek tensor indices $\mu, \nu, \rho, \sigma, \cdots$ in arbitrary coordinates.

Take for an example a self-interacting scalar field $\phi(x)$. Its behavior, in the absence of gravity, is described by the action

$$S[\phi]=\int \mathrm{d}^4x\mathcal{L}=\int \mathrm{d}^4x\left[\frac{1}{2}\partial_\mu\phi\partial^\nu\phi-V(\phi)\right]. \tag{3.4}$$

The derivative operators are given by

$$\partial_\mu=\frac{\partial}{\partial x^\mu}=\left(\frac{\partial}{\partial t},\nabla\right), \tag{3.5}$$

$x^\mu=(t,\vec{x})$ are the coordinates and $V(\phi)$ is the potential density. In the action, local quantities are the Lagrangian $\mathcal{L}$ and the volume element $\mathrm{d}x^4=\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$.

To include gravity, we have to interpret the variable x^μ and its derivative as being the "free fall" coordinates, so we identify

$$\{x^\mu\}\to\{\xi^m\},\quad m=0,1,2,3 \tag{3.6}$$

as the "flat" coordinates. In the flat system, the line element is

$$\mathrm{d}s^2=\eta_{mn}\mathrm{d}\xi^m\mathrm{d}\xi^n, \tag{3.7}$$

where η_{mn} is the metric of Special Relativity: $\eta_{00} = -\eta_{ii} = -1$, $\eta_{ij} = 0$ for $i \neq j$. It satisfies

$$\eta_{mn}\eta^{np} = \delta^p_m. \tag{3.8}$$

The new Lagrangian is now

$$\mathcal{L} \to \frac{1}{2}\eta^{mn}\partial_m\phi\partial_n\phi - V(\phi), \tag{3.9}$$

where $\partial_m \equiv \partial/\partial\xi^m$, and the volume element becomes the volume in terms of the flat coordinates

$$\mathrm{d}^4x \to \mathrm{d}\xi^0\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3. \tag{3.10}$$

We can now write the new action that is generalized to include the effects of gravitation [12]:

$$S[\phi] = \int \mathrm{d}^4\xi \left[\frac{1}{2}\eta^{mn}\partial_m\phi\partial_n\phi - V(\phi)\right]. \tag{3.11}$$

Although this looks like the action we started with, the difference lies in the integration, as this is to be integrated over a manifold which is labeled by the arbitrary coordinate system $\{x^\mu\}$. The information about the gravitational field is contained in the change of the flat coordinates from point to point.

We can express ξ^m as a local function of any non-inertial coordinates x^μ:

$$\mathrm{d}\xi^m = \frac{\partial\xi^m}{\partial x^\mu}\mathrm{d}x^\mu, \tag{3.12}$$

where the derivatives are evaluated at the point of interest. The transformation matrix between the flat and arbitrary coordinates is called the *vielbein*

$$e^m_\mu(x) \equiv \frac{\partial\xi^m}{\partial x^\mu}. \tag{3.13}$$

The inverse operation is

$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial\xi^m}\mathrm{d}\xi^m \equiv e^\mu_m\mathrm{d}\xi^m. \tag{3.14}$$

The vielbein satisfies the following equations:

$$e^m_\mu e^\mu_n = \delta^m_n \tag{3.15}$$

$$e^\mu_m e^m_\nu = \delta^\mu_\nu. \tag{3.16}$$

The transformation of the derivative operators are given by

$$\frac{\partial}{\partial \xi^m} = \frac{\partial x^\mu}{\partial \xi^m} \frac{\partial}{\partial x^\mu} = e^\mu_m \partial_\mu. \tag{3.17}$$

The Lagrangian rewritten in an arbitrary system becomes

$$\mathcal{L} = \frac{1}{2} \eta^{mn} e^\mu_m e^\nu_n \partial_\mu \phi \partial_\nu \phi - V(\phi) \tag{3.18}$$

from which we can identify the inverse metric

$$g^{\mu\nu}(x) = \eta^{mn} e^\mu_m(x) e^\nu_n(x). \tag{3.19}$$

The new metric appears in the line element in an arbitrary system

$$\mathrm{d}s^2 = \eta_{mn} e^m_\mu(x) e^n_\nu(x) \mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{3.20}$$

hence it's defined as

$$g_{\mu\nu}(x) = \eta_{mn} e^m_\mu(x) e^n_\nu(x). \tag{3.21}$$

It is easy to prove that

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho. \tag{3.22}$$

The volume element in an arbitrary system is

$$\mathrm{d}^4\xi = J(\xi, x) \mathrm{d}^4 x, \tag{3.23}$$

where it is easy to show that $J(\xi, x) = \sqrt{-\det g_{\mu\nu}}$ or $J(\xi, x) = \det e^m_\mu$.

Finally, the action for a scalar field in a gravitational field is given by

$$S = \int \mathrm{d}^4 x \sqrt{-\det g_{\mu\nu}} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \tag{3.24}$$

It is important to note that, unlike the derivatives ∂_μ, the derivatives ∂_m obey a non trivial algebra:

$$[\partial_m, \partial_n] = [\partial_m e^\mu_n - \partial_n e^\mu_m] e^p_\mu \partial_p. \tag{3.25}$$

3.2 Gauge theory

Symmetries in physics are described by groups G. Here we will focus on Lie groups that describe continuous symmetries. The elements $g \in G$ are then generated by a Lie algebra $\mathfrak{g}$. If T_A are the elements of $\mathfrak{g}$, where $A = \{1 \ldots N\}$ for some N, then $\mathfrak{g} = \mathrm{span}\{T_A\}$ and a general group element g is given by

$$g = e^{\theta^A T_A}, \tag{3.26}$$

where $\{\theta^A\}$ are parameters that can be real or complex, depending on the algebra. One of the group axioms is closure: if $g_1 \in G$ and $g_2 \in G$, then $g_1 g_2 \in G$. The group multiplication structure is given by the Lie bracket:

$$[T_A, T_B] = f_{AB}^{C} T_C. \tag{3.27}$$

The structure constants f_{AB}^{C} of the algebra $\mathfrak{g}$ are evidently antisymmetric in $\{AB\}$.

In gauge theory, we promote a global symmetry on a set of fields $\{\phi\}$ to a local symmetry, which brings up gauge fields $B_\mu{}^A$ on which the Lie algebra $\mathfrak{g}$ is realized. These gauge fields usually come from the kinetic terms of the fields $\{\phi\}$ because the kinetic terms are not invariant under the local transformations and need compensation. If the fields $\{\phi\}$ transform as

$$\delta_\epsilon \phi = \epsilon^A T_A \phi, \tag{3.28}$$

where ϵ^A is the transformation parameter, we can replace the ordinary derivative $\partial_\mu \phi$ in the kinetic terms by the covariant derivative

$$D_\mu \phi = \partial_\mu \phi - B_\mu{}^A T_A \phi, \tag{3.29}$$

which transforms just like the field:

$$\delta_\epsilon D_\mu \phi = \epsilon^A T_A D_\mu \phi, \tag{3.30}$$

and the gauge fields transform as

$$\delta_\epsilon B_\mu{}^A = \partial_\mu \epsilon^A + \epsilon^B B_\mu^C f_{BC}^A, \tag{3.31}$$

where summation over $\{BC\}$ is understood, so that the transformation 3.30 holds.

Gauge fields are mathematically connections, and the curvature of a gauge field is also called field strength:

$$R_{\mu\nu}{}^A = 2\partial_{[\mu} B_{\nu]}{}^A + B_\mu{}^B B_\nu{}^C f_{BC}{}^A, \tag{3.32}$$

and it transforms in a covariant way, without a derivative on the transformation parameter ϵ^A,

$$\delta_\epsilon R_{\mu\nu}{}^A = \epsilon^B R_{\mu\nu}{}^C f_{BC}^A. \tag{3.33}$$

The gauge fields carry both a spacetime index μ and an internal index A, which means that they transform under general coordinate transformations and the gauge transformations. The following relation holds [14]:

$$\delta_{gct}(\xi^\lambda) B_\mu^A + \xi^\lambda R_{\mu\lambda}^A - \sum_{\{C\}} \delta(\xi^\lambda B_\lambda^C) B_\mu^A = 0. \tag{3.34}$$

The gauge parameters in this relation are constructed out of the gauge fields $B_\mu{}^A$ and the parameter ξ^λ of the general coordinate transformation.

3.3 Gauging of the Poincaré algebra

The goal of this section is to show how to obtain the basic ingredients of Einstein gravity by gauging the Poincaré algebra. The commutation relations of a D-dimensional Poincaré algebra $\mathfrak{iso}(D-1,1)$ are

$$[P_a, P_b] = 0, \tag{3.35}$$

$$[M_{bc}, P_a] = -2\eta_{a[b}P_{c]}, \tag{3.36}$$

$$[M_{cd}, M_{ef}] = 4\eta_{[c[e}M_{f]d]}, \tag{3.37}$$

where P_a, M_{ab} $(a = 0, 1, \ldots, D-1)$ are the generators for translations and Lorentz transformations, respectively. To the local P-transformations we associate a gauge field ${e_\mu}^a$ with spacetime dependent parameters $\xi^a(x)$, and to the local Lorentz transformations we associate a gauge field ${\omega_\mu}^{ab}$ with spacetime dependent parameters $\lambda^{ab}(x)$. Firstly we choose a connection that takes values in the adjoint of the gauge group:

$$\mathcal{A}_\mu = {e_\mu}^a P_a + \frac{1}{2}{\omega_\mu}^{ab} M_{ab}. \tag{3.38}$$

The gauge transformation of $\mathcal{A}_\mu$ is given by

$$\delta\mathcal{A}_\mu = \partial_\mu \zeta + [\zeta, \mathcal{A}_\mu], \tag{3.39}$$

where ζ is the gauge parameter:

$$\zeta = \zeta^a P_a + \frac{1}{2}\lambda^{ab} M_{ab} \tag{3.40}$$

Using this along with the commutation relations of the algebra, we get the following transformation rules:

$$\delta {e_\mu}^a = \partial_\mu \zeta^a - {\omega_\mu}^{ab}\zeta^b + \lambda^{ab} {e_\mu}^b, \tag{3.41}$$

$$\delta {\omega_\mu}^{ab} = \partial_\mu \lambda^{ab} + 2\lambda^{c[a}{\omega_\mu}^{b]c}. \tag{3.42}$$

The curvatures are given by:

$$R_{\mu\nu}{}^a(P) = 2\partial_{[\mu} {e_{\nu]}}^a - 2{\omega_{[\mu}}^{ab} {e_{\nu]}}^b, \tag{3.43}$$

$$R_{\mu\nu}{}^{ab}(M) = 2\partial_{[\mu} {\omega_{\nu]}}^{ab} - 2{\omega_{[\mu}}^{ac} {\omega_{\nu]}}^{cb}. \tag{3.44}$$

We want to interpret the gauge fields ${e_\mu}^a$ and ${\omega_\mu}^{ab}$ as the vielbein and the spin-connection. Consider the general identity for a gauge algebra

$$0 = \delta_{gct}(\xi^\lambda) B_\mu^A + \xi^\lambda R_{\mu\lambda}^A - \sum_{\{C\}} \delta(\xi^\lambda B_\lambda^C) B_\mu^A. \tag{3.45}$$

The index A labels the gauge fields and the corresponding curvatures. We set $A = a$ for the P-transformations and choose the parameter ξ^λ to be $\xi^\lambda = {e_a}^\lambda \zeta^a$ and obtain

$$\delta_P(\zeta^b){e_\mu}^a = \delta_{gct}(\xi^\lambda){e_\mu}^a + \xi^\lambda R_{\mu\nu}{}^a(P) - \delta_M(\xi^\lambda {\omega_\lambda}^{ab}){e_\mu}^a. \tag{3.46}$$

The difference between a P-transformation and a general coordinate transformation is a curvature term and a Lorentz transformation. By imposing a constraint

$$R_{\mu\nu}{}^{a}(P) = 0, \tag{3.47}$$

the P-transformation of the vielbein, which is the only field that transforms under the P-transformations, can be replaced by a general coordinate transformation plus other symmetries of the algebra. This constraint allows us to solve for the Lorentz gauge field $\omega_\mu{}^{ab}$ in terms of the vielbein, its derivatives and the inverse. Writing

$$R_{\mu\nu}{}^{a} e_\rho{}^{a} + R_{\rho\mu}{}^{a} e_\nu{}^{a} - R_{\nu\rho}{}^{a} e_\mu{}^{a} = 0, \tag{3.48}$$

we get

$$\omega_\mu{}^{ab}(e, \partial e) = -2e^{\lambda[a}\partial_{[\mu} e_{\lambda]}{}^{b]} + e_\mu{}^{c} e^{\lambda a} e^{\rho b} \partial_{[\lambda} e_{\rho]}{}^{c}. \tag{3.49}$$

Now we are left with vielbein $e_\mu{}^{a}$ as the only independent field transforming under the local Lorentz transformations and general coordinate transformations and with $\omega_\mu{}^{ab}$ as the dependent spin connection field.

Next, we impose the vielbein postulate

$$\nabla_\mu e_\nu{}^{a} = \partial_\mu e_\nu{}^{a} - \Gamma^{\rho}_{\nu\mu} e_\rho{}^{a} - \omega_\mu{}^{ab} e_\nu{}^{b} = 0. \tag{3.50}$$

From the anti-symmetric part of the equation, together with the curvature constraint, we deduce that the anti-symmetric part of the Γ-connection is zero, i.e. there is no torsion. We can also solve 3.50 for the Γ-connection in terms of the vielbein and its inverse:

$$\Gamma^{\rho}_{\nu\mu} = e^{\rho}{}_{a} D_\mu e_\nu{}^{a}, \tag{3.51}$$

where D_μ is the Lorentz-covariant derivative. Lastly, a non-degenerate metric and its inverse are defined as

$$g_{\mu\nu} = e_\mu{}^{a} e_\nu{}^{b} \eta_{ab}, \qquad g^{\mu\nu} = e_a{}^{\mu} e_b{}^{\nu} \eta^{ab}. \tag{3.52}$$

The basic ingredients of off-shell Einstein gravity and the Poincaré algebra are an independent non-degenerate metric $g_{\mu\nu}$ and a dependent Γ-connection, or an independent vielbein $e_\mu^{\ a}$ and a dependent spin-connection $\omega_\mu^{\ ab}$ in the presence of flat indices.

4 Newton-Cartan Gravity

Newton-Cartan gravity is a geometric reformulation of Newtonian gravity. Consider the classical equations for the trajectories of massive particles, in Galilean space coordinates and the universal time t:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \frac{\partial \phi}{\partial x^i} = 0, \tag{4.1}$$

where ϕ is the Newtonian potential. Instead of interpreting these equations as describing the "curved paths" $x^i(t)$ along which particles fall in Euclidean space, we have to think of those trajectories as geodesics $[t(\lambda), x^i(\lambda)]$ in curved spacetime.

The Newtonian clocks always read the universal time or some multiple $\lambda = at + b$ of it, and the equation 4.1 can be rewritten as

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} = 0, \qquad \frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \frac{\partial \phi}{\partial x^i}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 = 0 \tag{4.2}$$

If we compare it to the geodesic equation

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha_{\ \beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} = 0, \tag{4.3}$$

we can identify the only non-vanishing connection coefficients as

$$\Gamma^i_{\ 00} = \frac{\partial \phi}{\partial x^i}. \tag{4.4}$$

Using 2.62 we see that the only non-vanishing components of the Riemann tensor are

$$R^i_{\ 0k0} = -R^i_{\ 00k} = \frac{\partial^2 \phi}{\partial x^i \partial x^k}. \tag{4.5}$$

The only non-vanishing coefficient of the Ricci tensor is

$$R_{00} = \sum_i \frac{\partial^2 \phi}{\partial x^{i2}}. \tag{4.6}$$

With it, we can rewrite the Poisson equation

$$\nabla^2\phi = 4\pi G\rho \tag{4.7}$$

in the geometric form

$$R_{00} = 4\pi G\rho. \tag{4.8}$$

By writing the equation 4.4 for $\Gamma^{\alpha}{}_{\beta\gamma}$, equation 4.5 for $R^{\alpha}{}_{\beta\gamma\delta}$, equation 4.8 for $R_{\alpha\beta}$ and the law of geodesic motion 4.3, we have effectively rewritten the full content of Newtonian gravity in geometric language.

A way to understand fitting the absolute space and the absolute time into Cartan's "Newtonian spacetime" is *stratification* [15]. We have to regard time as a function, or a scalar field, defined once and for all in Newtonian spacetime as its intrinsic property. Then the slices of constant t define the layers of spacetime and all of these "space slices" have an identical geometric structure: the old absolute space. The space slices are also endowed with a three-dimensional metric and an orthonormal Galilean coordinate basis. So what is the point of curvature and geodesic equation if we are dealing with flat space slices? These are actually properties of *spacetime*. Parallel transport of a vector around a closed curve that lies on a single space slice would actually return it to its starting point unchanged. However, transporting it in time by Δt, in space by Δx^k, back in time by $-\Delta t$ and back in space by $-\Delta x^k$ to its starting point, would actually change the vector by

$$\delta A^j = -R^j{}_{00k}A^0(\Delta t)(\Delta x^k) = \frac{\partial^2\Phi}{\partial x^j \partial x^k}A^0(\Delta t)(\Delta x^k). \tag{4.9}$$

4.1 Newton-Cartan as the non-relativistic limit

Consider the relativistic Minkowski metric and its inverse

$$\eta_{\mu\nu}/c^2 = \begin{bmatrix} -1 & 0 \\ 0 & \mathbb{I}_3/c^2 \end{bmatrix}, \qquad \eta^{\mu\nu} = \begin{bmatrix} -1/c^2 & 0 \\ 0 & \mathbb{I}_3 \end{bmatrix}. \tag{4.10}$$

The limit $c \to \inf$ leads to a degenerate covariant temporal metric $\tau_{\mu\nu}$ with three zero eigenvalues and a degenerate contravariant spatial metric $h^{\mu\nu}$ with one zero eigenvalue. Because $\tau_{\mu\nu}$ is effectively a 1×1 matrix, we can use its vielbein version

τ_μ defined by $\tau_{\mu\nu} = \tau_\mu \tau_\nu$. The degeneracy implies that

$$h^{\mu\nu}\tau_\nu = 0. \tag{4.11}$$

In order to introduce a symmetric connection $\Gamma^\rho_{\mu\nu}$ that depends on these metrics, we first impose metric compatibility:

$$\nabla_\rho h^{\mu\nu} = 0, \qquad \nabla_\rho \tau_\mu = 0, \tag{4.12}$$

where the covariant derivative is with respect to a connection $\Gamma^\rho_{\mu\nu}$. The second condition implies that

$$\tau_\mu = \partial_\mu f(x^\nu) \tag{4.13}$$

where $f(x^\nu)$ is a scalar function. In Newton-Cartan theory it is chosen to be the absolute time t

$$f(x^\nu) = t. \tag{4.14}$$

Our goal here is to write down the connection in terms of the metrics and their derivatives. First of all, the connection $\Gamma^\rho_{\mu\nu}$ is not uniquely determined by the metric compatibility conditions 4.12, as the conditions are preserved by the shift

$$\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} + h^{\rho\lambda}K_{\lambda(\mu}\tau_{\nu)} \tag{4.15}$$

for an arbitrary two-form (antisymmetric tensor with two indices) $K_{\mu\nu}$. We also need to introduce the spatial inverse metric $h_{\mu\nu}$ and the temporal inverse vielbein τ^μ defined by the following properties:

$$\begin{gathered} h^{\mu\nu}h_{\nu\rho} = \delta^\mu_\rho - \tau^\mu\tau_\rho, \qquad \tau^\mu\tau_\mu = 1, \\ h^{\mu\nu}t_\nu = 0, \qquad h_{\mu\nu}\tau^\nu = 0. \end{gathered} \tag{4.16}$$

With these conditions, we can prove that $\nabla_\rho h_{\mu\nu}$ does not vanish in general. By deriving covariantly the last relation in 4.16 and multiplying with τ_σ, we get

$$\nabla_\rho h_{\mu\nu}\tau^\nu\tau_\sigma + h_{\mu\nu}\nabla_\rho\tau^\nu\tau_\sigma = 0. \tag{4.17}$$

The rest of this proof is just a matter of manipulation using the relations in $\nabla_\rho h_{\mu\nu}$.

The final result is

$$\nabla_\rho h_{\mu\nu} = -2\tau_{(\mu} H_{\nu)\sigma} \nabla_\rho \tau^\sigma. \tag{4.18}$$

The most general connection that satisfies the metric compatibility condition is

$$\Gamma^\sigma_{\mu\nu} = \tau^\sigma \partial_{(\mu}\tau_{\nu)} + \frac{1}{2} h^{\sigma\rho} \left(\partial_\nu h_{\rho\mu} + \partial_\mu h_{\rho\nu} - \partial_\rho h_{\mu\nu}\right) + h^{\sigma\lambda} K_{\lambda(\mu}\tau_{\nu)}, \tag{4.19}$$

which can be proven by inserting it into equations 4.12.

In order to reproduce Newtonian gravity with the metric connection above, we will need to impose some extra conditions. From now on, we will use adapted coordinates $x^0 = t$. The conditions 4.16 imply

$$\begin{aligned} \tau_\mu = \delta^0_\mu, \qquad \tau^\mu = (1, \tau^i), \\ h^{\mu 0} = 0, \qquad h_{\mu 0} = -h_{\mu i}\tau^i. \end{aligned} \tag{4.20}$$

These conditions are preserved by the coordinate transformations

$$\begin{aligned} &x^0 \to x^0 + \xi^0, \\ &x^i \to x^i + \xi^i(x^\mu), \end{aligned} \tag{4.21}$$

where ξ^0 is a constant and the spatial transformation generated with $\xi^i(x^\mu)$ is invertible. The connection coefficients 4.19 rewritten using the adapted coordinates are

$$\begin{aligned} \Gamma^i_{00} &= h^{ij}\left(\partial_0 h_{j0} - \frac{1}{2}\partial_j h_{00} + K_{j0}\right) \equiv h^{ij}\Phi_j, \\ \Gamma^i_{0j} &= h^{ik}\left(\frac{1}{2}\partial_0 h_{jk} + \partial_{[j} h_{k]0} - \frac{1}{2}K_{jk}\right) \equiv h^{ik}\left(\frac{1}{2}\partial_0 h_{jk} + \omega_{jk}\right), \\ \Gamma^i_{jk} &= \frac{1}{2} h^{il}(\partial_k h_{lj} + \partial_j h_{lk} - \partial_l h_{jk}), \\ \Gamma^0_{\mu\nu} &= 0. \end{aligned} \tag{4.22}$$

We will now replace the equation of motion 4.8 with the following covariant ansatz:

$$R_{\mu\nu} = 4\pi G \rho \tau_\mu \tau_\nu \tag{4.23}$$

and show that this leads to Newtonian gravity. In adapted coordinates this means that

$$R_{ij} = R_{i0} = 0. \tag{4.24}$$

The fact that R_{ij} is zero implies that the spatial hypersurfaces are flat and a coordinate frame can be chosen with $\Gamma^i_{jk} = 0$ so that the spatial metric is given by

$$h_{ij} = \delta_{ij}, \quad h^{ij} = \delta^{ij}. \tag{4.25}$$

This implies

$$\begin{aligned} \Gamma^i_{0j} &= h^{ik}\omega_{jk} \leftrightarrow \omega_{ij} = h_{k[j}\Gamma^k_{i]0}, \\ \Gamma^i_{00} &= h^{ij}\Phi_j \leftrightarrow \Phi_i = h_{ij}\Gamma^j_{00}, \end{aligned} \tag{4.26}$$

where we made sure that ω_{ij} is antisymmetric as it should be per definition in 4.22. The choice of a flat metric reduces the allowed coordinate transformations 4.21 to

$$x^0 \to x^0 + \xi^0, \quad x^i \to A^i{}_j(t)x^j + a^i(t), \tag{4.27}$$

where $A^i{}_j$ is an element of $\mathrm{SO}(3)$.

We must impose two additional conditions in order to derive the Poisson equation from the ansatz 4.23, the first of which is the Trautman condition:

$$h^{\sigma[\lambda}R^{\mu]}{}_{(\nu\rho)\sigma} = 0. \tag{4.28}$$

Solving for $\nu = 0, \rho = 0$ and using the relations given in 4.26 gives

$$\partial_0\omega_{mi} - \partial_{[m}\Phi_{i]} = 0, \tag{4.29}$$

while solving for $\nu = 0, \rho = i$ gives

$$\partial_{[k}\omega_{mi]} = 0. \tag{4.30}$$

Although Φ_i and ω_{ij} are not tensors, equations 4.29 and 4.30 are covariant under transformation 4.27. Using the definitions for Φ_i and ω_{ij} in 4.22, we can rewrite the two equations above as

$$\partial_{[\rho}K_{\mu\nu]} = 0, \tag{4.31}$$

which indicates that

$$K_{\mu\nu} = 2\partial_{[\mu} m_{\nu]}, \tag{4.32}$$

where m_μ is a vector field determined up to the derivative of some scalar field.

The second condition we need to impose is that ω_{ij} depends only on time and not on space coordinates. There are three possible conditions called Ehlers conditions that lead to the desired restriction on ω_{ij}:

$$\begin{aligned} & h^{\rho\lambda} R^{\mu}{}_{\nu\rho\sigma} R^{\nu}{}_{\mu\lambda\alpha} = 0 \\ \text{or} \quad & \tau_{[\lambda} R^{\mu}{}_{\nu]\rho\sigma} = 0 \\ \text{or} \quad & h^{\sigma[\lambda} R^{\mu]}{}_{\nu\rho\sigma} = 0. \end{aligned} \tag{4.33}$$

Each of these conditions leads to $\partial_k \omega_{ij} = 0$ in adapted coordinates, which leaves ω_{ij} to be a function of time only $\omega_{ij} = \omega_{ij}(t)$. For example, consider the last possible condition in 4.33. Keeping in mind that $h^{00} = 0$, and $R^{0}{}_{\mu\nu\rho} = 0$ because $\Gamma^{0}_{\mu\nu} = 0$, we can rewrite the condition as

$$h^{i[j} R^{k]}{}_{\nu\rho i} = 0. \tag{4.34}$$

Solving for $\nu = 0, \rho = m$, which means that $R^{k}{}_{0pi} = \partial_p \Gamma^{k}_{0p} - \partial_i \Gamma^{k}_{0p}$, and using 4.26, with renamed indices we get

$$\partial_j \omega_{ki} - \partial_k \omega_{ji} - \partial_j \omega_{ik} + \partial_i \omega_{jk} = 0. \tag{4.35}$$

The first three terms are equal to zero according to 4.30, which leads the desired result $\partial_i \omega_{jk} = 0$. By a time-dependent rotation $x'^i = A^i{}_j(t) x^j$ to a new coordinate system, we can set $\omega_{ij} = 0$, which also implies that $\partial'_{[i} \Phi'_{j]} = 0$ from 4.29. It follows that $\Phi'_i = \partial'_i \phi$ for some scalar field ϕ, and

$$\Gamma'^{i}_{00} = \delta^{ij} \partial'_j \phi \tag{4.36}$$

in the new coordinate system. Finally, the covariant ansatz 4.23 gives the Poisson equation

$$R_{00} = \partial_i \Gamma^{i}_{00} = \delta^{ij} \partial_i \partial_j \phi = 4\pi G \rho. \tag{4.37}$$

Equations of motion can be recovered from the geodesic equation 4.3 by using adapted coordinates and performing the time-dependent rotation above:

$$\ddot{x}'^0(t) = 0, \qquad \ddot{x}'^i(t) + \partial'^i \phi = 0. \tag{4.38}$$

With this we have finished showing how the equations of Newtonian Gravity can be obtained from the Newton-Cartan gravity formulated in terms of two degenerate metrics and supplied with the Trautman and the Ehlers conditions.

5 Gauging the Bargmann Algebra

5.1 The Bargmann algebra

The Bargmann algebra is the Galilean algebra centrally extended with the mass generator M, such that the new generator commutes with everything and hence lies in the center. It can be obtained by extending the Poincaré algebra $\mathfrak{iso}(D-1,1)$ to the direct sum of the Poincaré algebra and a commutative subalgebra $\mathfrak{g}_M$ spanned by M:

$$\mathfrak{iso}(D-1,1) \to \mathfrak{iso}(D-1,1) \oplus \mathfrak{g}_M. \tag{5.1}$$

We contract the algebra

$$P_0 \to \frac{1}{\omega^2} M + H, \quad P_i \to \frac{1}{\omega} P_i, \quad J_{i0} \to \frac{1}{\omega^2} G_i, \quad \omega \to 0, \tag{5.2}$$

where the contraction of P_0 is motivated by the non-relativistic approximation of P_0 for a massive free particle

$$P_0 = \sqrt{c^2 P_i P^i + M^2 c^4} \approx M c^2 + \frac{P_i P^i}{2M}, \tag{5.3}$$

where c is the speed of light. The resulting algebra is called Bargmann algebra $\mathfrak{b}(D-1,1)$ and its non-vanishing commutation relations are

$$\begin{aligned} [J_{ij}, J_{kl}] &= 4\delta_{[i[k} J_{l]j]}, \quad [J_{ij}, P_k] = -2\delta_{k[i} P_{j]}, \\ [J_{ij}, G_k] &= -2\delta_{k[i} G_{j]}, \quad [G_i, H] = -P_i, \\ [G_i, P_j] &= -\delta_{ij} M. \end{aligned} \tag{5.4}$$

5.2 Gauging the Bargmann algebra

The procedure is the same as the one we used for gauging the Poincare algebra. We associate a gauge field $e_\mu{}^i$ and a parameter ξ^i to the generator of spatial translations P_i, $e_\mu{}^0 = \tau_\mu$ and $\xi^0 = \tau$ to the generator of time translations H, $\omega_\mu{}^{ij}$ and λ^{ij} to the generator of rotations J_{ij}, $\omega_\mu{}^{i0}$ and λ^{i0} to the generator of Galilean boosts G_i, and finally m_μ and σ to the generator M.

We choose a connection

$$\mathcal{A}_\mu = e_\mu{}^i P_i + \tau_\mu H + \frac{1}{2}\omega_\mu{}^{ij} J_{ij} + \omega_\mu{}^{i0} G_i + m_\mu M \tag{5.5}$$

and a gauge parameter

$$\zeta = \zeta^i P_i + \tau H + \frac{1}{2}\lambda^{ij} J_{ij} + \lambda^{i0} G_i + \sigma M, \tag{5.6}$$

so the gauge transformation of the connection is given by

$$\delta\mathcal{A}_\mu = \partial_\mu \zeta + [\zeta, \mathcal{A}_\mu]. \tag{5.7}$$

From this, the variations of the gauge fields are given by

$$\begin{aligned}
\delta e_\mu{}^i &= D_\mu \zeta^i + \lambda^{ij} e_\mu{}^j - \lambda^{i0}\tau_\mu + \tau\omega_\mu{}^{i0}, \\
\delta\tau_\mu &= \partial_\mu \tau, \\
\delta\omega_\mu{}^{ij} &= D_\mu \lambda^{ij}, \\
\delta\omega_\mu{}^{i0} &= D_\mu \lambda^{i0} + \lambda^{ij}\omega_\mu{}^{j0}, \\
\delta m_\mu &= \partial_\mu \sigma + \zeta^i \omega_\mu{}^{i0} - \lambda^{i0} e_\mu{}^i,
\end{aligned} \tag{5.8}$$

where the derivative D_μ is covariant with respect to the J transformations and containts only the $\omega_\mu^{\ ij}$ field. The curvatures of the gauge fields are

$$R_{\mu\nu}(P) = 2(D_{[\mu} e_{\nu]}^{\ i} - \omega_{[\mu}^{\ i0} \tau_{\nu]}), \tag{5.9}$$

$$R_{\mu\nu}(H) = 2\partial_{[\mu} \tau_{\nu]}, \tag{5.10}$$

$$R_{\mu\nu}(G) = 2D_{[\mu} \omega_{\nu]}^{\ i0}, \tag{5.11}$$

$$R_{\mu\nu}(J) = 2(\partial_{[\mu} \omega_{\nu]}^{\ i0}), \tag{5.12}$$

$$R_{\mu\nu}(M) = 2(\partial_{[\mu} m_{\nu]} + e_{[\mu}^{\ j} \omega_{\nu]}^{\ j0}). \tag{5.13}$$

We introduce the inverse spatial vielbein $e^\lambda_{\ i}$ and the inverse temporal vielbein τ^λ defined by conditions

$$\begin{aligned} e_\mu^{\ i} e^\mu_{\ j} &= \delta^i_j, \quad \tau^\mu \tau_\mu = 1, \\ \tau^\mu e_\mu^{\ i} &= 0, \quad \tau_\mu e^\mu_{\ i} = 0, \\ e_\mu^{\ i} e^\nu_{\ i} &= \delta^\nu_\mu - \tau_\mu \tau^\nu, \end{aligned} \tag{5.14}$$

which are just the vielbein version of the conditions in 4.16.

Only the gauge fields $e_\mu^{\ i}$, τ_μ and m_μ transform under the P and H transformations and those are the fields that we want to be independent, while the spin connections should be dependent fields. This is achieved by imposing following constraints:

$$R_{\mu\nu}^{\ \ i}(P) = R_{\mu\nu}(H) = R_{\mu\nu}(M) = 0. \tag{5.15}$$

Note that these constraints are conventional constraints, meaning that they do not reduce the fields of the theory, but they allow us to solve the dependent fields in terms of independent ones. Using 5.9 the following relations can be obtained:

$$R_{[\lambda\mu}^{\ \ ij}(J) e_{\nu]}^{\ j} = -R_{[\lambda\mu}^{\ \ i0}(G) \tau_{\nu]}, \qquad e_{[\lambda}^{\ i} R_{\mu\nu]}^{\ \ i0}(G) = 0. \tag{5.16}$$

The constraint $R_{\mu\nu}(H) = 0$ gives the condition $\partial_{[\mu} \tau_{\nu]} = 0$ which means we can take τ_μ as in 4.13. Using other conditions, we will solve for the spin connections $\omega_\mu^{\ ij}$, $\omega_\mu^{\ i0}$ in terms of other gauge fields, which leaves us with only $e_\mu^{\ i}$, τ_μ and m_μ as the independent fields.

To solve for $\omega_\mu{}^{ij}$, we begin by writing

$$R_{\mu\nu}{}^i(P)e_\rho{}^i + R_{\rho\mu}{}^i(P)e_\nu{}^i - R_{\nu\rho}{}^i(P)e_\mu{}^i = 0. \tag{5.17}$$

This equation is solved for $\omega_\mu{}^{ij}$ by contracting with $e^\nu{}_l$ and $e^\rho{}_k$:

$$\omega_\mu{}^{kl} = \partial_{[\mu}e_{\nu]}{}^k e^{\nu l} - \partial_{[\mu}e_{\nu]}{}^l e^{\nu k} + e_\mu{}^i \partial_{[\nu}e_{\rho]}{}^i e^{\nu k}e^{\rho l} - \tau_\mu e^{\rho[k}\omega_\rho{}^{l]0}. \tag{5.18}$$

To solve for $\omega_\mu{}^{i0}$, we first substitute 5.18 into $R_{\mu\nu}{}^i(P) = 0$ and contract it with $e^\mu{}_j$ and τ^ν to get the following condition:

$$e^{\mu(i}\omega_\mu{}^{j)0} = 2e^{\mu(i}\partial_{[\mu}e_{\nu]}{}^{j)}\tau^\nu. \tag{5.19}$$

By contracting $R_{\mu\nu}(M) = 0$ with e^μ_i we get

$$e^{\mu[i}\omega_\mu{}^{j]0} = e^{\mu i}e^{\nu j}\partial_{[\mu}m_{\nu]}, \tag{5.20}$$

while contracting the same equation with τ^μ gives

$$\tau^\mu\omega_\nu{}^{i0} = 2\tau^\mu e^{\nu i}\partial_{[\mu}m_{\nu]}. \tag{5.21}$$

To solve for $\omega_\mu{}^{i0}$, one of the ways is to start from equation $R_{\mu\nu}(M) = 0$ and use the equation for $e^{\mu i}\omega_\mu{}^{j0}$ obtained from adding 5.19 and 5.20, and just contract with e^μ

$$\omega_\mu{}^{i0} = e^{\nu i}\partial_{[\mu}m_{\nu]} + e^{\nu i}\tau^\rho e_\mu{}^j\partial_{[\nu}e_{\rho]}{}^j + \tau_\mu\tau^\nu e^{\rho i}\partial_{[\nu}m_{\rho]} + \tau^\nu\partial_{[\mu}e_{\nu]}{}^i \tag{5.22}$$

We have achieved our goal of having e_μ^i, τ_μ and m_μ as the independent fields. All that is left to do now is to impose equations of motion.

To show that gauging the Bargmann algebra leads to the formulation of Newton-Cartan gravity, we need to introduce a Γ connection. We will do this by imposing the vielbein postulate for the spatial vielbein

$$\partial_\mu e_\mu{}^i - \omega_\mu{}^{ij}e_\nu{}^j - \omega_\mu{}^{i0}\tau_\nu - \Gamma^\rho_{\nu\mu}e_\rho{}^i = 0 \tag{5.23}$$

and the vielbeing postulate for the temporal vielbein

$$\partial_\mu \tau_\nu - \Gamma^\lambda_{\nu\mu}\tau_\lambda = 0. \tag{5.24}$$

From these equations it follows that the Γ connection is given by

$$\Gamma^\rho_{\nu\mu} = \tau^\rho \partial_{(\mu}\tau_{\nu)} + e^\rho{}_i \left(\partial_{(\mu} e_{\nu)}{}^i - \omega_{(\mu}{}^{ij} e_{\nu)}{}^j - \omega_{(\nu}{}^{i0}\tau_{\nu)}\right). \tag{5.25}$$

The connection is symmetric because of the curvature constraints $R_{\mu\nu}{}^i(P) = 0$ and $R_{\mu\nu} = 0$. Note that the connection satisfies the metric compatibility conditions in 4.12 and that 5.23 and 5.24 define the connection Γ uniquely. From 4.19 and 5.25, it follows that

$$K_{\mu\nu} = 2\omega_{[\mu}{}^{i0} e_{\nu]}{}^i, \tag{5.26}$$

which according to $R(M) = 0$ equals

$$K_{\mu\nu} = 2\partial_{[\mu} m_{\nu]} \tag{5.27}$$

and hence satisfies the condition in 4.32. It can be shown that the Riemann tensor for connection 5.25 can be expressed in terms of the curvature tensors of the gauge algebra:

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma}(\Gamma) &= \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma} \\ &= -e^\mu{}_i \left(R_{\rho\sigma}{}^{i0}(G)\tau_\nu + R_{\rho\sigma}{}^{ij}(J) e_{\nu j}\right). \end{aligned} \tag{5.28}$$

Again we must impose the Ehlers conditions 4.33 to obtain Newton-Cartan formulation. Each of the three Ehlers conditions is equal to one curvature constraint

$$R_{\mu\nu}{}^{ij}(J) = 0. \tag{5.29}$$

Substituting this into 5.16 gets us

$$R_{[\lambda\mu}{}^{i0}(G)\tau_{\nu]} = 0, \qquad e_{[\lambda}{}^i R_{\mu\nu]}{}^{i0}(G) = 0. \tag{5.30}$$

Contracting the first equation with $e^\nu{}_j$ and $e^\mu{}_k$, and the second equation with τ^μ, $e^\lambda{}_j$

and $e^{\nu}{}_{k}$ and renaming indices gives

$$e^{\mu}{}_{k}e^{\nu}{}_{j}R_{\mu\nu}{}^{i0}(G) = 0, \quad \tau^{\mu}e^{\nu[i}R_{\mu\nu}{}^{j]0} = 0. \tag{5.31}$$

This implies that the only non-zero component of $R_{\mu\nu}{}^{i0}(G)$ is

$$\tau^{\mu}e^{\nu(i}R_{\mu\nu}{}^{j)0}(G) = \delta^{k(j}R^{i)}{}_{0k0}(\Gamma) \tag{5.32}$$

and gives exactly the only non-zero component of the Riemann tensor that occurs in Newton-Cartan gravity. These results lead to the Poisson equation and the geodesic equation of a massive free particle using the same procedure from the previous chapter.

6 Summary and outlook

6.1 Summary

We cite applications of geometrical reformulations of non-relativistic gravity as the main motivation to study Newton-Cartan gravity. First we go over the basics of differential geometry and introduce the vielbein formalism of general relativity. We show how to obtain the formulation by gauging the Poincaré algebra where we used a curvature constraint 3.47 in order to solve for spin connection $\omega_\mu{}^{ab}$ in terms of the independent vielbein $e_\mu{}^a$.

Newton-Cartan gravity is the geometric reformulation of Newtonian gravity. This formulation can also be obtained by a gauging procedure. The Lie algebra underlying this procedure is the Bargmann algebra 5.4, which is centrally extended Galilei algebra. The correct Newton-Cartan formulation is obtained by imposing curvature constraints 5.15 which allows us to solve the dependent fields $\omega_\mu{}^{ij}$ and $\omega_\mu{}^{i0}$ in terms of the independent fields $e_\mu{}^i$, τ_μ and m_μ. Next we impose the vielbein postulates 5.23 and 5.24 with which we can solve for the Γ connection. Finally, we impose an additional curvature constraint 5.29 which is equivalent to each of the three Ehlers conditions in 4.33. The Poisson equation can be recovered from the relation 5.32 between the curvature of the dependent field $\omega_\mu{}^{i0}$ and the Newton-Cartan Riemann tensor. The independent gauge fields $e_\mu{}^i$ and τ_μ describe the degenerate metrics of Newton-Cartan gravity.

6.2 Outlook

Applications of Newton-Cartan gravity cited in the introduction of this thesis, namely the AdS/CFT correspondence and the study of thermal transport, mostly use a generalized version of Newton-Cartan gravity that includes torsion. This gravity theory may be constructed through gauging or by dimensionally reducing general relativity in one dimension higher along a null isometry [13] [16] [17]. In holographic applications, it is common to consider the conformal extension of the Bargmann algebra, called the Schrödinger algebra. Field equations for Newtonian gravity with arbitrary torsion are then obtained by gauging the Schrödinger algebra or by performing null reduction of conformal algebra [13].

7 Prošireni sažetak

7.1 Uvod

Albert Einstein je svojom teorijom opće teorije relativnosti uspješno uskladio Newtonove zakone gravitacije sa specijalnom teorijom relativnosti. Opća teorija relativnosti opisuje gravitaciju kao geometrijsko svojstvo prostor-vremena te je neovisna o koordinatnom sustavu. Francuski matematičar Cartan uspio je geometrijski reformulirati nerelativističku Newtonovu gravitaciju te se ta teorija zove Newton-Cartanova gravitacija. Interes u nerelativističke teorije gravitacije je porastao zadnjih godina zbog njihove primjene. Koristi se u holografiji gdje je najpoznatiji primjer AdS/CFT korespondencija [1], te u proučavanju termalnog transporta u otpornom mediju [8].

7.2 Gravitacija i tenzorska algebra

Newtonov zakon gravitacije kaže da svaki objekt u svemiru privlači sve ostale objekte u svemiru silom koja je proporcionalna masi svakog objekta i obrnuto proporcionalna kvadratu njihove udaljenosti:

$$\mathbf{F}_{21} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^2}\hat{\mathbf{r}}_{12}. \tag{7.1}$$

Poissonova jednadžba za gravitaciju jest

$$\nabla^2\phi = 4\pi G\rho. \tag{7.2}$$

Newtonovi zakoni vrijede samo za inercijalne referentne sustave, tj sustave koji su međusobno povezani Galilei transformacijama. Prema Einsteinu, svi promatrači moraju biti ekvivalentni, bez obzira jesu li inercijalni ili ne. Prema tome, bilo koji koordinatni sustav se može koristiti. Ako teorije mora biti invarijantna na koordinante transformacije, dolazimo do zaključka da mora biti tenzorske forme.

Liejeva derivacija općenitog tenzorskog polja $T^{a\cdots}_{b\cdots}$ dana je s

$$\mathrm{L}_X T^{a\cdots}_{b\cdots} = X^c\partial_c T^{a\cdots}_{b\cdots} - T^{c\cdots}_{b\cdots}\partial_c X^a - \cdots + T^{a\cdots}_{c\cdots}\partial_b X^c + \cdots . \tag{7.3}$$

Kovarijantna derivacija općenitog tenzora je

$$\nabla_c T^{a\cdots}_{b\cdots} = \partial_c T^{a\cdots}_{b\cdots} + \Gamma^a_{dc} T^{d\cdots}_{b\cdots} + \cdots - \Gamma^d_{bc} T^{a\cdots}_{d\cdots} - \cdots \tag{7.4}$$

gdje je Γ^a_{bc} afina veza koja se transformira po sljedećoj jednadžbi

$$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \Gamma^d_{ef} - \frac{\partial x^d}{\partial x^b} \frac{\partial x^e}{\partial x^c} \frac{\partial^2 x^a}{\partial x^d \partial x^e} \tag{7.5}$$

Riemannov tenzor je tenzor tipa $(1,3)$ definiran s

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma^e_{bc} \Gamma^a_{ed} \tag{7.6}$$

Metrika je simetrični kovarijantni tenzor ranga 2 te se mnogostrukost koja ima metriku zove Riemannova mnogostrukost. Norma kontravarijantnog vektora X^a je

$$X^2 = g_{ab}(x) X^a X^b \tag{7.7}$$

a vektori X^a i Y^a su ortogonalni ako

$$g_{ab} X^a Y^b = 0 \tag{7.8}$$

Metrika g_{ab} i kontravarijantna metrika g^{ab} se koriste za spuštanje i dizanje tenzorskih indeksa:

$$T^{\cdots\ \cdots}_{\cdots a \cdots} = g_{ab} T^{\cdots b \cdots}_{\cdots\ \cdots} T^{\cdots a \cdots}_{\cdots\ \cdots} = g^{ab} T^{\cdots\ \cdots}_{\cdots b \cdots} \tag{7.9}$$

Kontrakcijom Riemannovog tenzora dobije se Riccijev tenzor:

$$R_{ab} = R^c_{acb} \tag{7.10}$$

te daljnjom kontrakcijom dobije se Riccijev skalar

$$R = g^{ab} R_{ab} \tag{7.11}$$

Einstenov tenzor je definiran s

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \tag{7.12}$$

7.3 Vielbein formalizam i baždarenje Poincaréove algebre

Razmatrajući klasični slučaj čestice u konstantnom gravitacijskom polju i pretpostavljajući da su gravitacijska te inercijalna masa ekvivalentne dolazi se do zaključka da, gledajući česticu iz sustava koji slobodno pada,

$$\mathbf{r}(t) \to \mathbf{r}'(t) = \mathbf{r}(t) - \frac{1}{2}\mathbf{g}t^2 \tag{7.13}$$

na česticu ne djeluje sila. Einstein je postulirao da u svakoj točki u vremenu i prostoru postoji transformacija kojom se može postići ovakvo izuzimanje gravitacijskog polja. Gravitaciju je moguće uključiti tako da koordinate u lokalnim vrijednostima prozovemo ”slobodno padajućim” koordinatama. Latinski indeksi $m, n, p, q, \cdots$ koristit će se za slobodno padajući sustav, a grčki indeksi $\mu, \nu, \rho, \sigma, \cdots$ za neki proizvoljni sustav.

Razmotrimo slučaj skalarnog polja $\phi(x)$. Akcija ovog polja bez gravitacije je

$$S[\phi] = \int \mathrm{d}^4x\mathcal{L} = \int \mathrm{d}^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\nu\phi - V(\phi)\right] \tag{7.14}$$

Da bismo dodali gravitaciju u sustav, varijablu x^μ moramo interpretirati kao slobodno padajuće koordinate

$$\{x^\mu\} \to \{\xi^m\}, \quad m = 0, 1, 2, 3 \tag{7.15}$$

te nova akcija koja uključuje efekte gravitacije je

$$S[\phi] = \int \mathrm{d}^4\xi \left[\frac{1}{2}\eta^{mn}\partial_m\phi\partial_n\phi - V(\phi)\right] \tag{7.16}$$

gdje treba uzeti u obzir da, iako ovo izgleda kao početna akcija, nova se integrira preko mnogostrukosti koja je određena proizvoljnim koordinatnim sustavom $\{x^\mu\}$. Transformacija između ravnih i proizvoljnih koordinata je zadana matricom koja se zove *vielbein*

$$e^m_\mu(x) \equiv \frac{\partial\xi^m}{\partial x^\mu} \tag{7.17}$$

te on zadovoljava jednadžbe

$$e_\mu^m e_n^\mu = \delta_n^m \tag{7.18}$$

$$e_m^\mu e_\nu^m = \delta_\nu^\mu. \tag{7.19}$$

Akcija se konačno može izraziti u proizvoljnom sustavu

$$S = \int \mathrm{d}^4 x \sqrt{-\det g_{\mu\nu}} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \tag{7.20}$$

gdje je $g_{\mu\nu}$ nova metrika

$$\mathrm{d}s^2 = \eta_{mn} e_\mu^m(x) e_\nu^n(x) \mathrm{d}x^\mu \mathrm{d}x^\nu \tag{7.21}$$

Komutatori Poincaréove algebre zadani su s

$$[P_a, P_b] = 0, \tag{7.22}$$

$$[M_{bc}, P_a] = -2\eta_{a[b} P_{c]}, \tag{7.23}$$

$$[M_{cd}, M_{ef}] = 4\eta_{[c[e} M_{f]d]}, \tag{7.24}$$

gdje su P_a generatori za translacije, a M_{ab} generatori Lorentzovih transformacija. Lokalnim P transformacijama pripadat će baždarno polje $e_\mu{}^a$ i parametri $\xi^a(x)$, a lokalnim Lorentz transformacijama pripadat će baždarno polje $\omega_\mu{}^{ab}$ i parametri λ^{ab}. Definiramo konekciju

$$\mathcal{A}_\mu = e_\mu{}^a P_a + \frac{1}{2} \omega_\mu{}^{ab} M_{ab}. \tag{7.25}$$

i baždarni parametar

$$\zeta = \zeta^a P_a + \frac{1}{2} \lambda^{ab} M_{ab} \tag{7.26}$$

Baždarna transformacija konekcije a je dana s

$$\mathcal{A}_\mu = e_\mu{}^a P_a + \frac{1}{2} \omega_\mu{}^{ab} M_{ab}. \tag{7.27}$$

iz čega se dobiju transformacije baždarnih polja

$$\delta e_\mu{}^a = \partial_\mu \zeta^a - \omega_\mu{}^{ab}\zeta^b + \lambda^{ab} e_\mu{}^b, \tag{7.28}$$

$$\delta \omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} + 2\lambda^{c[a}\omega_\mu{}^{b]c}. \tag{7.29}$$

Zakrivljenosti su

$$R_{\mu\nu}{}^a(P) = 2\partial_{[\mu} e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab} e_{\nu]}{}^b, \tag{7.30}$$

$$R_{\mu\nu}{}^{ab}(M) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{ac}\omega_{\nu]}{}^{cb}. \tag{7.31}$$

Želimo baždarna polja $e_\mu{}^a$ i $\omega_\mu{}^{ab}$ interpretirati kao vielbein i spin-konekciju. Primjećujemo da je vielbein jedino polje koje se transformira P transformacijama, te nam uvjet

$$R_{\mu\nu}{}^a(P) = 0 \tag{7.32}$$

omogućuje da izrazimo $\omega_\mu{}^{ab}$ preko vielbeina, njegovog inverza i derivacija:

$$\omega_\mu{}^{ab}(e, \partial e) = -2e^{\lambda[a}\partial_{[\mu} e_{\lambda]}{}^{b]} + e_\mu{}^c e^{\lambda a} e^{\rho b} \partial_{[\lambda} e_{\rho]}{}^c \tag{7.33}$$

Nametanjem vielbein postulata:

$$\nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma^\rho_{\nu\mu} e_\rho{}^a - \omega_\mu{}^{ab} e_\nu{}^b = 0. \tag{7.34}$$

možemo dobiti Γ vezu izraženu preko vielbeina i njegovog inverza

$$\Gamma^\rho_{\nu\mu} = e^\rho{}_a D_\mu e_\nu{}^a \tag{7.35}$$

Također, definiramo nedegeneriranu metriku i njen inverz:

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}, \qquad g^{\mu\nu} = e_a{}^\mu e_b{}^\nu \eta^{ab} \tag{7.36}$$

7.4 *Newton-Cartanova gravitacija*

Newton-Cartanova gravitacija je geometrijska reformulacije Newtonove gravitacije. Želimo klasičnu jednadžbu putanje čestice s masom

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \frac{\partial \phi}{\partial x^i} = 0 \tag{7.37}$$

usporediti s geodezijskom jednadžbom

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} = 0 \tag{7.38}$$

Jedini koeficijenti veze koji ne iščezavaju su

$$\Gamma^i{}_{00} = \frac{\partial \phi}{\partial x^i} \tag{7.39}$$

iz čega se dobije i da su jedini neiščezavajući koeficijenti Reimannovog tenzora

$$R^i{}_{0k0} = -R^i{}_{00k} = \frac{\partial^2 \phi}{\partial x^i \partial x^k} \tag{7.40}$$

te Riccijevog tenzora

$$R_{00} = \sum_i \frac{\partial^2 \phi}{\partial {x^i}^2} \tag{7.41}$$

Končano, Poissonovu jednadžbu možemo napisati u geometrijskom obliku:

$$R_{00} = 4\pi G \rho \tag{7.42}$$

Razmotrimo Minkowski metriku i njen inverz:

$$\eta_{\mu\nu}/c^2 = \begin{bmatrix} -1 & 0 \\ 0 & \mathbb{I}_3/c^2 \end{bmatrix}, \qquad \eta^{\mu\nu} = \begin{bmatrix} -1/c^2 & 0 \\ 0 & \mathbb{I}_3 \end{bmatrix}. \tag{7.43}$$

Limes $c \to \inf$ bi nam dao degeneriranu kovarijantnu vremensku metriku $\tau_{\mu\nu}$ čije su tri svojstvene vrijednosti nula, te degeneriranu kontravarijantnu prostornu metriku $h^{\mu\nu}$ s jednom svojstvenom vrijednosti koja je jednaka nuli. Metrika $\tau_{\mu\nu}$ je efektivno 1×1 matrica pa ćemo koristiti vielbein verziju τ_μ koja je definirana s $\tau_{\mu\nu} = \tau_\mu \tau_\nu$. Uvodimo i inverznu prostornu metriku $h_{\mu\nu}$ i invernu vremensku metriku τ^μ te za njih vrijedi

$$
\begin{aligned}
h^{\mu\nu}h_{\nu\rho} = \delta^{\mu}_{\rho} - \tau^{\mu}\tau_{\rho}, \qquad \tau^{\mu}\tau_{\mu} = 1,\\
h^{\mu\nu}t_{\nu} = 0, \qquad h_{\mu\nu}\tau^{\nu} = 0.
\end{aligned}
\tag{7.44}
$$

Primijenit ćemo uvjet kompatibilnosti:

$$
\nabla_{\rho}h^{\mu\nu} = 0, \qquad \nabla_{\rho}\tau_{\mu} = 0 \tag{7.45}
$$

gdje je kovarijantna derivacija s vezom $\Gamma^{\rho}_{\mu\nu}$. Najopćenitija veza koja zadovoljava ove jednadžbe je

$$
\Gamma^{\sigma}_{\mu\nu} = \tau^{\sigma}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\sigma\rho}\left(\partial_{\nu}h_{\rho\mu} + \partial_{\mu}h_{\rho\nu} - \partial_{\rho}h_{\mu\nu}\right) + h^{\sigma\lambda}K_{\lambda(\mu}\tau_{\nu)} \tag{7.46}
$$

gdje je $K_{\mu\nu}$ proizvoljni simetrični tenzor. Koristit ćemo nadalje adaptirane koordinate $x^0 = t$, te postaviti kovarijantni ansatz iz kojeg želimo dobiti Poissonovu jednadžbu:

$$
R_{\mu\nu} = 4\pi G\rho\tau_{\mu}\tau_{\nu} \tag{7.47}
$$

Ona se dobije postavljanjem još dva uvjeta, jedan je Trautmanov uvjet

$$
h^{\sigma[\lambda}R^{\mu]}{}_{(\nu\rho)\sigma} = 0. \tag{7.48}
$$

a drugi je bilo koji od Ehlerovih uvjeta

$$
\begin{aligned}
& h^{\rho\lambda}R^{\mu}{}_{\nu\rho\sigma}R^{\nu}{}_{\mu\lambda\alpha} = 0\\
\text{ili}\quad & \tau_{[\lambda}R^{\mu}{}_{\nu]\rho\sigma} = 0\\
\text{ili}\quad & h^{\sigma[\lambda}R^{\mu]}{}_{\nu\rho\sigma} = 0.
\end{aligned}
\tag{7.49}
$$

Pokaže se da su ovi uvjeti dovoljni da se dobije Poissonova jednažba

$$
R_{00} = \partial_i\Gamma^{i}_{00} = \delta^{ij}\partial_i\partial_j\phi = 4\pi G\rho. \tag{7.50}
$$

7.5 Baždarenje Bargmannove algebre

Bargmannova algebra je Galileieva algebra centralno proširena s generatorom M čiji su komutatori

$$\begin{aligned} [J_{ij}, J_{kl}] &= 4\delta_{[i[k}J_{l]j]}, \quad [J_{ij}, P_k] = -2\delta_{k[i}P_{j]}, \\ [J_{ij}, G_k] &= -2\delta_{k[i}G_{j]}, \quad [G_i, H] = -P_i, \\ [G_i, P_j] &= -\delta_{ij}M. \end{aligned} \tag{7.51}$$

Baždarenje Bargmannove algebre radi se na isti način kao što se baždarila i Poincaréova algebra. Generatoru prostornih translacija P_i pripadat će baždarno polje $e_\mu{}^i$ i parametar ξ^i, generatoru vremenskih translacija H polje $e_\mu{}^0 = \tau_\mu$ i parametar $\xi^0 = \tau$, generatoru rotacija J_{ij} polje $\omega_\mu{}^{ij}$ i parametar λ^{ij}, generatoru Galileievih potisaka polje $\omega_\mu{}^{i0}$ i parametar λ^{i0} te generatoru M polje m_μ i parametar σ. Varijacije baždarnih polja su

$$\begin{aligned} \delta e_\mu{}^i &= D_\mu \zeta^i + \lambda^{ij} e_\mu{}^j - \lambda^{i0}\tau_\mu + \tau\omega_\mu{}^{i0}, \\ \delta\tau_\mu &= \partial_\mu \tau, \\ \delta\omega_\mu{}^{ij} &= D_\mu \lambda^{ij}, \\ \delta\omega_\mu{}^{i0} &= D_\mu\lambda^{i0} + \lambda^{ij}\omega_\mu{}^{j0}, \\ \delta m_\mu &= \partial_\mu\sigma + \zeta^i\omega_\mu{}^{i0} - \lambda^{i0}e_\mu{}^i, \end{aligned} \tag{7.52}$$

gdje kovarijantna derivacija D_μ uključuje samo $\omega_\mu{}^{ij}$. Zakrivljenosti baždarnih polja su

$$R_{\mu\nu}(P) = 2(D_{[\mu}e_{\nu]}{}^i - \omega_{[\mu}{}^{i0}\tau_{\nu]}), \tag{7.53}$$

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]}, \tag{7.54}$$

$$R_{\mu\nu}(G) = 2D_{[\mu}\omega_{\nu]}{}^{i0}, \tag{7.55}$$

$$R_{\mu\nu}(J) = 2(\partial_{[\mu}\omega_{\nu]}{}^{i0}), \tag{7.56}$$

$$R_{\mu\nu}(M) = 2(\partial_{[\mu}m_{\nu]} + e_{[\mu}{}^j\omega_{\nu]}{}^{j0}). \tag{7.57}$$

Inverzni prostorni vielbein i inverzni vremenski vielbein definirani su uvjetima

$$
\begin{aligned}
e_\mu{}^i e^\mu{}_j &= \delta^i_j, \quad \tau^\mu \tau_\mu = 1, \\
\tau^\mu e_\mu{}^i &= 0, \quad \tau_\mu e^\mu{}_i = 0, \\
e_\mu{}^i e^\nu{}_i &= \delta^\nu_\mu - \tau_\mu \tau^\nu,
\end{aligned}
\tag{7.58}
$$

Jedino se polja $e_\mu{}^i$, τ_μ i m_μ transformiraju P i H transformacijama te želimo postići da navedena polja budu neovisna. Stoga postavljamo sljedeće uvjete:

$$
R_{\mu\nu}{}^i(P) = R_{\mu\nu}(H) = R_{\mu\nu}(M) = 0. \tag{7.59}
$$

Također, iz izraza za zakrivljenosti možemo dobiti sljedeće relacije:

$$
R_{[\lambda\mu}{}^{ij}(J) e_{\nu]}{}^j = -R_{[\lambda\mu}{}^{i0}(G)\tau_{\nu]}, \qquad e_{[\lambda}{}^i R_{\mu\nu]}{}^{i0}(G) = 0. \tag{7.60}
$$

Uvjet $R_{\mu\nu}(H) = 0$ daje $\partial_{[\mu}\tau_{\nu]} = 0$. Rješenje za $\omega_\mu{}^{ij}$ je

$$
\omega_\mu{}^{kl} = \partial_{[\mu} e_{\nu]}{}^k e^{\nu l} - \partial_{[\mu} e_{\nu]}{}^l e^{\nu k} + e_\mu{}^i \partial_{[\nu} e_{\rho]}{}^i e^{\nu k} e^{\rho l} - \tau_\mu e^{\rho[k} \omega_\rho{}^{l]0} \tag{7.61}
$$

a za $\omega_\mu{}^{i0}$

$$
\omega_\mu{}^{i0} = e^{\nu i}\partial_{[\mu} m_{\nu]} + e^{\nu i}\tau^\rho e_\mu{}^j \partial_{[\nu} e_{\rho]}{}^j + \tau_\mu \tau^\nu e^{\rho i}\partial_{[\nu} m_{\rho]} + \tau^\nu \partial_{[\mu} e_{\nu]}{}^i \tag{7.62}
$$

Ovim smo postigli da su $e_\mu{}^i$, τ_μ i m_μ neovisna polja, a preko njih su izražena ovisna polja $\omega_\mu{}^{ij}$ i $\omega_\mu{}^{i0}$.

Γ vezu uvodimo vielbein postulatom za prostorni vielbein

$$
\partial_\mu e_\mu{}^i - \omega_\mu{}^{ij} e_\nu{}^j - \omega_\mu{}^{i0}\tau_\nu - \Gamma^\rho_{\nu\mu} e_\rho{}^i = 0 \tag{7.63}
$$

te za vremenski vielbein

$$
\partial_\mu \tau_\nu - \Gamma^\lambda_{\nu\mu}\tau_\lambda = 0. \tag{7.64}
$$

Iz ovoga slijedi da je veza dana s

$$
\Gamma^\rho_{\nu\mu} = \tau^\rho \partial_{(\mu}\tau_{\nu)} + e^\rho{}_i \left(\partial_{(\mu} e_{\nu)}{}^i - \omega_{(\mu}{}^{ij} e_{\nu)}{}^j - \omega_{(\nu}{}^{i0}\tau_{\nu)} \right) \tag{7.65}
$$

Veza također zadovoljava uvjete kompatibilnosti iz 7.45 te je jedinstveno definirana.

Uspoređivanjem gornje jednadžbe s 7.46 dobije se

$$K_{\mu\nu} = 2\partial_{[\mu} m_{\nu]} \tag{7.66}$$

Riemannov tenzor može se napisati preko tenzora zakrivljenosti baždarne algebre:

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma}(\Gamma) &= \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\mu}_{\lambda\rho} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\mu}_{\lambda\sigma} \\ &= -e^{\mu}_{\ i}\left(R_{\rho\sigma}{}^{i0}(G)\tau_\nu + R_{\rho\sigma}{}^{ij}(J)e_{\nu j}\right). \end{aligned} \tag{7.67}$$

Da bismo došli do Newton-Cartanove formulacije gravitacije, moramo primijeniti Ehlersove uvjete 7.49. Ovi uvjeti istovjetni su jednom uvjetu za zakrivljenost

$$R_{\mu\nu}{}^{ij}(J) = 0. \tag{7.68}$$

Uvrštavanjem ovog uvjeta u 7.60 dobiju se relacije

$$R_{[\lambda\mu}{}^{i0}(G)\tau_{\nu]} = 0, \qquad e_{[\lambda}{}^{i} R_{\mu\nu]}{}^{i0}(G) = 0. \tag{7.69}$$

Konačno, iz ovoga slijedi da je jedini neiščezavajući član od $R_{\mu\nu}{}^{i0}(G)$

$$\tau^{\mu} e^{\nu(i} R_{\mu\nu}{}^{j)0}(G) = \delta^{k(j} R^{i)}{}_{0k0}(\Gamma) \tag{7.70}$$

te daje točno jedini neiščezavajući član Riemannovog tenzora Newton-Cartanove gravitacije. Ovakav Riemannov tenzor vodi ka Poissonovoj jednadžbi i geodezijskoj jednadžbi slobodne masivne čestice

7.6 Zaključak

U ovom diplomskom radu prvo prolazimo kroz osnove diferencijalne geometrije. Upoznajemo se s vielbein formalizmom opće teorije relativnosti te pokazujemo kako se on može dobiti baždarenjem Poincaréove algebre. Nametanjem uvjeta na zakrivljenost 7.32 omogućuje nam da ovisno baždardno polje $\omega_{\mu}^{\ ij}$ izrazimo preko neovisnog polja $e_{\mu}^{\ i}$. Nakon toga pokazujemo kako se formulacija Newton-Cartanove gravitacije može dobiti baždarenjem Bargmannove algebre. Za to su nam potrebni uvjeti na zakrivljenosti 7.59 s kojima rješavamo ovisna polja $\omega_{\mu}^{\ ij}$ i $\omega_{\mu}^{\ i0}$ preko neovisnih polja $e_{\mu}^{\ i}$,

τ_μ i m_μ. Vielbein postulatima 7.63 i 7.64 dobijemo Γ vezu, te još jednim uvjetom na zakrivljenosti 7.68 konačno dobijemo relaciju iz koje se dobije Poissonova jednadžba Newton-Cartanove teorije.

U primjenama Newton-Cartanove gravitacije u AdS/CFT korespondenciji i proučavanju termalnog transporta najčešće se koristi verzija Newton-Cartanove gravitacije s torzijom. Ova tema bi se, dakle, mogla proširiti izvodom takve gravitacije baždarenjem Schrödingerove algebre ili dimenzionalnom redukcijom opće teorije relativnosti u većoj dimneziji.

Bibliography

[1] J. M. Maldacena, Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] doi:10.1023/A:1026654312961, 10.4310/ATMP.1998.v2.n2.a1 [hep-th/9711200].

[2] J. Hartong, E. Kiritsis and N. A. Obers, JHEP **1508** (2015) 006 doi:10.1007/JHEP08(2015)006 [arXiv:1502.00228 [hep-th]].

[3] D. T. Son, Phys. Rev. D **78** (2008) 046003 doi:10.1103/PhysRevD.78.046003 [arXiv:0804.3972 [hep-th]].

[4] K. Balasubramanian and J. McGreevy, Phys. Rev. Lett. **101** (2008) 061601 doi:10.1103/PhysRevLett.101.061601 [arXiv:0804.4053 [hep-th]].

[5] M. Taylor, arXiv:0812.0530 [hep-th].

[6] M. H. Christensen, J. Hartong, N. A. Obers and B. Rollier, Phys. Rev. D **89** (2014) 061901 doi:10.1103/PhysRevD.89.061901 [arXiv:1311.4794 [hep-th]].

[7] M. H. Christensen, J. Hartong, N. A. Obers and B. Rollier, JHEP **1401** (2014) 057 doi:10.1007/JHEP01(2014)057 [arXiv:1311.6471 [hep-th]].

[8] J. M. Luttinger, Phys. Rev. **135** (1964) A1505. doi:10.1103/PhysRev.135.A1505

[9] D. T. Son, arXiv:1306.0638 [cond-mat.mes-hall].

[10] A. Gromov and A. G. Abanov, Phys. Rev. Lett. **114** (2015) 016802 doi:10.1103/PhysRevLett.114.016802 [arXiv:1407.2908 [cond-mat.str-el]].

[11] S. Moroz and C. Hoyos, Phys. Rev. B **91** (2015) no.6, 064508 doi:10.1103/PhysRevB.91.064508 [arXiv:1408.5911 [cond-mat.quant-gas]].

[12] Pierre, R. (2001). Field Theory: A Modern Primer (2nd ed.), Westview Press

[13] E. Bergshoeff, A. Chatzistavrakidis, L. Romano and J. Rosseel, JHEP **1710** (2017) 194 doi:10.1007/JHEP10(2017)194 [arXiv:1708.05414 [hep-th]].

[14] Andringa, R. (2016). Newton-Cartan gravity revisited. [Groningen]: Rijksuniversiteit Groningen.

[15] Misner, Charles W.; Thorne, Kip S.; Wheeler, John Archibald (1973), Gravitation, San Francisco: W. H. Freeman

[16] B. Julia and H. Nicolai, Nucl. Phys. B **439** (1995) 291 doi:10.1016/0550-3213(94)00584-2 [hep-th/9412002].

[17] C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, Phys. Rev. D **31** (1985) 1841. doi:10.1103/PhysRevD.31.1841

[18] R. Andringa, E. Bergshoeff, S. Panda and M. de Roo, Class. Quant. Grav. **28** (2011) 105011 doi:10.1088/0264-9381/28/10/105011 [arXiv:1011.1145 [hep-th]].

[19] d'Inverno, R. (1998). Introducing Einstein's Relativity, New York: Oxford University Press