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Izjava o izvornosti rada

Ja, Ana Grbac, studentica Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu, s prebivalištem na adresi [redacted], [redacted], ovim putem izjavljujem pod materijalnom i kaznenom odgovornošću da je moj doktorski rad pod naslovom: Samodualni i LCD kodovi iz asocijacijskih shema s dvije klase, isključivo moje autorsko djelo, koje je u potpunosti samostalno napisano uz naznaku izvora drugih autora i dokumenata korištenih u radu.

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SUMMARY

The main subjects of the thesis are LCD codes constructed from two class association schemes, *i.e.* from adjacency matrices of strongly regular graphs and doubly regular tournaments. In this thesis, we describe two methods of construction of self-dual codes. Firstly, a method of constuction of quadratic double circulant codes which was given by P. Gaborit in [28]. A method introduced by S. T. Dougherty, J.-L. Kim and P. Solé in [26] represents its generalization and it refers to the construction of self-dual codes from two class association schemes. We analyse some examples of self-dual codes obtained from Paley designs and Paley graphs. Further, we develop a method of constructing LCD codes from two class association schemes, which is the main contribution of the thesis. This method consists of pure and bordered construction. We show that LCD codes constructed using the pure construction are formally self-dual, and for these codes we present a decoding algorithm. We also give conditions under which the introduced construction method gives LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$. Then we analyse some LCD codes obtained using this method of construction from some families of strongly regular graphs which include line graphs of complete graphs and bipartite complete graphs, some special strongly regular graphs such as the Petersen, Shrikhande, Clebsch, Hoffman-Singleton, Gewirtz graph and the Chang graphs, block graphs of Steiner triple systems, graphs obtained from orthogonal arrays and rank three permutation groups. Lastly, we construct LCD codes from some doubly regular tournaments. All obtained codes are constructed and analysed using the software package Magma.

SAŽETAK

Predmet istraživanja doktorske disertacije su LCD kodovi konstruirani iz asocijacijskih shema s dvije klase tj. iz matrica susjedstva jako regularnih grafova i dvostruko regularnih turnira. U doktorskoj disertaciji opisane su dvije metode konstrukcija samodualnih kodova. Prva metoda prikazuje konstrukciju kvadratnih dvostruko cirkularnih kodova koji koriste kvadratne reziduale, a uveo ju je P. Gaborit u radu [28]. Druga metoda predstavlja generalizaciju prve metode, a odnosi se na konstrukciju samodualnih kodova iz asocijacijskih shema s dvije klase koju su uveli S. T. Dougherty, J.-L. Kim i P. Solé u radu [26]. U disertaciji analiziramo kodove dobivene iz Paleyevih dizajna i Paleyevih grafova primjenom tih dviju metoda. Nadalje, u disertaciji je razvijena metoda konstrukcije LCD kodova iz asocijacijskih shema s dvije klase, što čini glavni znanstveni doprinos rada. Metoda se sastoji od čiste i omeđene konstrukcije. Dokazano je da su LCD kodovi konstruirani iz čiste konstrukcije formalno samodualni i za te kodove je naveden algoritam dekodiranja. Osim toga, navedeni su uvjeti za konstrukciju LCD kodova nad poljima $\mathbb{F}_2$, $\mathbb{F}_3$ i $\mathbb{F}_4$. Posebno su analizirani LCD kodovi dobiveni primjenom razvijene metode koristeći se linijskim grafovima potpunih grafova i potpunih bipartitnih grafova, nekim istaknutim jako regularnim grafovima poput Petersenova, Shrikhandeova, Clebschova, Hoffman-Singletonova i Gewirtzova grafa te Changovih grafova, blokovnim grafovima Steinerovih sustava trojki i nekim grafovima dobivenih iz ortogonalnih područja i permutacijskih grupa ranga tri. Na kraju su analizirani LCD kodovi dobiveni iz nekih dvostruko regularnih turnira. Svi su kodovi konstruirani i analizirani nad poljima $\mathbb{F}_2$, $\mathbb{F}_3$ i $\mathbb{F}_4$ pomoću programskog paketa Magma.

KEYWORDS

LCD code, self-dual code, formally self-dual code, association scheme, strongly regular graph, doubly regular tournament, Paley design, Paley graph, Steiner triple system, orthogonal array, rank three permutation group.

Ključne riječi

LCD kod, samodualni kod, formalno samodualni kod, asocijacijska shema, jako regularni graf, dvostruko regularni turnir, Paleyev dizajn, Paleyev graf, Steinerov sustav trojki, ortogonalno područje, permutacijska grupa ranga tri.

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INTRODUCTION

Coding theory is a branch of discrete mathematics which deals with efficient and reliable data transmission from the sender to the receiver through a noisy communication channel as well as correcting errors that occur during the transmission. In 1948, C. E. Shannon in [54] set the foundations of the coding theory. In that paper, Shannon developed the concept of channel capacity and provided a mathematical model for computing it, known as Shannon's theorem, in which the so-called Shannon's limit is given. The theorem shows that with certain coding techniques it is possible to use the capacity of the communication channel to transmit data with minimal errors.

Linear codes have been studied more than non-linear codes. Because of their algebraic structure, they are easier to describe, encode and decode. An important class of linear codes are self-dual codes. One of the reasons is their close connection to other mathematical structures such as block designs. Furthermore, since many of the important codes have the property of being self-dual, the development of construction methods and the classification of self-dual codes of a given length and dimension is an active area of research in coding theory. Another important class of linear codes are LCD codes, *i.e.* linear codes with complementary duals. These are linear codes whose intersection with their duals are trivial and most of the dissertation will be devoted to the construction of this type of codes.

LCD codes are a relatively new class of codes introduced by J. L. Massey in [46], and since then they have been widely applied in data storage, communication systems, electronics and cryptography. In his paper, Massey presented LCD codes as codes that provide an optimum linear coding solution for the two user binary adder channel. In [53], N. Sendrier showed that LCD codes are asymptotically good by proving that they meet the asymptotic Gilbert-Varshamov bound. In addition to being used in communication systems and data storage, in [16], C. Carlet and S. Guilley studied applications of binary LCD codes against side-channel attacks (SCA) and fault injection attacks (FIA). Carlet and Guilley also presented several constructions of LCD codes and showed that non-binary LCD codes in characteristic 2 can be transformed into binary LCD codes by expansion. A multisecret-sharing scheme based on LCD codes is given in [1].

There are many papers dealing with constructions of LCD codes. In [63], X. Yang and J. L. Massey provided necessary and sufficient conditions under which a cyclic code has a complementary dual. In [23], C. Ding, C. Li and S. Li constructed several families of LCD cyclic codes over finite fields and analysed their parameters. In [25], S. T. Dougherty, J.-L. Kim, B. Ozkaya, L. Sok and P. Solé constructed binary LCD codes using orthogonal matrices, self-dual codes, combinatorial designs and Gray map from codes over a family of non-chain rings of characteristic 2. They also showed that a single self-dual code, or a single orthogonal matrix gives rise to several LCD codes. In [18], C. Carlet, S. Mesnager, C. Tang and Y. Qi introduced a general construction of LCD codes from any linear codes and showed that any linear code over $\mathbb{F}_q$ $(q > 3)$ is equivalent to an Euclidean LCD code and any linear code over $\mathbb{F}_{q^2}$ $(q > 2)$ is equivalent to a Hermitian LCD code. In [21], D. Crnković, R. Egan, B. G. Rodrigues and A. Švob constructed LCD codes from weighing matrices, including the Paley conference matrices and Hadamard matrices. They also extended the construction to Hermitian LCD codes over the finite field $\mathbb{F}_4$.

Recently, some optimal LCD codes have been constructed, namely codes that have the highest possible minimum distance for a given length and dimension. In [55], L. Sok, M. Shi and P. Solé constructed optimal LCD codes over large finite fields from optimal self-orthogonal codes and orthogonal matrices. Using generalized Reed-Solomon codes, in [19] and [36], B. C. Chen, H. W. Liu and L. F. Jin proved the existence of optimal LCD codes over finite fields with some conditions on lengths and the field sizes. Moreover, in [36], Jin constructed optimal LCD codes over finite fields of even characteristic. Further, in [17], C. Carlet, S. Mesnager, C. Tang and Y. Qi completely solved the problem of existence of q-ary $[n,k]$ optimal LCD codes for the Euclidean case. They also provided several constructions of Euclidean and Hermitian optimal LCD codes for which they used linear codes with small dimension, self-orthogonal codes and generalized Reed-Solomon codes.

The most important motivation for the topic of this dissertation was research given by [26] in which S. T. Dougherty, J.-L. Kim and P. Solé developed a method of constructing self-dual codes from two class association schemes. Their construction represents a generalization of the construction of ternary self-dual codes by V. Pless [48] and the construction of the self-dual codes presented by P. Gaborit in [28]. The main subject of the research of this dissertation is the construction of LCD codes from two class association schemes, *i.e.* the construction of LCD codes from the adjacency matrices of strongly regular graphs and doubly regular tournaments.

The dissertation is thematically divided into five chapters. In the first chapter, we give an insight into the basics of group theory, design theory, graph theory and coding theory. In the second chapter, we describe two construction methods of self-dual codes. The first method is Gaborit's construction of quadratic double circulant codes that uses quadratic residues. The second method is a generalization of the first method and it refers to the construction of self-dual codes from two class association schemes presented by Dougherty, Kim and Solé. We also apply these methods to construct self-dual codes from Paley designs and Paley graphs.

The aim of the third chapter is to develop a method of constructing LCD code from two class association schemes, which is inspired by the construction in the previous chapter. This method consists of pure and bordered construction. We also show that LCD codes constructed using the pure construction are formally self-dual, and for these codes we deduce a decoding algorithm. Furthermore, we give conditions under which the introduced construction method gives LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$.

In the fourth chapter, we apply the developed method to construct LCD codes obtained from some families of strongly regular graphs. It includes line graphs of complete graphs and bipartite complete graphs, some special strongly regular graphs such as the Petersen, Shrikhande, Clebsch, Hoffman-Singleton, Gewirtz graph and the Chang graphs, block graphs of Steiner triple systems, graphs obtained from orthogonal arrays and rank three permutation groups.

The last chapter is devoted to the construction of LCD codes from some doubly regular tournaments. All obtained codes in this dissertation are constructed over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$ using the software package Magma [10].

1. Preliminaries

This chapter will be useful for understanding the dissertation. Here we describe some elementary concepts and results from group theory, design theory, graph theory and coding theory. Definitions and results not included here will be introduced later on as they are needed. It is presumed that the reader is already familiar with the basics of linear algebra, groups, rings and fields, as well as combinatorics. For additional reading on these topics, we suggest the reader [51, 52] for groups, [42, 59] for designs, [5, 7] for graphs and [35, 43, 45] for codes.

1.1. Groups

We start with the basic terminology concerning permutation groups, which we will require later in the dissertation. More about these groups can be found in [6, 24].

Definition 1.1.1. Let Ω be a non-empty set. A **permutation** on Ω is a bijective function $\sigma : \Omega \to \Omega$. Under the operation of composition the set of all permutations of Ω forms a group $S(\Omega)$ which is called the **symmetric group** on Ω. A **permutation group** on Ω is a subgroup of $S(\Omega)$.

If $\Omega = \{1, 2, \ldots, n\}$, then the symmetric group on Ω is called the **symmmetric group of degree** n. This group is denoted by S_n and has the order $|S_n| = n!$.

Definition 1.1.2. Let G be a group and Ω a set. A **group action** of G on Ω is a function $\varphi : G \times \Omega \to \Omega$, where we write $\varphi(g, x) = g \cdot x$, that satisfies the following two axioms:

1. $1 \cdot x = x$ for all $x \in \Omega$ and 1 is the identity element of G,

2. $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in \Omega$.

The set Ω is called a ***G*-set**. The group G is said **to act on** Ω.

Example 1.1.3.

1. Let $\Omega = \{1, 2, \dots, n\}$, $x \in \Omega$ and $\sigma \in S_n$. S_n acts on Ω via $\sigma \cdot x = \sigma(x)$.

2. Let $\Omega = \mathbb{R}^n$, $G = GL_n(\mathbb{R})$, $A \in G$ and $x \in \Omega$. G acts on Ω by the usual rule $A \cdot x = Ax$, which is the multiplication of matrix A and vector x.

Definition 1.1.4. Let G be a group acting on a set Ω and $x \in \Omega$. The **orbit** Gx of x under the action of G is the set of elements in Ω to which x can be moved by the elements of G, *i.e.*

$$Gx = \{g \cdot x \mid g \in G\} \subseteq \Omega.$$

Proposition 1.1.5. Let G be a group acting on a set Ω and let the relation $\sim_G$ be defined by

$$x \sim_G y \Longleftrightarrow (\exists g \in G)\ g \cdot x = y.$$

Then $\sim_G$ is an equivalence relation and the equivalence class containing x is the orbit Gx. Thus, two orbits under the action of G are either disjoint or identical.

The proof can be found in [24].

Definition 1.1.6. A group action of G on a set Ω is **transitive** if it has a single orbit, *i.e.* $Gx = \Omega$ for all $x \in \Omega$. In this case, for any two elements $x, y \in \Omega$, there is a group element $g \in G$ such that $g \cdot x = y$.

Example 1.1.7. S_n acts trasitively on $\Omega = \{1, 2, \dots, n\}$, *i.e.* for all $x \in \Omega$, there exists an element σ from S_n such that $\sigma(1) = x$. Therefore, $S_n 1 = \Omega$ and there is just one orbit.

Definition 1.1.8. Let G be a group acting on a set Ω and $x \in \Omega$. The **stabilizer** G_x of x in G is the set of elements $g \in G$ that fix x, *i.e.*

$$G_x = \{g \in G \mid g \cdot x = x\} \subseteq G.$$

The next result is the most important basic result in the theory of group actions, and the proof can be found in [29].

Theorem 1.1.9 (Orbit-stabilizer theorem)**.** Suppose G is a finite group which acts on a set Ω. For any $x \in \Omega$, we have

$$|G| = |G_x||Gx|.$$

1.2. DESIGNS

Design theory is a field of combinatorial mathematics that studies the possibility of arranging elements of a finite set into subsets according to some defined rules of balance and symmetry. Like most areas of combinatorics, design theory has grown up with computer science, especially in the last 30 years. It has become of great importance due to its connection to other fields in mathematics such as group theory, the theory of finite fields, the theory of finite geometries, number theory, graph theory and coding theory. Besides, it has a wide application in non-mathematical areas such as information theory, computer science, biology and engineering.

Design theory has its roots in recreational mathematics. Many types of designs that are studied today were first considered in the context of mathematical puzzles or brain-teasers in the eighteenth and nineteenth centuries. The study of design theory as a mathematical discipline really began in the twentieth century due to applications in the design and analysis of statistical experiments. Designs have many other applications as well, such as tournament scheduling, lotteries, mathematical biology, algorithm design and analysis, networking, group testing and cryptography [59].

Among the main subjects of design theory are balanced incomplete block designs (BIBDs), Hadamard matrices and Hadamard designs, which will be used in the dissertation to construct self-dual codes. In the following, we give a definition of design and BIBD using the notion of incidence structure. Later, as needed, we will define Hadamard matrix and Hadamard design.

Definition 1.2.1. An **incidence structure** is an ordered triple $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$.
The elements of the set $\mathcal{P}$ are called **points**, the elements of the set $\mathcal{B}$ are called **blocks** and $\mathcal{I}$ is called an **incidence relation**. If the ordered pair (p, B) is in $\mathcal{I}$, we say that p is **incident** with B or that B contains p, and we can also write $p \in B$.
The incidence structure is **finite** if the sets $\mathcal{P}$ and $\mathcal{B}$ are finite.

Definition 1.2.2. A $\boldsymbol{t}$**-**$\boldsymbol{(v,k,\lambda)}$ **design**, or briefly a $\boldsymbol{t}$**-design**, is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}| = v$,

2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,

3. every t distinct elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

The non-negative integers t, v, k and λ are refered to as the **parameters** of a design.

This definition allows $\mathcal{B}$ to be a multiset, which means blocks may be repeated.

Definition 1.2.3. A t-design without repeated blocks is called a **simple design**.

Remark 1.2.4. We will consider only simple t-designs.

Proposition 1.2.5. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a t-(v,k,λ) design. Then for every integer s such that $0 \leq s \leq t$, $\mathcal{D}$ is also an s-(v,k,λ_s) design, where

$$\lambda_s \binom{k-s}{t-s} = \lambda \binom{v-s}{t-s}.$$

From the previous proposition, the proof of which can be found in [42], it follows that λ_0 is the number of blocks in a design which is usually denoted by b and λ_1 is the number of blocks containing any given point which is usually denoted by r. Since any t-design for $t \geq 1$ can be regarded as a 1-design, we obtain

$$bk = vr,$$

which indicates that the parameters of a design are not independent.

Definition 1.2.6. The **degree of a point** $p \in \mathcal{P}$ is the number r, and this is the number of blocks to which it is incident. The **degree of a block** $B \in \mathcal{B}$ is the number k, and this is the number of points to which it is incident.

Definition 1.2.7. A t-design is called **trivial** if every set of k points is incident with a block, in which case $b = \binom{v}{k}$.

Definition 1.2.8. A t-(v,k,λ) design is called **symmetric** if the number of blocks b equals the number of points v.

Remark 1.2.9. Non-trivial symmetric t-designs exist only for $t \leq 2$ [35].

Definition 1.2.10. A 2-(v,k,λ) design is called a **block design**.

Remark 1.2.11. In the literature, 2-design is often called *balanced incomplete block design*, abbreviated to BIBD. Balanced refers to the constancy of the parameter λ and incomplete to the fact that $k < v$.

The following theorems describe two basic properties of 2-designs that can be obtained from Proposition 1.2.5 (the proofs can be found in [59]).

Theorem 1.2.12. In a 2-(v,k,λ) design every point is incident with exactly

$$r = \frac{\lambda(v-1)}{k-1}$$

blocks.

Theorem 1.2.13. The number of blocks in a 2-(v,k,λ) design is

$$b = \frac{vr}{k} = \frac{\lambda(v^2-v)}{k^2-k}.$$

Example 1.2.14. Figure 1.1 shows a symmetric 2-$(7,3,1)$ design given by a set of points

$$\mathcal{P} = \{1,2,3,4,5,6,7\}$$

and a set of blocks

$$\mathcal{B} = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\}.$$

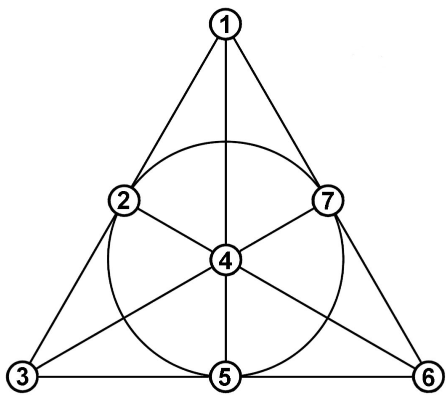

Figure 1.1: The Fano plane

This design is known as the Fano plane. As we can see in the picture, points of the design are the seven points of the plane, and blocks of the design are the six lines and a circle in the plane.

Definition 1.2.15. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a t-(v,k,λ) design with $\mathcal{P} = \{p_1, \ldots, p_v\}$ and $\mathcal{B} = \{B_1, \ldots, B_b\}$. The **incidence matrix** of the t-design $\mathcal{D}$ is the $b \times v$ matrix $M = \left[m_{i,j}\right]$ with entries 0 and 1 such that

$$m_{i,j} = \begin{cases} 1, & p_j \in B_i, \\ 0, & p_j \notin B_i. \end{cases}$$

Example 1.2.16. The 2-$(7,3,1)$ design presented in Example 1.2.14 has the following incidence matrix:

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

If $\mathcal{D}$ is a 2-(v,k,λ) design, then the incidence matrix M has precisely k entries equal to 1 in each row, r entries equal to 1 in each column, and any pair of distinct columns has λ common entries equal to 1. These conditions can be expressed in terms of matrix equations:

$$\begin{aligned} MJ &= kJ, \\ JM &= rJ, \\ M^T M &= (r-\lambda)I + \lambda J, \end{aligned}$$

where I denotes an identity matrix and J is a matrix with 1 in every entry, both of the appropriate size. The labels I and J for these matrices will be used throughout the dissertation.

1.3. GRAPHS

Graph theory is a branch of discrete mathematics that studies graphs, which are mathematical structures used to model pairwise relations between objects. The origins of graph theory can be found in 1735, when the Swiss mathematician L. Euler solved the Königsberg bridge problem. The solution was presented in [27], and it refers to the possibility of finding a walk over every one of seven bridges of Königsberg but without crossing any bridge twice. Since then, graph theory has become an important part of mathematics which has been applied to many problems in mathematics, computer science, and other scientific and not-scientific areas.

In this section, we give some basic definitions and properties for graphs and introduce a number of families of graphs which will be used in later chapters.

Definition 1.3.1. A finite incidence structure $G = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ is a **graph** if each element of $\mathcal{E}$ is incident with exactly two, not necessarily distinct, elements of $\mathcal{V}$. $\mathcal{V}$ is a non-empty set of elements called **vertices**, $\mathcal{E}$ is a set disjoint from $\mathcal{V}$ with elements called **edges**, and $\mathcal{I} \subseteq \mathcal{V} \times \mathcal{E}$ is called an **incidence relation**.

Definition 1.3.2. If $e = \{u, v\} = uv$ is an edge of G, then e is said to **join** the vertices u and v, and these vertices are said to be **adjacent**. In other words, e is **incident** to u and v, and v is a **neighbour** of u. Similarly, two edges of G incident to the same vertex are called **adjacent edges**.

Definition 1.3.3. A **loop** is an edge that joins a vertex to itself, *i.e.* an edge incident to a single vertex. Two or more edges joining the same pair of vertices are called **multiple edges**.

Definition 1.3.4. A graph with no loops and no multiple edges is called a **simple graph**.

Remark 1.3.5. In this dissertation, all graphs are simple graphs.

Definition 1.3.6. The **complement** of a graph G is the graph $\overline{G}$ with the same vertex set as G, where two vertices are adjacent if and only if they are not adjacent in G.

Definition 1.3.7. A **complete graph** is a simple graph whose each pair of vertices is adjacent. The complete graph on n vertices and $\frac{1}{2}n(n-1)$ edges is denoted by K_n.

Definition 1.3.8. A **bipartite graph** is one whose set of vertices can be partitioned into two sets in such a way that each edge joins a vertex of the first set to a vertex of the second set. A **complete bipartite graph** is a bipartite graph in which every vertex in the first set is adjacent to every vertex in the second set. If the two sets contain n and m vertices respectively, then the complete bipartite graph is denoted by $K_{n,m}$.

Definition 1.3.9. A graph $G = (\mathcal{V},\mathcal{E},\mathcal{I})$ is **isomorphic** to a graph $G' = (\mathcal{V}',\mathcal{E}',\mathcal{I}')$, denoted by $G \cong G'$, if and only if there exist bijections $f_V : \mathcal{V} \to \mathcal{V}'$ and $f_E : \mathcal{E} \to \mathcal{E}'$ that preserve the incidence in the sense that

$$(v,e) \in \mathcal{I} \;\Leftrightarrow\; (f_V(v), f_E(e)) \in \mathcal{I}', \quad \forall v \in \mathcal{V},\, \forall e \in \mathcal{E}.$$

A pair of functions (f_V, f_E) is called an **isomorphism** of graphs G and G'. An **automorphism** of a graph G is an isomorphism of G to itself. The automorphism group $\mathrm{Aut}(G)$ is the group of all automorphisms of a graph G.

Definition 1.3.10. Let $G = (\mathcal{V},\mathcal{E},\mathcal{I})$ be a graph with the vertex set $\mathcal{V} = \{v_1,\ldots,v_n\}$ and the edge set $\mathcal{E} = \{e_1,\ldots,e_m\}$. The **adjacency matrix** of G is the $n \times n$ matrix $A = \left[a_{i,j}\right]$ such that

$$a_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The **incidence matrix** of G is the $n \times m$ matrix $B = \left[b_{i,j}\right]$ such that

$$b_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ is incident to } e_j, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.3.11. Let $v \in \mathcal{V}$ be a vertex in a graph G. The **degree** (or **valency**) of v is the number of edges incident to v.

Definition 1.3.12. A graph G is called ***k*-regular** ($k \in \mathbb{N}_0$) if all the vertices of G have the same degree k.

Definition 1.3.13. Let G be a k-regular graph with v vertices. G is called a **strongly regular graph** with parameters (v,k,λ,μ) if any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours. A strongly regular graph with parameters (v,k,λ,μ) is denoted by $\mathrm{SRG}(v,k,\lambda,\mu)$.

Theorem 1.3.14. The complement of a strongly regular graph with parameters (v,k,λ,μ) is again strongly regular graph, with parameters

$$(v, v-k-1, v-2-2k+\mu, v-2k+\lambda).$$

The parameters of a strongly regular graph are not independent. Their relationship is described in the following theorem.

Theorem 1.3.15. The parameters (v,k,λ,μ) of a strongly regular graph satisfy the equation

$$k(k-\lambda-1) = (v-k-1)\mu.$$

The proofs of the previous two theorems can be found in [5].

Definition 1.3.16. Let $q \in \mathbb{N}$ and $n \in \mathbb{Z}$ such that $\gcd(q,n) = 1$. Then n is called a **quadratic residue** modulo q if and only if $x^2 \equiv n \pmod q$ has a solution. Likewise, n is called a **quadratic non-residue** modulo q if and only if $x^2 \equiv n \pmod q$ has no solution.

Definition 1.3.17. Let q be a prime power such that $q \equiv 1 \pmod 4$ and let $(\mathbb{F}_q^*)^2$ be a set of non-zero quadratic residues in $\mathbb{F}_q$. The **Paley graph** of order q, denoted by $P(q)$, has as vertex set the elements of a finite field of order q, *i.e.* $\mathcal{V} = \mathbb{F}_q$, and two vertices are adjacent if and only if their difference is in $(\mathbb{F}_q^*)^2$, which means $\mathcal{E} = \{\{u,v\} \mid u-v \in (\mathbb{F}_q^*)^2,\ u,v \in \mathbb{F}_q\}$.

Remark 1.3.18. The Paley graph $P(q)$ is strongly regular with parameters

$$\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right).$$

The Paley graph is *self-complementary*, which means the complement of any Paley graph is isomorphic to it.

Example 1.3.19. Let $P(13) = (\mathcal{V},\mathcal{E})$ be the Paley graph of order 13. Here we have

$$\mathcal{V} = \mathbb{Z}_{13} = \{0,1,2,3,4,5,6,7,8,9,10,11,12\} \text{ and } (\mathbb{Z}_{13}^*)^2 = \{1,3,4,9,10,12\}.$$

Each vertex $v \in \mathcal{V}$ is adjacent to 6 vertices: $v+i \pmod{13}$, $\forall i \in (\mathbb{Z}_{13}^*)^2$. It follows that

$$\mathcal{E} = \{\{v, v+i \pmod{13}\} \mid \forall v \in \mathbb{Z}_{13},\ \forall i \in (\mathbb{Z}_{13}^*)^2\}.$$

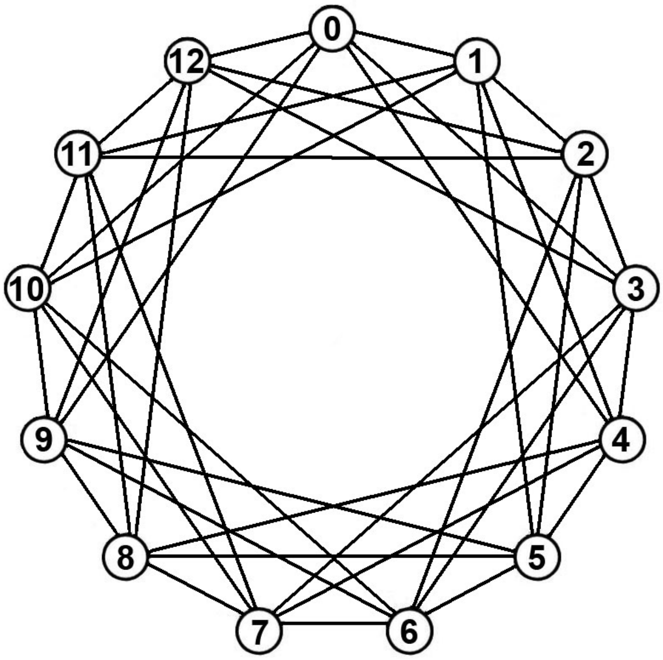

Figure 1.2: The Paley graph of order 13

The adjacency matrix of the Paley graph of order 13 is given by

$$
A = \begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}.
$$

Definition 1.3.20. A **walk** in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ is a non-trivial finite sequence of alternating vertices and edges such as

$$W = v_0 e_1 v_1 e_2 v_2 \dots e_k v_k,$$

where $v_0, \dots, v_k \in \mathcal{V}$, $e_1, \dots, e_k \in \mathcal{E}$ and vertices v_{i-1} and v_i are incident with the edge e_i for $i \in \{1, \dots, k\}$. A walk W is called a (v_0, v_k)-walk or a walk from v_0 to v_k, where v_0 and v_k are known as the endpoints.
The **length** of a walk is the number of edges in the walk. A walk is said to be **closed** if it has positive lenght and its endpoints are the same.

Definition 1.3.21. A **trail** is a walk with no repeated edges. A **path** is a trail in which all the vertices are distinct.

Definition 1.3.22. In a graph G two vertices u and v are said to be **connected** if there is a path between them. A graph in which each pair of vertices is connected is a **connected graph**.

Definition 1.3.23. The **distance** $d(u, v)$ between two vertices u and v of a finite graph G is the minimum length of the (u, v)-paths connecting them in G. If no such path exists, *i.e.* if the vertices lie in different connected components, then the distance is set equal to ∞.

Definition 1.3.24. The **diameter** $\operatorname{diam}(G)$ of a graph G is the maximal distance between any two vertices in G.

Definition 1.3.25. A **cycle** is a closed trail in which all the vertices are distinct, except for the first and the last which are identical.
A **cyclic graph** is a graph containing at least one cycle. A graph not containing any cycle is called an **acyclic graph**.

Definition 1.3.26. The **girth** of a graph G is the length of the shortest cycle contained in a given graph. If G is an acyclic graph, its girth is defined to be ∞.

Definition 1.3.27. A **cycle graph** or ***n*-cycle** C_n ($n \geq 3$) is a graph with n vertices $v_0, v_1, v_2, \dots, v_{n-1}$ and n edges $v_0v_1, v_1v_2, \dots, v_{n-1}v_0$ forming a cycle of length n.

1.4. Digraphs

Many problems can be presented with graphs, but sometimes this concept is not acceptable. The reason is that in some problems the relations between objects are not symmetric. For these cases we need directed graphs. In the previous chapter, we introduced undirected graphs. If we assign an orientation to each edge of the undirected graph, we obtain a directed graph. This is usually indicated with an arrow on the edge.

This section describes direct graphs and here we introduce tournaments, for us the most important example of direct graphs. Later, they will have an important role in the construction of LCD codes. For more information about digraphs, we refer the reader to [3].

Definition 1.4.1. A **directed graph** or a **digraph** $G_D = (\mathcal{V}_D, \mathcal{E}_D)$ is a graph in which edges have a direction. It consists of two finite sets: a non-empty set $\mathcal{V}_D$ of **vertices** and a set $\mathcal{E}_D \subseteq \mathcal{V}_D \times \mathcal{V}_D$ of **directed edges** or **arcs**, where each arc is associated with an ordered pair of vertices called its **endpoints**.

Definition 1.4.2. If e is an edge of G_D associated with the pair of vertices (u,v), then e is said to be the directed edge from **initial vertex** u to **terminal vertex** v. In this case, u is **adjacent to** v or u **dominates** v and v is an **out-neighbour** of u. Similarly, v is **adjacent from** u or v **is dominated** by u and u is an **in-neighbour** of v.

Definition 1.4.3. An arc is a **loop** if its endpoints are equal, *i.e.* if it directly connects vertex to itself. **Multiple arcs** are arcs with identical initial and terminal vertices.

Definition 1.4.4. A **simple directed graph** is a directed graph that has no loops and no multiple arcs.

Definition 1.4.5. An arc (u,v) is called the **inverted arc** of (v,u). A pair of inverse arcs (u,v) and (v,u) is called a **symmetric pair** of arcs.

Definition 1.4.6. A **complete directed graph** is a simple directed graph where each pair of vertices is joined by a symmetric pair of arcs.

Remark 1.4.7. A complete directed graph is equivalent to an undirected complete graph with the edges replaced by pairs of inverse arcs.

Definition 1.4.8. A **symmetric directed graph** is a directed graph where for every arc that belongs to the digraph, the corresponding inversed arc also belongs to it.

Definition 1.4.9. A directed graph is called an **oriented graph** (or **asymmetric directed graph**) if none of its pairs of vertices is linked by a pair of symmetric arcs.

Remark 1.4.10. In this dissertation, by a directed graph we refer to a simple oriented graph.

Definition 1.4.11. Let $G_D = (\mathcal{V}_D, \mathcal{E}_D)$ be a directed graph with the vertex set $\mathcal{V}_D = \{v_1, \dots, v_n\}$. The **adjacency matrix** of G_D is the $n \times n$ matrix $A = \left[a_{i,j}\right]$ such that

$$a_{i,j} = \begin{cases} 1, & \text{if } (v_i, v_j) \in \mathcal{E}_D, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.4.12. Let $v \in \mathcal{V}_D$ be a vertex in a directed graph G_D. The set of out-neighbours of v is the set $N^+(v) = \{u \in \mathcal{V}_D \mid (v,u) \in \mathcal{E}_D\}$, and the **out-degree** (or **out-valency** or **score**) $d^+(v)$ of the vertex v is the number of vertices that are dominated by v, *i.e.* $d^+(v) = |N^+(v)|$. The set of in-neighbours of v is the set $N^-(v) = \{u \in \mathcal{V}_D \mid (u,v) \in \mathcal{E}_D\}$, and the **in-degree** (or **in-valency** or **co-score**) $d^-(v)$ of the vertex v is the number of vertices that dominate v, *i.e.* $d^-(v) = |N^-(v)|$.

Definition 1.4.13. A vertex with $d^-(v) = 0$ is called a **source**, as it is the origin of each of its outcoming arcs. Similarly, a vertex with $d^+(v) = 0$ is called a **sink**, since it is the end of each of its incoming arcs. If a vertex is neither a source nor a sink, it is called an **internal vertex**.

Proposition 1.4.14. Let G_D be a directed graph. The *degree sum formula* states that

$$\sum_{v \in \mathcal{V}_d} d^+(v) = \sum_{v \in \mathcal{V}_d} d^-(v) = |\mathcal{E}_D|$$

The proof can be found in [3].

Definition 1.4.15. A directed graph G_D is said to be **regular of degree *k*** (or ***k*-regular**) if each vertex dominates k vertices and is dominated by k vertices. In other words, G_D is a k-regular directed graph if every vertex in G_D has in-degree and out-degree k.

Definition 1.4.16. A **tournament** $T = (\mathcal{V}_T, \mathcal{E}_T)$ of order n or an ***n*-tournament** is a directed graph where the vertex set $\mathcal{V}_T$ consists of n elements and the edge set $\mathcal{E}_T \subset \mathcal{V}_T \times \mathcal{V}_T$ such that each pair of vertices u and v is joined by exactly one of the arcs (u,v) or (v,u).

Remark 1.4.17. Tournaments are oriented graphs obtained by choosing a direction for each edge in undirected complete graphs.

In terms of the adjacency matrix A, an n-tournament is a directed graph with the property that $A + A^T = J - I$. Thus, $J - I$ is the adjacency matrix of a complete graph on n vertices.

Remark 1.4.18. Let T be a k-regular tournament with v vertices. If every vertex in $\mathcal{V}_T$ has out-degree k, then every vertex in $\mathcal{V}_T$ has in-degree $v-k-1$. It follows $k = v-k-1$, *i.e.* $v = 2k+1$. This implies that v must be an odd number.

Definition 1.4.19. Let T be a k-regular tournament with v vertices. T is called a **doubly regular tournament** with parameters (v,k,λ,μ) if any two adjacent vertices have λ common out-neighbours and each of these two vertices has additional μ out-neighbours which are not common to them.
A doubly regular tournament with parameters (v,k,λ,μ) is denoted by DRT(v,k,λ,μ).

For a DRT, relations among the parameters are given in the following lemma, which can be found in [26].

Lemma 1.4.20. If T is a DRT with parameters (v,k,λ,μ), then $v = 4\lambda+3$, $k = 2\lambda+1$ and $\mu = \lambda+1$.

Definition 1.4.21. Let q be a prime power such that $q \equiv 3 \pmod 4$ and let $(\mathbb{F}_q^*)^2$ be a set of non-zero quadratic residues in $\mathbb{F}_q$. The **Paley tournament** of order q, denoted by $P_T(q)$, has as vertex set the elements of a finite field of order q, *i.e.* $\mathcal{V}_T = \mathbb{F}_q$, and two vertices are adjacent if and only if their difference is in $(\mathbb{F}_q^*)^2$, which means $\mathcal{E}_T = \{(u,v) \mid u-v \in (\mathbb{F}_q^*)^2,\ u,v \in \mathbb{F}_q\}$. The congruence condition implies that -1 is not a square in $\mathbb{F}_q$, so for each pair (u,v) of distinct elements of $\mathbb{F}_q$, either $u-v$ or $v-u$ is a square, but not both.

Remark 1.4.22. The Paley tournament $P_T(q)$ is doubly regular with parameters

$$\left(q, \frac{q-1}{2}, \frac{q-3}{4}, \frac{q+1}{4}\right).$$

Example 1.4.23. Let $P_T(7) = (\mathcal{V}_T, \mathcal{E}_T)$ be the Paley tournament of order 7. Here we have

$$\mathcal{V}_T = \mathbb{Z}_7 = \{0,1,2,3,4,5,6\} \text{ and } (\mathbb{Z}_7^*)^2 = \{1,2,4\}.$$

Each vertex $v \in \mathcal{V}_T$ is adjacent to 3 vertices: $v + i \pmod 7$, $\forall i \in (\mathbb{Z}_7^*)^2$. It follows that

$$\mathcal{E}_T = \{(v, v+i \pmod 7) \mid \forall v \in \mathbb{Z}_7,\ \forall i \in (\mathbb{Z}_7^*)^2\}.$$

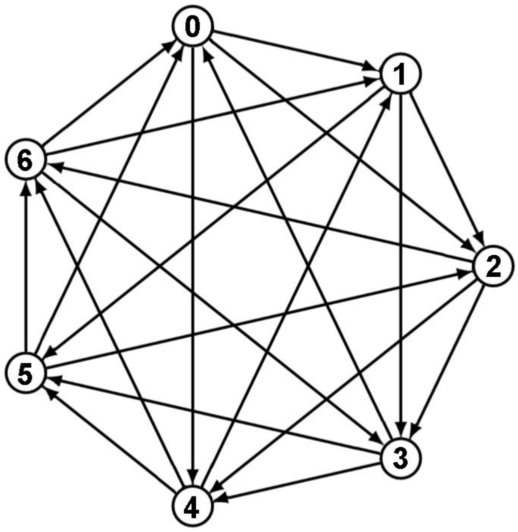

Figure 1.3: The Paley tournament of order 7

The adjacency matrix of the Paley tournament of order 7 is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

1.5. CODES

Coding theory or the theory of error-correcting codes is a branch of discrete mathematics which deals with efficient and reliable data transmission from the sender to the receiver through a noisy communication channel as well as correcting errors that occur during the transmission. In other words, coding theory is used for making messages easy to read, which is the opposite of cryptography. In the process of communication the message is first encoded, which means converted to an error-correcting code by adding redundancy. Then the code is sent through the channel. After accepting the message, the recevier tries to decode the message so that it resembles the original message as close as possible. The redundancy allows the receiver to detect a limited number of errors that may occur anywhere in the message. The number of errors generated in the transmission will depend on how good the code is in relation to the channel.

In 1948 C. Shannon published “A mathematical theory of communication” [54], that signified the beginning of both information theory and coding theory. Given a communication channel which may corrupt information sent over it, Shannon identified a number called the capacity of the channel and proved that arbitrarily reliable communication is possible at any rate below the channel capacity. Shannon’s results guarantee that the data can be encoded before transmission so that the altered data can be decoded to the specified degree of accuracy. No specific codes were produced in the proof that give the desired accuracy for a given channel. Shannon’s theorem only guarantees their existence. The goal of research in coding theory is to produce codes that fulfill the conditions of Shannon’s theorem [35].

Linear codes have been studied more than non-linear codes. Because of their algebraic structure, they are easier to describe, encode and decode. An important class of linear codes are self-dual codes. One of the reasons is their close connection to other mathematical structures such as block designs. Another important class of linear codes are LCD codes, *i.e.* linear codes with complementary duals. The construction of these two types of linear codes is the main task of this dissertation. Below, we outline the basic concepts in coding theory.

Definition 1.5.1. Let F_q be a finite set called an **alphabet** containing q elements called

symbols. A ***q*-ary code** C of length n over F_q is a subset C of the set F_q^n of all n-letter words with components from F_q. The elements in C are called **codewords**. A code is **binary** if $q = 2$, **ternary** if $q = 3$, and **quaternary** if $q = 4$.

Linear codes, because of their algebraic properties, are the most studied codes from a mathematical point of view. In this dissertation, we will work only with linear codes.

Definition 1.5.2. Let $\mathbb{F}_q$ be a finite field of order q, where q is a prime power. An $[\mathbf{n}, \mathbf{k}]$ **linear code** C of length n and rank k is a k-dimensional subspace of the n-dimensional vector space $\mathbb{F}_q^n$. The **size** of a linear code is the number of codewords and equals q^k.

Definition 1.5.3. The **Hamming distance** $d(x,y)$ between two vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$ is the number of coordinate positions in which x and y differ, *i.e.*

$$d(x,y) = |\{i \mid x_i \neq y_i,\ 1 \leq i \leq n\}|.$$

Definition 1.5.4. The **minimum distance** d of a code C is the smallest of the Hamming distances between distinct codewords from C, *i.e.*

$$d = \min\{d(x,y) \mid x, y \in C,\ x \neq y\}.$$

Definition 1.5.5. The **Hamming weight** $w(x)$ of a vector $x \in \mathbb{F}_q^n$ is the number of its non-zero components, *i.e.*

$$w(x) = |\{i \mid x_i \neq 0\}|.$$

Definition 1.5.6. The **minimum weight** $w(C)$ of a code C is the smallest among the weights of all non-zero codewords in C, *i.e.*

$$w(C) = \min\{w(x) \mid x \in C,\ x \neq 0\}.$$

Remark 1.5.7. From previous definitions follows $d(x,y) = w(x-y)$. In particular, $w(x) = d(x,0)$, where 0 is the all-zero vector.

In general, finding the minimum distance of a code requires comparing every pair of distinct elements. This is not necessary for a linear code. If codewords x and y belong to a linear code C, $x - y$ is also a codeword in C. Now remark 1.5.7 implies the following result [45].

Proposition 1.5.8. The minimum distance of a linear code is equal to the minimal weight among all non-zero codewords.

We use the notation $[n,k,d]$ for a linear $[n,k]$ code with minimum distance, or equivalently, minimum weight d.

A code can be thought as a collection of messages that are being transmitted over a communication channel. If the channel is subject to noise, some of the components of a message $x = (x_1,\ldots,x_n) \in C$ may be corrupted. Thus, the received message $y = (y_1,\ldots,y_n)$ may differ from x and the distance $d(x,y)$ counts the number of errors in y. The process of recovering the original message x from the received message y is called *decoding*.

The following result shows the importance of the concept minimum distance and it can be found in [2].

Theorem 1.5.9. A linear $[n,k,d]$ code can detect at most $d-1$ errors in one codeword and correct at most $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ errors.

Definition 1.5.10. Two linear codes are **equivalent** if one can be obtained from the other by permuting the coordinate positions in all the codewords and multiplying any specific coordinate by a non-zero field element.
Two linear codes are **isomorphic** if a permutation of the coordinate positions suffices to take one code to the other.
Any isomorphism of a code C onto itself is called an **automorphism** of C. The set of all automorphisms of C is the **automorphism group** of C and it is denoted by $\mathrm{Aut}(C)$.

Two most common ways to present a linear code are with either a generator matrix or a parity check matrix.

Definition 1.5.11. A **generator matrix** G of a linear $[n,k,d]$ code C is a $k \times n$ matrix obtained from any k linearly independent codewords of C which form a basis for code C. A generator matrix of the form $G = [I_k \mid A]$, where I_k is the identity matrix of order k and A is a $k \times (n-k)$ matrix, is called a **generator matrix in standard form**.

Remark 1.5.12. Every linear code is equivalent to a code with a generator matrix in standard form.

Definition 1.5.13. The **inner product** of vectors $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$ is given by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

Definition 1.5.14. Given a linear $[n,k,d]$ code $C \subset \mathbb{F}_q^n$, the **dual code** $C^\perp \subset \mathbb{F}_q^n$ is the set of all codewords which are orthogonal to every codeword in C, *i.e.*

$$C^\perp = \{x \in \mathbb{F}_q^n \mid x \cdot y = 0,\ \forall y \in C\}.$$

A code C is **self-orthogonal** if $C \subseteq C^\perp$ and **self-dual** if $C = C^\perp$.

Remark 1.5.15. If C is a linear $[n,k]$ code, then $C^\perp$ is a linear $[n,n-k]$ code.

Proposition 1.5.16. Let C be a linear code of length n. Then

$$\dim(C) + \dim(C^\perp) = n.$$

From this proposition, the proof of which can be found in [2], follows that the length n of a self-dual code is even and the dimension is $n/2$.

The following result shows that we can easily determine if a code is self-orthogonal by studying its generator matrix.

Lemma 1.5.17. A linear code C with generator matrix G is self-orthogonal if and only if $GG^T = 0$.

Hence, to show that a linear $[n,k]$ code C is self-dual, we verify that C has length $n = 2k$ and that its generator matrix G satisfies $GG^T = 0$. From this we can easily obtain one more result.

Lemma 1.5.18. If C is a self-dual $[n,k]$ code over a field $\mathbb{F}_q$ with generator matrix $G = [I_k \mid B]$, then $BB^T = -I_k$.

Proofs of the two preceding lemmas can be found in [61].

Definition 1.5.19. Let w_i denote the number of codewords of weight i in a linear code C of length n. Then the **weight distribution** of C is the list $[w_i \mid 0 \leq i \leq n]$.

Definition 1.5.20. A code C is called **even** if the weights of all codewords of C are even.

Definition 1.5.21. A code C is **formally self-dual** if C and $C^{\perp}$ have identical weight distributions.

In [39], it is shown that formally self-dual even $[2n,n]$ codes may occasionally have a larger minimum weight than any $[2n,n]$ self-dual code.

Definition 1.5.22. A **parity-check matrix** H of a linear code C is a generator matrix of the dual code $C^{\perp}$.

If H is a parity-check matrix for a linear code C, the codewords of C can be recovered from H because they must be orthogonal to every row of H. Thus, the code C given by the parity-check matrix H is as follows

$$C = \{x \in \mathbb{F}_q^n \mid H \cdot x^T = 0\}.$$

Theorem 1.5.23. Let H be a parity-check matrix for a linear $[n,k]$ code C. Then every set of $d-1$ columns of H is linearly independent if and only if C has minimum distance at least d.

It follows from the theorem, the proof of which can be found in [2], that a linear code C with a parity-check matrix H has minimum distance d if and only if every set of $d-1$ columns of H is linearly independent and some set of d columns is linearly dependent. Hence, this theorem could be used to determine the minimum distance of a linear code, given a parity-check matrix.

When working with linear codes it is often desirable to be able to convert from the generator matrix to the parity-check matrix and vice-versa. This can easily be done using the following theorem. The proof can be found in [35].

Theorem 1.5.24. If $G = [I_k \mid A]$ is a generator matrix for an $[n,k]$ code C in standard form, then $H = \left[-A^T \mid I_{n-k}\right]$ is a parity check matrix for C.

Example 1.5.25. The *Hamming codes* are an important family of linear codes which are easy to encode and decode. A *binary Hamming code* $\mathcal{H}_r$ of length $n = 2^r - 1$ $(r \geq 2)$ has parity check matrix H whose columns consist of all non-zero binary vectors of length r, each used once. $\mathcal{H}_r$ is a $[2^r-1,\ 2^r-r-1,\ 3]$ code. Adding an overall parity check coordinate to each vector of its generator matrix and thus to each codeword of $\mathcal{H}_r$ gives

the *extended binary Hamming code* $\hat{\mathcal{H}}_r$, which is a $[2^r,\ 2^r - r - 1,\ 4]$ code.
For example, $\hat{\mathcal{H}}_3$ is a $[8,4,4]$ binary Hamming code with generator matrix G and parity check matrix H given by

$$G = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right], \quad H = \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

It is easy to verify that $GG^T = 0$, so this code is self-dual.

The following theorem describes the relationship between the parameters of a linear code which is known as the *Singleton bound*. The proof can be found in [45].

Theorem 1.5.26. If C is a linear $[n,k,d]$ code, then $d \leq n-k+1$.

The Singleton bound is one of the upper bounds on d for given parameters n and k. For more examples of upper bounds for minimum distance we refer to [35].

Definition 1.5.27. An $[n,k]$ linear code C is **optimal** if the minimum weight of C achieves a theoretical upper bound on the minimum weight of $[n,k]$ linear codes. C is **near-optimal** if its minimum weight is at most 1 less than the largest possible value.
An $[n,k]$ linear code C is said to be a **best known** linear $[n,k]$ code if C has the highest minimum weight among all known $[n,k]$ linear codes.

A catalogue of best known codes is maintained at [30], to which we compare the minimum weight of all codes constructed in this dissertation.

1.5.1. LCD codes

In this chapter, we define LCD codes, which are the main interest of this dissertation.

Definition 1.5.28. A **linear code with a complementary dual** (or an **LCD code**) is a linear code C whose dual code $C^\perp$ satisfies $C \cap C^\perp = \{0\}$.

It other words, C is an LCD code when $\mathbb{F}_q^n = C \oplus C^\perp$, *i.e.* when every vector in $\mathbb{F}_q^n$ can be written uniquely as the sum of a vector in C and a vector in $C^\perp$. This follows from the fact that the $n-k$ vectors in a basis for $C^\perp$, when adjoined to a basis for C, yield a set of n linearly independent vectors (and hence a basis for $\mathbb{F}_q^n$) if and only if $C \cap C^\perp = \{0\}$.

Remark 1.5.29. If C is an LCD code, then $C^{\perp}$ is also an LCD code because $(C^{\perp})^{\perp} = C$.

The following two lemmas characterize LCD codes, and the proofs can be found in [25,46].

Lemma 1.5.30. Let $u_1, u_2, \ldots, u_k$ be vectors over a commutative ring R of characteristic 2, such that $u_i \cdot u_i = 1$ for each $i = 1, \ldots, k$ and $u_i \cdot u_j = 0$ for $i \neq j$. Then the $n \times k$ matrix $G = [u_i]_{1 \leq i \leq k}$ generates an LCD code over R.

Lemma 1.5.31. Let G be a generator matrix for a code over a field. Then $\det(GG^{\top}) \neq 0$ if and only if G generates an LCD code.

Furthermore, in [46] Massey showed that the *nearest-codeword* (or *maximum-likelihood*) decoding problem for an LCD code may be simpler than that for a general linear code. The decoding method proposed by Massey is based on the following theorem.

Theorem 1.5.32. Let C be an LCD code of length n over the field $\mathbb{F}_q$, and let φ be a map $\varphi : C^{\perp} \to C$ such that $u \in C^{\perp}$ maps to one of the closest codewords v to it in C. Further, let Π_C and $\Pi_{C^{\perp}}$ be the orthogonal projectors from $\mathbb{F}_q^n$ onto C and $C^{\perp}$, respectively. Then the map $\tilde{\varphi} : \mathbb{F}_q^n \to C$ such that

$$\tilde{\varphi}(w) = \Pi_C(w) + \varphi(\Pi_{C^{\perp}}(w))$$

maps each $w \in \mathbb{F}_q^n$ to one of it closest neighbours in C.

1.6. ASSOCIATION SCHEMES

Association schemes are the most important unifying concept in algebraic combinatorics. They provide a common viewpoint for the treatment of problems in several fields, such as coding theory, design theory, algebraic graph theory and finite group theory [12]. The theory of association schemes first appeared in statistics, in the theory of experimental design. It was introduced by R.C. Bose and T. Shimamoto [9]. The most important contribution to the theory was given by P. Delsarte [22], who applied association schemes to coding theory. He showed that association schemes can be used for problems ranging from error-correcting codes to combinatorial designs. To study permutation groups, D. G. Higman [33] generalized association schemes further to the theory of coherent configurations.

An association scheme is a set with relations defined on it satisfying certain properties. Following the approach from [22, 33] we define an association scheme with d classes.

Definition 1.6.1. Let X be a finite set of size $v \geq 2$ and d is a positive integer. A ***d*-class association scheme** on X is a sequence of $d+1$ binary relations $R_0, R_1, \ldots, R_d$ defined on X which satisfy:

1. $X \times X = R_0 \cup R_1 \cup \ldots \cup R_d$, $R_i \cap R_j = \emptyset$ for $i \neq j$,
2. $R_0 = \{(x,x) \mid x \in X\}$,
3. for every $i \in \{0,1,\ldots,d\}$, there exists $j \in \{0,1,\ldots,d\}$ such that $R_i^T = R_j$, where $R_i^T = \{(y,x) \mid (x,y) \in R_i\}$,
4. for any triple i, j, k, the number of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant p_{ij}^k which does not depend on the choice of x and y that satisfy $(x,y) \in R_k$,
5. $p_{ij}^k = p_{ji}^k$ for all $i, j, k \in \{0,1,\ldots,d\}$.

The relations $R_0, R_1, \ldots, R_d$ are called the **associate classes** of the scheme. Two points x and y are called ***i*th associates** if $(x,y) \in R_i$. The numbers p_{ij}^k are called the **intersection numbers** of the association scheme.

The definition states that the set of relations $R_0, R_1, \ldots, R_d$ is a partition of $X \times X$, which means that every pair of points are ith associates for exactly one i. Each point is its own zeroth associate while distinct points are never zeroth associates. For each R_i, there is a unique R_i^T for which $(x,y) \in R_i$ if and only if $(y,x) \in R_i^T$. If x and y are kth associates, then the number of points z which are both ith associates of x and jth associates of y is a constant p_{ij}^k. An association scheme is commutative, *i.e.* for any triple of indices i, j, k, $p_{ij}^k = p_{ji}^k$.

A **symmetric association scheme** is one in which $R_i^T = R_i$ for each $i \in \{1, \ldots, d\}$. In this case, an association scheme can be visualized as a complete graph with labeled edges. The graph has v vertices, one for each point of X. The edge joining vertices x and y is labeled i if x and y are ith associates. Each edge has a unique label, and the number of triangles with a fixed base labeled k having the other two edges labeled i and j is a constant p_{ij}^k, depending on i, j, k but not on the choice of the base. In particular, each vertex is incident with exactly $p_{ii}^0 = v_i$ edges labeled i, and the number v_i is called the *valency* of the relation R_i. It follows that for each $i \in \{1, \ldots, d\}$, (X, R_i) is a simple graph which is regular of degree v_i.

The relations $R_i,\ i = 0, 1, 2, \ldots, d$, of an association scheme can be described by their adjacency matrices $A_i,\ i = 0, 1, 2, \ldots, d$, whose rows and columns are indexed by the elements of X and whose entries satisfy

$$(A_i)_{x,y} = \begin{cases} 1, & \text{if } (x,y) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1.6.2. Let $A_i,\ i = 0, 1, 2, \ldots, d$, be the adjacency matrices of relations $R_i,\ i = 0, 1, 2, \ldots, d$. Then the axioms of the association scheme are equivalent to:

1. $\sum\limits_{i=0}^{d} A_i = J$,
2. $A_0 = I$,
3. for every $i \in \{0, 1, \ldots, d\}$, there exists $j \in \{0, 1, \ldots, d\}$ such that $A_i^T = A_j$,
4. $A_i A_j = \sum\limits_{k=0}^{d} p_{ij}^k A_k$,
5. $A_i A_j = A_j A_i$ for all $i, j \in \{0, 1, \ldots, d\}$.

The adjacency matrices $A_i,\ i = 0,1,2,\ldots,d$, generate a $(d+1)$-dimensional commutative and associative algebra $\mathcal{A}$ (over the real or complex numbers). This algebra was first studied by R. C. Bose and D. M. Mesner [8] and is called the **Bose-Mesner algebra** of the association scheme.

Two important classes of association schemes are the Hamming scheme and the Johnson scheme. The Hamming scheme $H(n,q)$ is defined on the set F_q^n of all words of length n (ordered n-tuples) over an alphabet F of q symbols. Two words are ith associates if and only if the Hamming distance between them is i. The Johnson scheme $J(n,k)$, with $k \le \frac{1}{2}n$, is defined on the set of all k-subsets of a set of size n. Two k-sets X and Y are ith associates if and only if $|X \cap Y| = k - i$. More about these association schemes can be found in [15,45].

In this dissertation, we are interested in two class association schemes, *i.e.* the case $d = 2$. Let $A_0 = I, A_1$ and A_2 be the adjacency matrices of a two class association scheme. As stated in [26], two cases may occur. Either $A_1^T = A_1$ and $A_2^T = A_2$ in which case the undirected graph (X,R_1) is a strongly regular graph with parameters $(v,\ k := p_{11}^0,\ \lambda := p_{11}^1,\ \mu := p_{11}^2)$ or $A_1^T = A_2$ and $A_2^T = A_1$ in which case the directed graph (X,R_1) is a doubly regular tournament with parameters $(v,\ k := p_{12}^0,\ \lambda := p_{11}^1,\ \mu := p_{11}^2)$. In both cases (SRGs and DRTs) we have that $A_2 = J - I - A_1$. Let $\overline{A}_1 = A_2$ and $A = A_1$. Then the matrix A satisfies $AJ = JA = kJ$. In the first case, for SRGs we have

$$A^2 = kI + \lambda A + \mu(J - I - A)$$

and in the second case, for DRTs we have

$$A^2 = \lambda A + \mu(J - I - A).$$

Another characteristic of SRGs and DRTs is given in the following lemma, the proof of which can be found in [26].

Lemma 1.6.3. If G is an SRG, we have

$$AA^T = A^2 = kI + \lambda A + \mu\overline{A}.$$

If G is a DRT, we have

$$AA^T = kI + (k - 1 - \lambda)A + (k - \mu)\overline{A}.$$

2. CONSTRUCTIONS OF SELF-DUAL CODES

In this chapter, we introduce constructions of self-dual codes from two class association schemes. These constructions were given by S. T. Dougherty, J.-L. Kim and P. Solé in [26]. In fact, they are a generalization of P. Gaborit's constructions of quadratic double circulant codes which were presented in [28]. Following a similar approach to that of Dougherty, Kim and Solé, we will give constructions of LCD codes from two class association schemes in the next chapter. For a better understanding of the constructions of LCD codes, we briefly describe both of the above-mentioned constructions here. We also apply the specified constructions to give examples of self-dual codes obtained from Paley designs and Paley graphs over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$. Working over $\mathbb{F}_{p^n}$, we interpret integers as their value modulo p.

2.1. QUADRATIC DOUBLE CIRCULANT CODES

Gaborit studied known constructions of double circulant codes which use quadratic residues and provided a general scheme for constructing them over any field. The codes obtained with these constructions are called *quadratic double circulant codes* or QDC codes. Here we present the results given by Gaborit in [28].

Let R be a commutative ring containing 1 and $r, s, t \in R$ are arbitrary scalars. Further, let q be a power of an odd prime number and let a be a one-to-one mapping of $\{0,1,\ldots,q-1\} \cup \{\infty\}$ onto $GF(q) \cup \{\infty\}$ such that $a(0) = 0$, $a(1) = 1$ and $a(\infty) = \infty$. Then the inverse mapping a^{-1} is a one-to-one mapping from $GF(q) \cup \{\infty\}$ onto $\{0,1,\ldots,q-1\} \cup \{\infty\}$. If q is a prime, the elements of $GF(q)$ can be represented as integers, *i.e.* $a(0)=0$, $a(1)=1$, $a(2)=2$, $\dots$, $a(q-1)=q-1$ and we can let a be the identity. Now we define the matrix $Q_q(r,s,t)=[q_{i,j}]$ to be the $q\times q$ matrix over R with rows and columns labeled by the elements of $GF(q)$: $a_0=0,\ a_1=1,\ a_2=a(2),\ \dots\ ,\ a_{q-1}=a(q-1)$. The entries $q_{i,j}$ are defined by the indicator function χ of the quadratic residues of $GF(q)$ such that

$$\chi(a_i)=\begin{cases} r, & \text{if } a_i=a_0=0,\\ s, & \text{if } a_i \text{ is a quadratic residue in } GF(q),\\ t, & \text{if } a_i \text{ is a quadratic non-residue in } GF(q),\end{cases}$$

and we let $q_{i,j}=\chi(a_j-a_i)$. In the case where q is a prime, we have $a_0=0,\ a_1=1,\ a_2=2,\ \dots\ ,\ a_{q-1}=q-1$ and this leads to a circulant $Q_q(r,s,t)$ matrix. Further, we define $Q=Q_q(0,1,0)$ to be the matrix of quadratic residues not including 0 and $N=Q_q(0,0,1)$ to be the matrix of quadratic non-residues also not including 0. From this follows $Q_q(r,s,t)=rI+sQ+tN$.

We define two constructions of quadratic double circulant codes over R. In the first construction, a code is given by a double circulant generator matrix

$$P_q(r,s,t)=\left[\begin{array}{c|c} I & Q_q(r,s,t)\end{array}\right]$$

and this form is called *pure quadratic double circulant form*. In the second construction, for any scalars $\alpha,\beta,\gamma\in R$, a code is given by a double circulant generator matrix

$$B_q(r,s,t)=\left[\begin{array}{c|c|c|c} 1 & 0\dots0 & \alpha & \beta\dots\beta\\ \hline 0 & & \gamma & \\ \vdots & I & \vdots & Q_q(r,s,t)\\ 0 & & \gamma & \end{array}\right]$$

and this form is called *bordered quadratic double circulant form*. In the case where $R=GF(p)$, for p a power of a prime, the two codes are respectively $[2q,q]$ and $[2q+2,q+1]$ codes over $GF(p)$.

In the case where q is not a prime, the matrix $Q_q(r,s,t)$ is not circulant. It corresponds to the case of generalized quadratic residue codes which can be found in [44]. Regardless, for simplicity's sake, Gaborit kept the same name for his codes. The following theorem represents Gaborit's main result.

Theorem 2.1.1. Let q be a power of an odd prime, and let $Q_q(r,s,t)$ be a quadratic residue circulant matrix with r, s and t elements of the ring R.
If $q = 4l+1$, then

$$\begin{aligned} Q_q(r,s,t)Q_q(r,s,t)^T &= (r^2+2l(s^2+t^2))I \\ &+ (2rs-s^2+l(s+t)^2)Q \\ &+ (2rt-t^2+l(s+t)^2)N. \end{aligned}$$

If $q = 4l+3$, then

$$\begin{aligned} Q_q(r,s,t)Q_q(r,s,t)^T &= (r^2+(2l+1)(s^2+t^2))I \\ &+ (rs+rt+l(s^2+t^2)+(2l+1)st)Q \\ &+ (rs+rt+l(s^2+t^2)+(2l+1)st)N. \end{aligned}$$

The proof can be found in [28]. Using the results of the preceding theorem with $R = GF(p)$, for p a power of a prime, Gaborit constructed numerous self-dual codes over $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{F}_5, \mathbb{F}_7$ and $\mathbb{F}_9$.

2.2. Self-dual codes from two class association schemes

Dougherty, Kim and Solé [26] gave constructions of double circulant codes from two class association schemes, generalizing Gaborit's constructions of quadratic double circulant codes. The pure and the bordered constructions given by Gaborit correspond to their constructions, so they have used the same terms. They also gave general self-duality conditions valid over any commutative alphabet ring. Here we outline the basics of constructions from [26].

Definition 2.2.1. Let R be a finite commutative ring with identity. A **linear code** of length n over R is an R-submodule of R^n. A linear code over R is **free** if it has rank r and $|R|^r$ elements.

Remark 2.2.2. If R is a field, then all linear codes are free.

Let A be the adjacency matrix of a two class association scheme, as defined in Section 1.6. A is the adjacency matrix of a graph G with v vertices, degree k and parameters λ and μ. The above authors considered two cases. In the first case, G is an SRG and $A^T = A$. In the second case, G is a DRT and $A^T = \overline{A} = J - I - A$.

Let $r,s,t \in R$ and let $Q_R = (rI + sA + t\overline{A})$. Further, let P_R be a $v \times 2v$ matrix over R and suppose

$$P_R = \left[\begin{array}{c|c} I & Q_R(r,s,t) \end{array}\right]$$

generates a $[2v, v]$ free code over R. Lastly, let B_R be a $(v+1) \times (2v+2)$ matrix over R with scalars $\alpha,\ \beta,\ \gamma \in R$, and suppose

$$B_R = \left[\begin{array}{c|c|c|c} 1 & 0\ldots 0 & \alpha & \beta\ldots\beta \\ \hline 0 & & \gamma & \\ \vdots & I & \vdots & Q_R(r,s,t) \\ 0 & & \gamma & \end{array}\right]$$

generates a $[2v+2, v+1]$ free code over R.

We want to show that codes generated by matrices P_R and B_R are self-dual. In order to do that, it suffices to show that these codes are self-orthogonal. For the code $P_R(r,s,t)$ to be self-orthogonal, we need $P_R P_R^T = \mathbf{0}$, *i.e.*

$$Q_R(r,s,t)Q_R(r,s,t)^T = -I. \tag{2.1}$$

For the code $B_R(r,s,t)$ to be self-orthogonal, we need $B_R B_R^T = \mathbf{0}$, *i.e.*

$$\begin{aligned} 1+\alpha^2+v\beta^2 &= 0 \\ \alpha\gamma+\beta(r+sk+t(v-k-1)) &= 0 \\ I+\gamma^2 J+Q_R(r,s,t)Q_R(r,s,t)^T &= \mathbf{0}. \end{aligned} \tag{2.2}$$

The first equation is the inner product of the top row with itself. The second equation is the inner product of the top row with any other row. The third equation indicates that all the other rows are orthogonal to each other. In both cases we need to compute the product $Q_R(r,s,t)Q_R(r,s,t)^T$, which is done in the following lemma.

Lemma 2.2.3. For SRGs we have

$$\begin{aligned} Q_R(r,s,t)Q_R(r,s,t)^T &= (r^2+s^2k-t^2-t^2k+t^2v)I \\ &+ (2rs+s^2\lambda-2st-2st\lambda+t^2\lambda+2stk+t^2v-2t^2k)A \\ &+ (2rt+s^2\mu-2st\mu+t^2\mu+2stk+t^2v-2t^2-2t^2k)\overline{A}. \end{aligned}$$

For DRTs we have

$$\begin{aligned} Q_R(r,s,t)Q_R(r,s,t)^T &= (r^2+(s^2+t^2)k)I \\ &+ (rt+sr+(s^2+t^2)(k-1-\lambda)+st\lambda+st\mu)A \\ &+ (rt+sr+(s^2+t^2)(k-\mu)+st\mu+st\lambda)\overline{A}. \end{aligned}$$

From all the above follow the main results in the form of two theorems, and all the proofs can be found in [26].

Theorem 2.2.4. The code $P_R(r,s,t)$ formed from an SRG(v,k,λ,μ) is Euclidean self-dual code over R if and only if

$$\begin{aligned} r^2+s^2k-t^2-t^2k+t^2v &= -1 \\ 2rs+s^2\lambda-2st-2st\lambda+t^2\lambda+2stk+t^2v-2t^2k &= 0 \\ 2rt+s^2\mu-2st\mu+t^2\mu+2stk+t^2v-2t^2-2t^2k &= 0. \end{aligned}$$

The code $P_R(r,s,t)$ formed from a DRT(v,k,λ,μ) is Euclidean self-dual code over R if and only if

$$\begin{aligned} r^2+(s^2+t^2)k &= -1 \\ rt+sr+(s^2+t^2)(k-1-\lambda)+st\lambda+st\mu &= 0 \\ rt+sr+(s^2+t^2)(k-\mu)+st\mu+st\lambda &= 0. \end{aligned}$$

Theorem 2.2.5. The code $B_R(r,s,t)$ formed from an SRG(v,k,λ,μ) is Euclidean self-dual code over R if and only if

$$\begin{aligned} r^2+s^2k-t^2-t^2k+t^2v &= -1-\gamma^2 \\ 2rs+s^2\lambda-2st-2st\lambda+t^2\lambda+2stk+t^2v-2t^2k &= -\gamma^2 \\ 2rt+s^2\mu-2st\mu+t^2\mu+2stk+t^2v-2t^2-2t^2k &= -\gamma^2 \\ 1+\alpha^2+v\beta^2 &= 0 \\ \alpha\gamma+\beta(r+sk+t(v-k-1)) &= 0. \end{aligned}$$

The code $B_R(r,s,t)$ formed from a DRT(v,k,λ,μ) is Euclidean self-dual code over R if and only if

$$\begin{aligned} r^2+(s^2+t^2)k &= -1-\gamma^2 \\ rt+sr+(s^2+t^2)(k-1-\lambda)+st\lambda+st\mu &= -\gamma^2 \\ rt+sr+(s^2+t^2)(k-\mu)+st\mu+st\lambda &= -\gamma^2 \\ 1+\alpha^2+v\beta^2 &= 0 \\ \alpha\gamma+\beta(r+sk+t(v-k-1)) &= 0. \end{aligned}$$

The matrices A and $\overline{A}$ in these constructions correspond to respectively matrices Q and N of Gaborit's constructions. From the description of Gaborit's constructions, we have two cases. If $q=4l+1$, then A is the adjacency matrix of an SRG, which is actually a Paley graph with the following parameters:

$$v=q,\ k=2l,\ \lambda=l-1,\ \mu=l.$$

If $q=4l+3$, then A is the adjacency matrix of a DRT, which is actually a Paley tournament with the following parameters:

$$v=q,\ k=2l+1,\ \lambda=l,\ \mu=l+1.$$

2.3. Paley designs

We will use the constructions introduced in the previous sections to find self-dual codes constructed from Paley designs. These designs are examples of Hadamard designs, which are associated with Hadamard matrices. Here, we start by reviewing basic facts about Hadamard matrices and designs.

Definition 2.3.1. A **Hadamard matrix** of order n is an $n \times n$ matrix H with all entries equal to 1 or -1 and with mutually orthogonal rows so that $HH^T = nI_n$.

Definition 2.3.2. A Hadamard matrix H is **skew** if $H + H^T = 2I$.

Remark 2.3.3. Multiplying by -1 all the entries in any row or column of a Hadamard matrix gives as the result a Hadamard matrix again. By a sequence of multiplications of this type, we can transform any Hadamard matrix into a Hadamard matrix in which every entry in the first row or column is a 1. Such a Hadamard matrix is called *normalized* (or *standardized*).

Example 2.3.4. Hadamard matrices of orders 1, 2 and 4:

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The following result provides a necessary condition for the existence of a Hadamard matrix of order n. The proof can be found in [59] .

Theorem 2.3.5. If there exists a Hadamard matrix of order $n > 2$, then $n \equiv 0 \pmod 4$.

The most important open conjecture in the theory of Hadamard matrices is about their existence. The Hadamard conjecture proposes that there exists a Hadamard matrix of every order $n \equiv 0 \pmod 4$. In fact, the smallest order for which a Hadamard matrix is not currently known to exist is 668.

The following theorem shows a connection between Hadamard matrices and certain symmetric block designs. The proof can also be found in [59].

Theorem 2.3.6. There exists a Hadamard matrix of order $4n$, $n > 1$, if and only if there exists a symmetric 2-$(4n-1, 2n-1, n-1)$ design.

Definition 2.3.7. A symmetric 2-$(4n-1, 2n-1, n-1)$ design is called a **Hadamard design**.

Example 2.3.8. The incidence matrix of a symmetric 2-$(7,3,1)$ design is presented in Example 1.2.16. If we change that matrix by putting -1 instead of 0, adding a row of 1s as the first row and a column of 1s as the first column of the matrix, we get the following Hadamard matrix of order 8:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

The *Paley matrices* are a well-known family of Hadamard matrices which can be obtained from the *Paley construction*. This is a method for constructing Hadamard matrices using quadratic residues in a finite field $GF(q)$ where q is a power of an odd prime number. There are two versions of the construction depending on whether $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$ which give Hadamard matrices of order $2(q+1)$ and $q+1$ respectively. We will use the Paley construction where $q \equiv 3 \pmod 4$.

Definition 2.3.9. The **quadratic character** of $GF(q)$ is the function $\chi_q : GF(q) \rightarrow \{-1, 0, 1\}$ given by

$$\chi(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \text{ is a quadratic residue,} \\ -1, & \text{if } x \text{ is a quadratic non-residue.} \end{cases}$$

Let $Q = [q_{i,j}]$ be a matrix whose rows and columns are indexed by the elements of $GF(q)$ having $q_{i,j} = \chi(j-i)$. The matrix Q is skew-symmetric, with zero diagonal and

± 1 elsewhere. If we replace the diagonal zeros by -1s and border Q with a row and column of 1s, we obtain a Hadamard matrix of order $q+1$ called a *Paley Type I Hadamard matrix*.

From the Theorem 2.3.6, we can use these matrices to obtain Paley designs. Deleting the first row and column of the Paley type I Hadamard matrix and replacing -1 by 0 throughout, we obtain an incidence matrix of a Paley design. The point set is $GF(q)$ and the blocks are the sets $S+x = \{s+x \mid s \in S\}$ for $x \in GF(q)$, where S is the set of non-zero squares in $GF(q)$.

2.4. SELF-DUAL CODES FROM PALEY DESIGNS

In this section, we give self-dual codes constructed from Paley designs. The incidence matrix of a Paley design is the $q \times q$ matrix of quadratic residues, not including 0, where $q = 4l + 3$ is a power of an odd prime. This matrix corresponds to matrix Q in the Gaborit's construction, so we can apply his pure and bordered quadratic double circulant forms to construct codes over an arbitrary ring R. Using the Theorem 2.1.1 for the product $Q_q Q_q^T$ and conditions 2.1 and 2.2, we obtain conditions for these codes to be self-dual.

The code $P_R(r,s,t)$ formed from a Paley design is self-dual over R if and only if

$$\begin{aligned} r^2 + (2l+1)(s^2+t^2) &= -1 \\ rs + rt + l(s^2+t^2) + (2l+1)st &= 0. \end{aligned} \tag{2.3}$$

The code $B_R(r,s,t)$ formed from a Paley design is self-dual over R if and only if

$$\begin{aligned} r^2 + (2l+1)(s^2+t^2) &= -1 - \gamma^2 \\ rs + rt + l(s^2+t^2) + (2l+1)st &= -\gamma^2 \\ 1 + \alpha^2 + v\beta^2 &= 0 \\ \alpha\gamma + \beta(r + sk + t(v-k-1)) &= 0. \end{aligned} \tag{2.4}$$

These codes are constructed over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$ using the software package Magma [10].

2.4.1. Self-dual codes over the field $\mathbb{F}_2$

Over the field $\mathbb{F}_2$, we can simplify the preceding conditions. Using the fact that $\mathbb{F}_2$ has characteristic 2 and by Fermat little theorem, the equations (2.3) and (2.4) can be reduced to

$$\begin{aligned} r + s + t &= 1 \\ rs + rt + l(s+t) + st &= 0 \end{aligned}$$

for the pure construction and

$$\begin{aligned} r + s + t &= 1 + \gamma \\ rs + rt + l(s+t) + st &= \gamma \\ 1 + \alpha + v\beta &= 0 \\ \alpha\gamma + \beta(r + sk + t(v+k+1)) &= 0 \end{aligned}$$

for the bordered construction. We use these equations to examine when the pure and the bordered constructions give self-dual codes over the field $\mathbb{F}_2$. Below, we give the list of all self-dual codes constructed from Paley designs.

Pure case

- $P_{\mathbb{F}_2}(1,0,0)$ is a self-dual code.
- If $l \equiv 0 \pmod 2$, then $P_{\mathbb{F}_2}(0,0,1)$ and $P_{\mathbb{F}_2}(0,1,0)$ are self-dual codes.

Bordered case

- $B_{\mathbb{F}_2}(0,1,1)$, with $\alpha = 0,\ \beta = \gamma = 1$, is a self-dual code. $B_{\mathbb{F}_2}(1,0,0)$, with $\alpha = 1,\ \beta = \gamma = 0$, is a self-dual code.
- If $l \equiv 0 \pmod 2$, then $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$, with $\alpha = 1,\ \beta = \gamma = 0$, are self-dual codes. $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, with $\alpha = 0,\ \beta = \gamma = 1$, are self-dual codes.

2.4.2. Self-dual codes over the field $\mathbb{F}_3$

Over the field $\mathbb{F}_3$, we cannot simplify conditions 2.3 and 2.4. Below, we give the list of all self-dual codes constructed from Paley designs.

Pure case
There are no self-dual codes in this case.

Bordered case $(a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b)$

- If $l \equiv 1 \pmod 3$, then $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, with $\alpha\gamma + a\beta = 0,\ 1+\alpha^2+\beta^2 = 0,\ \alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are self-dual codes.
- If $l \equiv 2 \pmod 3$, then $B_{\mathbb{F}_3}(0,a,b)$, with $\alpha = 0,\ \beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are self-dual codes.

2.4.3. Self-dual codes over the field $\mathbb{F}_4$

In the field $\mathbb{F}_4$, we use the Hermitian inner product. Therefore, we need to compute an analog of the product $Q_qQ_q^T$ from the Theorem 2.1.1. In order to do this, we need the following results which can be found in [28].

Lemma 2.4.1. Let q be a power of an odd prime.
If $q = 4l+1$, then

$$Q = Q^T \text{ and } N = N^T.$$

If $q = 4l+3$, then

$$Q = N^T \text{ and } Q+Q^T = J-I.$$

Lemma 2.4.2. Let q be a power of an odd prime.
If $q = 4l+1$, then

$$QQ^T = 2lI+(l-1)Q+lN.$$

If $q = 4l+3$, then

$$QQ^T = (l+1)I+lJ.$$

Corollary 2.4.3. Let q be a power of an odd prime.
If $q = 4l+1$, then

$$\begin{aligned}
QN^T &= NQ^T = Q(J-I-Q^T) = 2lJ-Q-QQ^T = lJ-lI,\\
NN^T &= (J-I-Q)(J-I-Q^T) = lQ+(l-1)N+2lI.
\end{aligned}$$

If $q = 4l+3$, then

$$\begin{aligned}
QN^T &= Q(J-I-Q^T) = 2lJ-Q-QQ^T = lQ+(l+1)N,\\
NQ^T &= (l+1)Q+lN,\\
NN^T &= (J-I-Q)(J-I-Q^T) = (l+1)I+lJ.
\end{aligned}$$

The following lemma represents an analog of the computation in Theorem 2.1.1, but only for the case when $q = 4l+3$.

Lemma 2.4.4. Let q be a power of an odd prime, and let $Q_q(r,s,t)$ be a quadratic residue circulant matrix with $r,s,t \in \mathbb{F}_4$. Over $\mathbb{F}_4$ with involution ϕ and $q = 4l+3$ we have

$$
\begin{aligned}
Q_q(r,s,t)\phi(Q_q(r,s,t)) &= (r^3+s^3+t^3)I \\
&+ (rt^2+sr^2+ts^2+l(s^3+ts^2+st^2+t^3))Q \\
&+ (rs^2+tr^2+st^2+l(s^3+ts^2+st^2+t^3))N.
\end{aligned}
$$

Proof. From Lemma 2.4.1 for $q=4l+3$ we obtain

$$
\begin{aligned}
Q_q(r,s,t)\phi(Q_q(r,s,t)) &= (rI+sQ+tN)(\phi(r)\phi(I)+\phi(s)\phi(Q)+\phi(t)\phi(N)) \\
&= (rI+sQ+tN)(\phi(r)I+\phi(s)N+\phi(t)Q).
\end{aligned}
$$

In $\mathbb{F}_4$, we have an involution given by $\phi(x)=x^2$, therefore we have

$$
Q_q(r,s,t)\phi(Q_q(r,s,t)) = (rI+sQ+tN)(r^2I+s^2N+t^2Q).
$$

Then using the results in Lemma 2.4.1, Lemma 2.4.2 and Corollary 2.4.3 as well as the fact that the characteristic of $\mathbb{F}_4$ equals 2, the computation follows.

□

Using the preceding lemma and conditions (2.1) and (2.2), we determine when the pure and the bordered constructions give self-dual codes over the field $\mathbb{F}_4$. Below, we give the list of all self-dual codes constructed from Paley designs.

Pure case $(a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b)$

- $P_{\mathbb{F}_4}(a,0,0)$ and $P_{\mathbb{F}_4}(a,b,b)$ are self-dual codes.
- If $l \equiv 0 \pmod 2$, then $P_{\mathbb{F}_4}(a,a,b)$, $P_{\mathbb{F}_4}(a,b,a)$, $P_{\mathbb{F}_4}(0,a,0)$ and $P_{\mathbb{F}_4}(0,0,a)$ are self-dual codes.

Bordered case $(a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b)$

- $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, with $\alpha=1,\ \beta=\gamma=0$, are self-dual codes. $B_{\mathbb{F}_4}(0,a,a)$, with $\alpha=0,\ \beta=\gamma=1$, are self-dual codes.
- If $l \equiv 0 \pmod 2$, then $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, with $\alpha=0,\ \beta=\gamma=1$, are self-dual codes. $B_{\mathbb{F}_4}(a,b,a)$, $B_{\mathbb{F}_4}(a,a,b)$, $B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, with $\alpha=1,\ \beta=\gamma=0$, are self-dual codes.

2.4.4. Examples computed in Magma

We used the software package Magma [10] to construct self-dual codes from the $q \times q$ incidence matrices of Paley designs and then to determine their minimum distances. These incidence matrices were also constructed using programmes written in Magma. By applying the pure and the bordered constructions, we constructed self-dual codes over the field $\mathbb{F}_2$ for all $q = 4l+3$, $l = 1, \ldots, 20$, where q is a power of an odd prime number. Furthermore, we constructed self-dual codes over the field $\mathbb{F}_3$ for all $q = 4l+3$, $l = 1, \ldots, 11$, and over the field $\mathbb{F}_4$ for all $q = 4l+3$, $l = 1, \ldots, 14$, where q is a power of an odd prime number.

1. $q = 7$ gives 2-$(7,3,1)$ Paley design.
   $P_{\mathbb{F}_2}(1,0,0)$ is a $[14,7,2]$ self-dual code.
   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[16,8,4]$ self-dual code, and this code is near-optimal.
   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[16,8,2]$ self-dual code.
   $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma + a\beta = 0,\ 1+\alpha^2+\beta^2 = 0,\ a,b,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[16,8,6]$ self-dual codes, and these codes are optimal.
   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[14,7,2]$ self-dual codes.
   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[14,7,4]$ self-dual codes.
   $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[16,8,2]$ self-dual codes.
   $B_{\mathbb{F}_4}(0,a,a)$, where $\alpha = 0,\ \beta = \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[16,8,4]$ self-dual codes.

2. $q = 11$ gives 2-$(11,5,2)$ Paley design.
   $P_{\mathbb{F}_2}(0,0,1)$ and $P_{\mathbb{F}_2}(0,1,0)$ are $[22,11,6]$ self-dual codes, and these codes are near-optimal.
   $P_{\mathbb{F}_2}(1,0,0)$ is a $[22,11,2]$ self-dual code.
   $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, are $[24,12,2]$ self-dual codes.
   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[24,12,4]$ self-dual code.
   $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha = 0,\ \beta = \gamma = 1$, are $[24,12,8]$ self-dual codes,

and these codes are optimal.

$B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=0,\ a,b,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[24,12,9]$ self-dual codes, and these codes are optimal.

$P_{\mathbb{F}_4}(a,0,0)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[22,11,2]$ self-dual codes.

$P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(a,b,a)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[22,11,8]$ self-dual codes, and these codes are near-optimal and best known.

$P_{\mathbb{F}_4}(a,b,b)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[22,11,4]$ self-dual codes.

$P_{\mathbb{F}_4}(0,a,0)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[22,11,6]$ self-dual codes.

$B_{\mathbb{F}_4}(a,0,0), B_{\mathbb{F}_4}(a,a,b), B_{\mathbb{F}_4}(a,b,a), B_{\mathbb{F}_4}(a,b,b), B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\alpha=1,\ \beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[24,12,2]$ self-dual codes.

$B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha=0,\ \beta=\gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[24,12,8]$ self-dual codes, and these codes are near-optimal.

$B_{\mathbb{F}_4}(0,a,a)$, where $\alpha=0,\ \beta=\gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[24,12,4]$ self-dual codes.

3. $q=19$ gives 2-$(19,9,4)$ Paley design.

$P_{\mathbb{F}_2}(0,0,1)$ and $P_{\mathbb{F}_2}(0,1,0)$ are $[38,19,8]$ self-dual codes, and these codes are near-optimal and best known.

$P_{\mathbb{F}_2}(1,0,0)$ is a $[38,19,2]$ self-dual code.

$B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha=1,\ \beta=\gamma=0$, are $[40,20,2]$ self-dual codes.

$B_{\mathbb{F}_2}(0,1,1)$, where $\alpha=0,\ \beta=\gamma=1$, is a $[40,20,4]$ self-dual code.

$B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=0,\ \beta=\gamma=1$, are $[40,20,8]$ self-dual codes.

$B_{\mathbb{F}_3}(a,a,0), B_{\mathbb{F}_3}(a,0,a), B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+a\beta=0,\ 1+\alpha^2+\beta^2=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[40,20,12]$ self-dual codes, and these codes are best known.

$P_{\mathbb{F}_4}(a,0,0)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[38,19,2]$ self-dual codes.

$P_{\mathbb{F}_4}(a,a,b)$, $P_{\mathbb{F}_4}(a,b,a)$, $P_{\mathbb{F}_4}(0,a,0)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[38,19,8]$ self-dual codes.

$P_{\mathbb{F}_4}(a,b,b)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[38,19,4]$ self-dual codes.

$B_{\mathbb{F}_4}(a,0,0), B_{\mathbb{F}_4}(a,a,b), B_{\mathbb{F}_4}(a,b,a), B_{\mathbb{F}_4}(a,b,b), B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\alpha=1,\ \beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[40,20,2]$ self-dual codes.

$B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha=0,\ \beta=\gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[40,20,8]$

self-dual codes.

$B_{\mathbb{F}_4}(0,a,a)$, where $\alpha = 0,\ \beta = \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[40,20,4]$ self-dual codes.

4. $q = 23$ gives 2-$(23,11,5)$ Paley design.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[46,23,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[48,24,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[48,24,2]$ self-dual code.

   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = 0,\ a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[48,24,15]$ self-dual codes, and these codes are near-optimal and best known.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[46,23,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[46,23,4]$ self-dual codes.

   $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[48,24,2]$ self-dual codes.

   $B_{\mathbb{F}_4}(0,a,a)$, where $\alpha = 0,\ \beta = \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[48,24,4]$ self-dual codes.

5. $q = 27$ gives 2-$(27,13,6)$ Paley design.

   $P_{\mathbb{F}_2}(0,0,1)$ and $P_{\mathbb{F}_2}(0,1,0)$ are $[54,27,10]$ self-dual codes.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[54,27,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, are $[56,28,2]$ self-dual codes.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[56,28,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha = 0,\ \beta = \gamma = 1$, are $[56,28,12]$ self-dual codes, and these codes are best known.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[54,27,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(a,b,a)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[54,27,12]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[54,27,4]$ self-dual codes.

   $P_{\mathbb{F}_4}(0,a,0)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[54,27,10]$ self-dual codes.

   $B_{\mathbb{F}_4}(a,0,0)$, $B_{\mathbb{F}_4}(a,a,b)$, $B_{\mathbb{F}_4}(a,b,a)$, $B_{\mathbb{F}_4}(a,b,b)$, $B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[56,28,2]$ self-dual codes.

   $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha = 0,\ \beta = \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[56,28,12]$

self-dual codes.

$B_{\mathbb{F}_4}(0,a,a)$, where $\alpha=0,\ \beta=\gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[56,28,4]$ self-dual codes.

6. $q=31$ gives 2-$(31,15,7)$ Paley design.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[62,31,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha=0,\ \beta=\gamma=1$, is a $[64,32,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha=1,\ \beta=\gamma=0$, is a $[64,32,2]$ self-dual code.

   $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+a\beta=0,\ 1+\alpha^2+\beta^2=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[64,32,18]$ self-dual codes, and these codes are best known.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[62,31,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[62,31,4]$ self-dual codes.

   $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, where $\alpha=1,\ \beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[64,32,2]$ self-dual codes.

   $B_{\mathbb{F}_4}(0,a,a)$, where $\alpha=0,\ \beta=\gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[64,32,4]$ self-dual codes.

7. $q=43$ gives 2-$(43,21,10)$ Paley design.

   $P_{\mathbb{F}_2}(0,0,1)$ and $P_{\mathbb{F}_2}(0,1,0)$ are $[86,43,14]$ self-dual codes.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[86,43,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha=1,\ \beta=\gamma=0$, are $[88,44,2]$ self-dual codes.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha=0,\ \beta=\gamma=1$, is a $[88,44,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=0,\ \beta=\gamma=1$, are $[88,44,16]$ self-dual codes.

   $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+a\beta=0,\ 1+\alpha^2+\beta^2=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[88,44,21]$ self-dual codes, and these codes are best known.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[86,43,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(a,b,a)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[86,43,16]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[86,43,4]$ self-dual codes.

   $P_{\mathbb{F}_4}(0,a,0)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[86,43,14]$ self-dual codes.

$B_{\mathbb{F}_4}(a,0,0), B_{\mathbb{F}_4}(a,a,b), B_{\mathbb{F}_4}(a,b,a), B_{\mathbb{F}_4}(a,b,b), B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[88,44,2]$ self-dual codes.

$B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha = 0,\ \beta = \gamma = 1,\ a \in \mathbb{F}_4 \backslash \{0\}$, are $[88,44,16]$ self-dual codes.

$B_{\mathbb{F}_4}(0,a,a)$, where $\alpha = 0,\ \beta = \gamma = 1,\ a \in \mathbb{F}_4 \backslash \{0\}$, are $[88,44,4]$ self-dual codes.

8. $q = 47$ gives 2-$(47,23,11)$ Paley design.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[94,47,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[96,48,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[96,48,2]$ self-dual code.

   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = 0,\ a,b,\beta,\gamma \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[96,48,24]$ self-dual codes, and these codes are best known.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4 \backslash \{0\}$, are $[94,47,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[94,47,4]$ self-dual codes.

   $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[96,48,2]$ self-dual codes.

   $B_{\mathbb{F}_4}(0,a,a)$, where $\alpha = 0,\ \beta = \gamma = 1,\ a \in \mathbb{F}_4 \backslash \{0\}$, are $[96,48,4]$ self-dual codes.

9. $q = 59$ gives 2-$(59,29,14)$ Paley design.

   $P_{\mathbb{F}_2}(0,0,1)$ and $P_{\mathbb{F}_2}(0,1,0)$ are $[118,59,18]$ self-dual codes.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[118,59,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, are $[120,60,2]$ self-dual codes.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[120,60,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha = 0,\ \beta = \gamma = 1$, are $[120,60,20]$ self-dual codes, and these codes are best known.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4 \backslash \{0\}$, are $[118,59,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(a,b,a)$, where $a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[118,59,20]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[118,59,4]$ self-dual codes.

   $P_{\mathbb{F}_4}(0,a,0)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a \in \mathbb{F}_4 \backslash \{0\}$, are $[120,60,18]$ self-dual codes.

$B_{\mathbb{F}_4}(a,0,0), B_{\mathbb{F}_4}(a,a,b), B_{\mathbb{F}_4}(a,b,a), B_{\mathbb{F}_4}(a,b,b), B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[120,60,2]$ self-dual codes.

$B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha = 0,\ \beta = \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[120,60,20]$ self-dual codes.

$B_{\mathbb{F}_4}(0,a,a)$, where $\alpha = 0,\ \beta = \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[120,60,4]$ self-dual codes.

10. $q = 67$ gives 2-$(67,33,16)$ Paley design.
    $P_{\mathbb{F}_2}(0,0,1)$ and $P_{\mathbb{F}_2}(0,1,0)$ are $[134,67,22]$ self-dual codes.
    $P_{\mathbb{F}_2}(1,0,0)$ is a $[134,67,2]$ self-dual code.
    $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, are $[136,68,2]$ self-dual codes.
    $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[136,68,4]$ self-dual code.
    $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha = 0,\ \beta = \gamma = 1$, are $[136,68,24]$ self-dual codes, and these codes are best known.

11. $q = 71$ gives 2-$(71,35,17)$ Paley design.
    $P_{\mathbb{F}_2}(1,0,0)$ is a $[142,71,2]$ self-dual code.
    $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[144,72,4]$ self-dual code.
    $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[144,72,2]$ self-dual code.

12. $q = 79$ gives 2-$(79,39,19)$ Paley design.
    $P_{\mathbb{F}_2}(1,0,0)$ is a $[158,79,2]$ self-dual code.
    $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[160,80,4]$ self-dual code.
    $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[160,80,2]$ self-dual code.

13. $q = 83$ gives 2-$(83,41,20)$ Paley design.
    $P_{\mathbb{F}_2}(0,0,1)$ and $P_{\mathbb{F}_2}(0,1,0)$ are $[166,83,24]$ self-dual codes, and these codes are best known. $P_{\mathbb{F}_2}(1,0,0)$ is a $[166,83,2]$ self-dual code.
    $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, are $[168,84,2]$ self-dual codes.
    $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[168,84,4]$ self-dual code.
    $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha = 0,\ \beta = \gamma = 1$, are $[168,84,24]$ self-dual codes, and these codes are best known.

2.5. Self-dual codes from Paley graphs

In this section, we give self-dual codes constructed from Paley graphs. If p divides $(q+1)/2$, where q is the number of vertices of a Paley graph, then under some conditions the self-dual code over $\mathbb{F}_p$ obtained by the bordered construction is equivalent to the self-dual code obtained from that Paley graph by the construction given in [20], as shown in [32].

Paley graphs are defined in Section 1.3, so we know that the adjacency matrix of a Paley graph is the $q \times q$ matrix of quadratic residues, not including 0, where $q = 4l+1$ is a power of an odd prime. We also know that Paley graphs are strongly regular graphs with parameters

$$\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right),$$

so we can construct self-dual codes using the construction from Dougherty, Kim and Solé described in Section 2.2.

As in the previous section, we want to construct self-dual codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$ using the software package Magma [10].

2.5.1. Self-dual codes over the field $\mathbb{F}_2$

Over $\mathbb{F}_2$ and by using Fermat's little theorem, the equation for SRGs from Lemma 2.2.3 can be reduced to

$$Q_{\mathbb{F}_2}(r,s,t)Q_{\mathbb{F}_2}(r,s,t)^T = (r+sk+t+tk+tv)I + (s\lambda+t\lambda+tv)A + (s\mu+t\mu+tv)\overline{A}.$$

We use this equation and conditions (2.1) and (2.2) to examine when the pure and the bordered constructions give self-dual codes over the field $\mathbb{F}_2$. Below, we give the list of all self-dual codes constructed from Paley graphs.

Pure case

- $P_{\mathbb{F}_2}(1,0,0)$ is a self-dual code.

Bordered case

- $B_{\mathbb{F}_2}(0,1,1)$, with $\alpha = 0$, $\beta = \gamma = 1$, is a self-dual code.
- $B_{\mathbb{F}_2}(1,0,0)$, with $\alpha = 1$, $\beta = \gamma = 0$, is a self-dual code.

2.5.2. Self-dual codes over the field $\mathbb{F}_3$

Over the field $\mathbb{F}_3$, we cannot simplify the equation for SRGs from Lemma 2.2.3, so we use conditions stated in Theorems 2.2.4 and 2.2.5 to examine when the pure and the bordered constructions give self-dual codes. Below, we give the list of all self-dual codes constructed from Paley graphs.

Pure case

There are no self-dual codes in this case.

Bordered case $(a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b)$

- If $l \equiv 1 \pmod 3$, then $B_{\mathbb{F}_3}(0,a,b)$, with $\alpha = 0,\ \beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are self-dual codes.

2.5.3. Self-dual codes over the field $\mathbb{F}_4$

In the field $\mathbb{F}_4$, we use the Hermitian inner product. The following lemma represents an analog of the computation in Lemma 2.2.3, which can be found in [26].

Lemma 2.5.1. Over $\mathbb{F}_4$ with involution ϕ for SRGs we have

$$\begin{aligned} Q_{\mathbb{F}_4}(r,s,t)\phi(Q_{\mathbb{F}_4}(r,s,t)) &= (r^3+s^3k+t^3+t^3k+t^3v)I \\ &+ (rs^2+sr^2+s^3\lambda+t^3\lambda+(ts^2+st^2)(k+1+\lambda)+t^3v)A \\ &+ (rt^2+tr^2+s^3\mu+t^3\mu+(ts^2+st^2)(k+\mu)+t^3v)\overline{A}. \end{aligned}$$

Over $\mathbb{F}_4$ with involution ϕ for DRTs we have

$$\begin{aligned} Q_{\mathbb{F}_4}(r,s,t)\phi(Q_{\mathbb{F}_4}(r,s,t)) &= (r^3+(s^3+t^3)k)I \\ &+ (rt^2+sr^2+(s^3+t^3)(k+1+\lambda)+st^2\lambda+ts^2\mu)A \\ &+ (rs^2+tr^2+(s^3+t^3)(k+\mu)+st^2\mu+ts^2\lambda)\overline{A}. \end{aligned}$$

Using the preceding lemma and conditions (2.1) and (2.2), we determine when the pure and the bordered constructions give self-dual codes over the field $\mathbb{F}_4$. Below, we give the list of all self-dual codes constructed from Paley graphs.

Pure case $(a,b,c \in \mathbb{F}_4\backslash\{0\},\ a \neq b \neq c,\ a \neq c)$

- $P_{\mathbb{F}_4}(a,0,0)$ and $P_{\mathbb{F}_4}(a,b,b)$ are self-dual codes.
- If $l \equiv 0 \pmod 2$, then $P_{\mathbb{F}_4}(a,b,0)$, $P_{\mathbb{F}_4}(a,0,b)$ and $P_{\mathbb{F}_4}(a,b,c)$ are self-dual codes.

Bordered case $(a,b,c \in \mathbb{F}_4 \backslash \{0\},\ a \neq b \neq c,\ a \neq c)$

- $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, with $\alpha = 1,\ \beta = \gamma = 0$, are self-dual codes.
- If $l \equiv 0 \pmod 2$, then $B_{\mathbb{F}_4}(a,b,0)$, $B_{\mathbb{F}_4}(a,0,b)$ and $B_{\mathbb{F}_4}(a,b,c)$, with $\alpha = 1,\ \beta = \gamma = 0$, are self-dual codes.

2.5.4. Examples computed in Magma

We used the software package Magma [10] to construct self-dual codes from the $q \times q$ adjacency matrices of Paley graphs and then to determine their minimum distances. These adjacency matrices were also obtained using the built-in functions in Magma. By applying the pure and the bordered construction, we constructed self-dual codes over the fields $\mathbb{F}_2$ and $\mathbb{F}_4$ for all $q = 4l+1,\ l = 1,\ldots,15$, where q is a power of an odd prime number. Furthermore, we constructed self-dual codes over the field $\mathbb{F}_3$ for all $q = 4l+1,\ l = 1,\ldots,13$, where q is a power of an odd prime number.

1. $q = 5$ gives the Paley SRG$(5,2,0,1)$.
   $P_{\mathbb{F}_2}(1,0,0)$ is a $[10,5,2]$ self-dual code.
   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[12,6,4]$ self-dual code, and this code is optimal.
   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[12,6,2]$ self-dual code.
   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = 0,\ a,b,\beta,\gamma \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[12,6,6]$ self-dual codes, and these codes are optimal.
   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4 \backslash \{0\}$, are $[10,5,2]$ self-dual codes.
   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[10,5,4]$ self-dual codes, and these codes are near-optimal.
   $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[12,6,2]$ self-dual codes.

2. $q = 9$ gives the Paley SRG$(9,4,1,2)$.
   $P_{\mathbb{F}_2}(1,0,0)$ is a $[18,9,2]$ self-dual code.
   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[20,10,4]$ self-dual code.
   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[20,10,2]$ self-dual code.

$P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4 \backslash \{0\}$, are $[18,9,2]$ self-dual codes.

$P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[18,9,4]$ self-dual codes.

$P_{\mathbb{F}_4}(a,b,0)$, $P_{\mathbb{F}_4}(a,0,b)$ and $P_{\mathbb{F}_4}(a,b,c)$, where $a,b,c \in \mathbb{F}_4 \backslash \{0\},\ a \neq b \neq c,\ a \neq c$, are $[18,9,6]$ self-dual codes.

$B_{\mathbb{F}_4}(a,0,0)$, $B_{\mathbb{F}_4}(a,b,0)$, $B_{\mathbb{F}_4}(a,0,b)$, $B_{\mathbb{F}_4}(a,b,b)$ and $B_{\mathbb{F}_4}(a,b,c)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b,c \in \mathbb{F}_4 \backslash \{0\},\ a \neq b \neq c,\ a \neq c$, are $[20,10,2]$ self-dual codes.

3. $q = 13$ gives the Paley SRG$(13,6,2,3)$.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[26,13,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[28,14,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[28,14,2]$ self-dual code.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4 \backslash \{0\}$, are $[26,13,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[26,13,4]$ self-dual codes.

   $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[28,14,2]$ self-dual codes.

4. $q = 17$ gives the Paley SRG$(17,8,3,4)$.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[34,17,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[36,18,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[36,18,2]$ self-dual code.

   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = 0,\ a,b,\beta,\gamma \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[36,18,12]$ self-dual codes, and these codes are optimal.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4 \backslash \{0\}$, are $[34,17,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[34,17,4]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,0)$, $P_{\mathbb{F}_4}(a,0,b)$ and $P_{\mathbb{F}_4}(a,b,c)$, where $a,b,c \in \mathbb{F}_4 \backslash \{0\},\ a \neq b \neq c,\ a \neq c$, are $[34,17,10]$ self-dual codes.

   $B_{\mathbb{F}_4}(a,0,0)$, $B_{\mathbb{F}_4}(a,b,0)$, $B_{\mathbb{F}_4}(a,0,b)$, $B_{\mathbb{F}_4}(a,b,b)$ and $B_{\mathbb{F}_4}(a,b,c)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b,c \in \mathbb{F}_4 \backslash \{0\},\ a \neq b \neq c,\ a \neq c$, are $[36,18,2]$ self-dual codes.

5. $q = 25$ gives the Paley SRG$(25,12,5,6)$.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[50,25,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[52,26,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[52,26,2]$ self-dual code.

$P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[50,25,2]$ self-dual codes.

$P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[50,25,4]$ self-dual codes.

$P_{\mathbb{F}_4}(a,b,0)$ and $P_{\mathbb{F}_4}(a,0,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[50,25,10]$ self-dual codes.

$P_{\mathbb{F}_4}(a,b,c)$, where $a,b,c \in \mathbb{F}_4\backslash\{0\},\ a \neq b \neq c,\ a \neq c$, are $[50,25,12]$ self-dual codes.

$B_{\mathbb{F}_4}(a,0,0), B_{\mathbb{F}_4}(a,b,0), B_{\mathbb{F}_4}(a,0,b), B_{\mathbb{F}_4}(a,b,b)$ and $B_{\mathbb{F}_4}(a,b,c)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b,c \in \mathbb{F}_4\backslash\{0\},\ a \neq b \neq c,\ a \neq c$, are $[52,26,2]$ self-dual codes.

6. $q = 29$ gives the Paley SRG$(29,14,6,7)$.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[58,29,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[60,30,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[60,30,2]$ self-dual code.

   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = 0,\ a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[60,30,18]$ self-dual codes, and these codes are best known.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[58,29,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[58,29,4]$ self-dual codes.

   $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[60,30,2]$ self-dual codes.

7. $q = 37$ gives the Paley SRG$(37,18,8,9)$.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[74,37,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[76,38,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[76,38,2]$ self-dual code.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[74,37,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[74,37,4]$ self-dual codes.

   $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[76,38,2]$ self-dual codes.

8. $q = 41$ gives the Paley SRG$(41,20,9,10)$.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[82,41,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[84,42,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[84,42,2]$ self-dual code.

$B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = 0$, $a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[84,42,21]$ self-dual codes, and these codes are best known.

$P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[82,41,2]$ self-dual codes.

$P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, are $[82,41,4]$ self-dual codes.

$P_{\mathbb{F}_4}(a,b,0)$ and $P_{\mathbb{F}_4}(a,0,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, are $[82,41,18]$ self-dual codes.

$P_{\mathbb{F}_4}(a,b,c)$, where $a,b,c \in \mathbb{F}_4\backslash\{0\}$, $a \neq b \neq c$, $a \neq c$, are $[82,41,20]$ self-dual codes.

$B_{\mathbb{F}_4}(a,0,0), B_{\mathbb{F}_4}(a,b,0), B_{\mathbb{F}_4}(a,0,b), B_{\mathbb{F}_4}(a,b,b)$ and $B_{\mathbb{F}_4}(a,b,c)$, where $\alpha = 1$, $\beta = \gamma = 0$, $a,b,c \in \mathbb{F}_4\backslash\{0\}$, $a \neq b \neq c$, $a \neq c$, are $[84,42,2]$ self-dual codes.

9. $q = 49$ gives the Paley SRG$(49,24,11,12)$.

   $P_{\mathbb{F}_2}(1,0,0)$ is a $[98,49,2]$ self-dual code.

   $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0$, $\beta = \gamma = 1$, is a $[100,50,4]$ self-dual code.

   $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1$, $\beta = \gamma = 0$, is a $[100,50,2]$ self-dual code.

   $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[98,49,2]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, are $[98,49,4]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,0)$ and $P_{\mathbb{F}_4}(a,0,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, are $[98,49,14]$ self-dual codes.

   $P_{\mathbb{F}_4}(a,b,c)$, where $a,b,c \in \mathbb{F}_4\backslash\{0\}$, $a \neq b \neq c$, $a \neq c$, are $[98,49,16]$ self-dual codes.

   $B_{\mathbb{F}_4}(a,0,0), B_{\mathbb{F}_4}(a,b,0), B_{\mathbb{F}_4}(a,0,b), B_{\mathbb{F}_4}(a,b,b)$ and $B_{\mathbb{F}_4}(a,b,c)$, where $\alpha = 1$, $\beta = \gamma = 0$, $a,b,c \in \mathbb{F}_4\backslash\{0\}$, $a \neq b \neq c$, $a \neq c$, are $[100,50,2]$ self-dual codes.

10. $q = 53$ gives the Paley SRG$(53,26,12,13)$.

    $P_{\mathbb{F}_2}(1,0,0)$ is a $[106,53,2]$ self-dual code.

    $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0$, $\beta = \gamma = 1$, is a $[108,54,4]$ self-dual code.

    $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1$, $\beta = \gamma = 0$, is a $[108,54,2]$ self-dual code.

    $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = 0$, $a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[108,54,24]$ self-dual codes.

    $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[106,53,2]$ self-dual codes.

    $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, are $[106,53,4]$ self-dual codes.

$B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[108,54,2]$ self-dual codes.

11. $q = 61$ gives the Paley SRG$(61,30,14,15)$.
    $P_{\mathbb{F}_2}(1,0,0)$ is a $[122,61,2]$ self-dual code.
    $B_{\mathbb{F}_2}(0,1,1)$, where $\alpha = 0,\ \beta = \gamma = 1$, is a $[124,62,4]$ self-dual code.
    $B_{\mathbb{F}_2}(1,0,0)$, where $\alpha = 1,\ \beta = \gamma = 0$, is a $[124,62,2]$ self-dual code.
    $P_{\mathbb{F}_4}(a,0,0)$, where $a \in \mathbb{F}_4 \backslash \{0\}$, are $[122,61,2]$ self-dual codes.
    $P_{\mathbb{F}_4}(a,b,b)$, where $a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[122,61,4]$ self-dual codes.
    $B_{\mathbb{F}_4}(a,0,0)$ and $B_{\mathbb{F}_4}(a,b,b)$, where $\alpha = 1,\ \beta = \gamma = 0,\ a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[124,62,2]$ self-dual codes.

3. CONSTRUCTIONS OF LCD CODES

The purpose of this chapter is to present two constructions of LCD codes, which are defined in Section 1.5.1. In the previous chapter, we presented constructions of self-dual codes from two class association schemes. Here we use a similar approach in order to construct LCD codes from two class association schemes. For this reason, we took the phrases *pure* and *bordered* constructions from [26]. Further, we show that LCD codes constructed using the pure construction are formally self-dual, and for these codes we deduce a decoding algorithm. We also give conditions for constructing LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$. Working over $\mathbb{F}_{p^n}$, we interpret integers as their value modulo p.

3.1. PURE AND BORDERED CONSTRUCTIONS

3.1.1. Pure construction

In the first case, we consider the pure construction. The following theorem shows how to obtain LCD codes from the adjacency matrices of SRGs and DRTs in this construction.

Theorem 3.1.1. Let $r,s,t \in \mathbb{F}_q$ and let $Q_{\mathbb{F}_q} = (rI + sA + t\overline{A})$. Further, let $P_{\mathbb{F}_q}$ be an $n \times 2n$ matrix over $\mathbb{F}_q$, and suppose $P_{\mathbb{F}_q} = \left[\begin{array}{c|c} I & Q_{\mathbb{F}_q}(r,s,t) \end{array}\right]$ generates a $[2n,n]$ code C over $\mathbb{F}_q$. The code $P_{\mathbb{F}_q}(r,s,t)$ formed from an SRG(v,k,λ,μ), with the adjacency matrix A, is Euclidean LCD code if $x \neq 0,\ x \in \mathbb{F}_q$, and

$$\begin{aligned}
r^2 + s^2k - t^2 - t^2k + t^2v &= x - 1 \\
2rs + s^2\lambda - 2st - 2st\lambda + t^2\lambda + 2stk + t^2v - 2t^2k &= 0 \\
2rt + s^2\mu - 2st\mu + t^2\mu + 2stk + t^2v - 2t^2 - 2t^2k &= 0.
\end{aligned}$$

The code $P_R(r,s,t)$ formed from a DRT(v,k,λ,μ), with the adjacency matrix A, is Euclidean LCD code if $x \neq 0,\ x \in \mathbb{F}_q$, and

$$\begin{aligned} r^2+(s^2+t^2)k &= x-1 \\ rt+sr+(s^2+t^2)(k-1-\lambda)+st\lambda+st\mu &= 0 \\ rt+sr+(s^2+t^2)(k-\mu)+st\mu+st\lambda &= 0. \end{aligned}$$

Proof. From Lemma 1.5.31, we have that $P_{\mathbb{F}_q}(r,s,t)$ generates an LCD code if and only if $\det(P_{\mathbb{F}_q}P_{\mathbb{F}_q}^\top) \neq 0$. Following the results given in Lemma 2.2.3, we get $Q_{\mathbb{F}_q}Q_{\mathbb{F}_q}^\top = aI + bA + c\overline{A}$. By construction, this gives $P_{\mathbb{F}_q}P_{\mathbb{F}_q}^\top = (a+1)I + bA + c\overline{A}$, and so $\det(P_{\mathbb{F}_q}P_{\mathbb{F}_q}^\top) \neq 0$ if $a+1 = x \neq 0$, $b=0$, $c=0$ over $\mathbb{F}_q$. Now the conditions follow.

□

We will use the following lemma, that can be found in [21], to prove that the LCD codes from Theorem 3.1.1 are formally self-dual.

Lemma 3.1.2. Let M be an $n\times n$ matrix over $\mathbb{F}_q$, and suppose $G = \left[\begin{array}{c|c} M & I \end{array}\right]$ generates a $[2n,n]_q$ code C. Further, suppose that $\overline{G} = \left[\begin{array}{c|c} M & \alpha I \end{array}\right]$ generates the dual code $C^\perp$ for some $\alpha \in \mathbb{F}_q$. Then C is formally self-dual.

Theorem 3.1.3. Let $r,s,t \in \mathbb{F}_q$ and let $Q_{\mathbb{F}_q} = (rI + sA + t\overline{A})$. Suppose $P_{\mathbb{F}_q} = \left[\begin{array}{c|c} I & Q_{\mathbb{F}_q}(r,s,t) \end{array}\right]$ and $Q_{\mathbb{F}_q}Q_{\mathbb{F}_q}^\top = aI$, $a+1 \neq 0$. Then $P_{\mathbb{F}_q}$ generates a formally self-dual LCD code over $\mathbb{F}_q$.

Proof. The code C generated by $P_{\mathbb{F}_q}$ over $\mathbb{F}_q$ is an LCD code, by Theorem 3.1.1. The matrix $\overline{P}_{\mathbb{F}_q} = \left[\begin{array}{c|c} -aI & Q_{\mathbb{F}_q}(r,s,t) \end{array}\right]$ is a generator matrix of the dual code $C^\perp$. Hence, by Lemma 3.1.2, the code C is formally self-dual.

□

Below, we give a decoding algorithm that can be feasible for the LCD codes obtained using the pure construction. This algorithm is obtained similarly to the approach employed in [40].

Lemma 3.1.4. Let $r,s,t \in \mathbb{F}_q$, $Q_{\mathbb{F}_q} = (rI + sA + t\overline{A})$, and let $P_{\mathbb{F}_q} = \left[\begin{array}{c|c} I & Q_{\mathbb{F}_q}(r,s,t) \end{array}\right]$ generates an LCD code C over the field $\mathbb{F}_q$. Further, let $\overline{P}_{\mathbb{F}_q}$ be a generator matrix of the dual code $C^\perp$. Denote the ith row of the matrix $P_{\mathbb{F}_q}$ by r_i, and the ith row of $\overline{P}_{\mathbb{F}_q}$ by $\overline{r}_i$. Suppose C has minimum distance d and $e = \left\lfloor \frac{d-1}{2} \right\rfloor$. Then the following hold.

1. Let J be a subset of the set of indices of the rows of $P_{\mathbb{F}_q}$. If $|J| \leq e$, then the codeword in C closest to $\sum_{i\in J}\lambda_i\overline{r}_i$, where $\lambda_i \in \mathbb{F}_q$ for $i \in J$, is $\sum_{i\in J}\lambda_i r_i$, at a distance $|J|$ from the vector $\sum_{i\in J}\lambda_i\overline{r}_i$.

2. For $|J| \leq e$ the map φ of Theorem 1.5.32 can be uniquely defined by $\varphi(\sum_{i\in J}\lambda_i\overline{r}_i) = \sum_{i\in J}\lambda_i r_i$.

3. If $w = \sum_{i\in J}\lambda_i\overline{r}_i$ and $w = \sum_{i\in K}\mu_i\overline{r}_i$, then $J = K$ and $\lambda_i = \mu_i$ for all $i \in J$.

Proof. The proof of parts (1) and (2) follows from the definition of the matrices $P_{\mathbb{F}_q}$ and $\overline{P}_{\mathbb{F}_q}$ and the fact that C has minimum distance d, and that of part (3) follows from the structure of the matrix $\overline{P}_{\mathbb{F}_q}$.

□

Theorem 3.1.5. Let $r,s,t \in \mathbb{F}_q$, $Q_{\mathbb{F}_q} = (rI + sA + t\overline{A})$, and C be the $[2n,n]_q$ LCD code generated by the matrix $P_{\mathbb{F}_q} = \left[\begin{array}{c|c} I & Q_{\mathbb{F}_q}(r,s,t)\end{array}\right]$. Suppose C has minimum distance d and $e = \left\lfloor \frac{d-1}{2} \right\rfloor$. If the transmitted codeword from C has no more than e errors, it can be correctly decoded.

Proof. Suppose a codeword c is sent and $w = c + w'$ is received, where w' has no more than e non-zero coordinates. Then $w' = \sum_{i\in K_1}\lambda_i' r_i + \sum_{i\in K_2}\mu_i'\overline{r}_i$, where $|K_1| + |K_2| \leq e$. Hence, $w = \sum_{i\in J_1}\lambda_i r_i + \sum_{i\in J_2}\mu_i\overline{r}_i = c + \sum_{i\in K_1}\lambda_i' r_i + \sum_{i\in K_2}\mu_i'\overline{r}_i$. It follows that $\sum_{i\in J_1}\lambda_i r_i = c + \sum_{i\in K_1}\lambda_i' r_i$ and $\sum_{i\in J_2}\mu_i\overline{r}_i = \sum_{i\in K_2}\mu_i'\overline{r}_i$. By Lemma 3.1.4 (3), $J_2 = K_2$ and $\mu_i = \mu_i'$ for $i \in J_2$. Hence, $|J_2| \leq e$ and $\tilde{\varphi}(w) = \sum_{i\in J_1}\lambda_i r_i + \varphi(\sum_{i\in J_2}\mu_i\overline{r}_i) = \sum_{i\in J_1}\lambda_i r_i + \sum_{i\in J_2}\mu_i r_i$. Since $c = \sum_{i\in J_1}\lambda_i r_i - \sum_{i\in K_1}\lambda_i' r_i$, it follows that $\tilde{\varphi}(w) - c = \sum_{i\in J_1}\lambda_i r_i + \sum_{i\in J_2}\mu_i r_i - \sum_{i\in J_1}\lambda_i r_i + \sum_{i\in K_1}\lambda_i' r_i = \sum_{i\in J_2}\mu_i r_i + \sum_{i\in K_1}\lambda_i' r_i = \sum_{i\in K_2}\mu_i' r_i + \sum_{i\in K_1}\lambda_i' r_i$. Hence, $\tilde{\varphi}(w) - c$ is a codeword of C that has weight $|K_1| + |K_2| \leq e$, so $\tilde{\varphi}(w) = c$.

□

Lemma 3.1.4 and Theorem 3.1.5 show that the decoding method proposed by Massey, when implemented for codes obtained by the pure construction, taking the map φ from Lemma 3.1.4 (2), defined using the partial definition for linear combinations of at most $e = \left\lfloor \frac{d-1}{2} \right\rfloor$ rows of $\overline{G}$, can correct up to e errors.

3.1.2. Bordered construction

In the second case, we consider the bordered construction. The following theorem shows how to obtain LCD codes from the adjacency matrices of SRGs and DRTs in this construction.

Theorem 3.1.6. Let $r,s,t \in \mathbb{F}_q$ and let $Q_{\mathbb{F}_q} = (rI + sA + t\overline{A})$. Further, let $B_{\mathbb{F}_q}$ be an $(n+1)\times(2n+2)$ matrix over $\mathbb{F}_q$ and α,β and γ are scalars, and suppose

$$B_{\mathbb{F}_q} = \left(\begin{array}{c|c|c|c} 1 & 0\ldots 0 & \alpha & \beta\ldots\beta \\ \hline 0 & & \gamma & \\ \vdots & I & \vdots & Q_{\mathbb{F}_q}(r,s,t) \\ 0 & & \gamma & \end{array}\right)$$

generates a $[2n+2,n+1]$ code C over $\mathbb{F}_q$. The code $B_{\mathbb{F}_q}(r,s,t)$ formed from an SRG(v,k,λ,μ), with the adjacency matrix A, is Euclidean LCD code if $x\neq 0,\ y\neq 0,\ x,y\in\mathbb{F}_q$, and

$$\begin{aligned} r^2+s^2k-t^2-t^2k+t^2v &= x-1-\gamma^2 \\ 2rs+s^2\lambda-2st-2st\lambda+t^2\lambda+2stk+t^2v-2t^2k &= -\gamma^2 \\ 2rt+s^2\mu-2st\mu+t^2\mu+2stk+t^2v-2t^2-2t^2k &= -\gamma^2 \\ 1+\alpha^2+v\beta^2 &= y \\ \alpha\gamma+\beta(r+sk+t(v-k-1)) &= 0. \end{aligned}$$

The code $B_{\mathbb{F}_q}(r,s,t)$ formed from a DRT(v,k,λ,μ), with the adjacency matrix A, is Euclidean LCD code if $x\neq 0,\ y\neq 0,\ x,y\in\mathbb{F}_q$, and

$$\begin{aligned} r^2+(s^2+t^2)k &= x-1-\gamma^2 \\ rt+sr+(s^2+t^2)(k-1-\lambda)+st\lambda+st\mu &= -\gamma^2 \\ rt+sr+(s^2+t^2)(k-\mu)+st\mu+st\lambda &= -\gamma^2 \\ 1+\alpha^2+v\beta^2 &= y \\ \alpha\gamma+\beta(r+sk+t(v-k-1)) &= 0. \end{aligned}$$

Proof. From Lemma 1.5.31, we have that $B_{\mathbb{F}_q}(r,s,t)$ generates an LCD code if and only if $\det(B_{\mathbb{F}_q}B_{\mathbb{F}_q}^\top)\neq 0$. By construction, we have $\det(B_{\mathbb{F}_q}B_{\mathbb{F}_q}^\top)\neq 0$ if we get that $B_{\mathbb{F}_q}B_{\mathbb{F}_q}^\top$ is a diagonal matrix. So we need the following equations obtained from $B_{\mathbb{F}_q}B_{\mathbb{F}_q}^\top$. The inner product of the top row of $B_{\mathbb{F}_q}$ with itself must be different from 0, *i.e.* $1+\alpha^2+v\beta^2 = y\neq 0$. The inner product of the top row of $B_{\mathbb{F}_q}$ with any other row must be equal to 0, *i.e.* $\alpha\gamma+\beta(r+sk+t(v-k-1))=0$. The inner products of all the other rows of $B_{\mathbb{F}_q}$ must give a diagonal matrix, *i.e.* $I+\gamma^2J+Q_{\mathbb{F}_q}(r,s,t)Q_{\mathbb{F}_q}(r,s,t)^T = xI$. The last equation gives $Q_{\mathbb{F}_q}(r,s,t)Q_{\mathbb{F}_q}(r,s,t)^T = (x-1-\gamma^2)I-\gamma^2A-\gamma^2\overline{A}$. From this and Lemma 2.2.3, the conditions follow.

□

3.2. Conditions for constructing LCD codes over finite fields

In this section, we simplify the preceding conditions, and we give conditions for constructing LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$.

3.2.1. LCD codes over the field $\mathbb{F}_2$

SRGs

As mentioned in Subsection 2.5.1, over $\mathbb{F}_2$ and by using Fermat's little theorem, the equation for SRGs from Lemma 2.2.3 can be reduced to

$$Q_{\mathbb{F}_2}(r,s,t)Q_{\mathbb{F}_2}(r,s,t)^T = (r+sk+t+tk+tv)I + (s\lambda+t\lambda+tv)A + (s\mu+t\mu+tv)\overline{A}.$$

Then, because $x \neq 0$ and $y \neq 0$, we have $x = y = 1$, so the equations from Theorem 3.1.1 for SRGs are

$$\begin{aligned} r+sk+t+tk+tv &= 0 \\ s\lambda+t\lambda+tv &= 0 \\ s\mu+t\mu+tv &= 0, \end{aligned}$$

and the equations from Theorem 3.1.6 for SRGs are

$$\begin{aligned} r+sk+t+tk+tv &= \gamma \\ s\lambda+t\lambda+tv &= \gamma \\ s\mu+t\mu+tv &= \gamma \\ \alpha+v\beta &= 0 \\ \alpha\gamma+\beta(r+sk+t(v+k+1)) &= 0. \end{aligned}$$

We use these equations to examine when the pure and the bordered constructions give LCD codes obtained from SRGs. In the table below, we give conditions on v, k, λ, μ, γ which are necessary for the results, and all equalities are given in the field $\mathbb{F}_2$. For the bordered construction, we have two more conditions given for α and β:

$$\alpha+v\beta = 0\ ,\ \alpha\gamma+\beta\gamma = 0.$$

Table 3.1: LCD codes from SRGs over $\mathbb{F}_2$

r	s	t	Pure construction	Bordered construction
0	0	1	$v=\lambda=\mu=k+1$	$\lambda=\mu,\ k+v+1=\gamma$
0	1	0	$k=\lambda=\mu=0$	$k=\lambda=\mu=\gamma$
0	1	1	Never	Never
1	0	0	Never	Never
1	0	1	$k=v=\lambda=\mu$	$k=\lambda=\mu=v+\gamma$
1	1	0	$k=1,\ \lambda=\mu=0$	$k+1=\lambda=\mu=\gamma$
1	1	1	$v=0$	$v=\gamma$

DRTs

The equation for DRTs from Lemma 2.2.3 can be reduced by using Fermat's little theorem. As in the previous case, because $x\neq 0$ and $y\neq 0$, we have $x=y=1$, so the equations from Theorem 3.1.1 for DRTs are

$$\begin{aligned} r+(s+t)k &= 0\\ rt+sr+(s+t)(k+1+\lambda)+st\lambda+st\mu &= 0\\ rt+sr+(s+t)(k+\mu)+st\mu+st\lambda &= 0, \end{aligned}$$

and the equations from Theorem 3.1.6 for DRTs are

$$\begin{aligned} r+(s+t)k &= \gamma\\ rt+sr+(s+t)(k+1+\lambda)+st\lambda+st\mu &= \gamma\\ rt+sr+(s+t)(k+\mu)+st\mu+st\lambda &= \gamma\\ \alpha+v\beta &= 0\\ \alpha\gamma+\beta(r+sk+t(v+k+1)) &= 0. \end{aligned}$$

From Lemma 1.4.20, the parameters of a DRT satisfy $v=4\lambda+3$, $k=2\lambda+1$ and $\mu=\lambda+1$. This gives that $P_{\mathbb{F}_2}(0,1,1)$, $B_{\mathbb{F}_2}(0,1,1)$ and $B_{\mathbb{F}_2}(1,1,1)$ are never LCD codes since the equations require that $\mu=\lambda$ which contradicts that $\mu=\lambda+1$. In the table below, we give conditions on v, k, λ, μ, γ which are necessary for the results, and all equalities are given in the field $\mathbb{F}_2$. Additionally, it must satisfy the necessary conditions given for α and β.

Table 3.2: LCD codes from DRTs over $\mathbb{F}_2$

r	s	t	Pure construction	Bordered construction
0	0	1	$k=0,\ \lambda=1$	$k=\gamma,\ \lambda=1$
0	1	0	$k=0,\ \lambda=1$	$k=\gamma,\ \lambda=1$
0	1	1	Never	Never
1	0	0	Never	$\gamma=1$
1	0	1	$k=\lambda=1$	$k=\gamma+1,\ \lambda=1$
1	1	0	$k=\lambda=1$	$k=\gamma+1,\ \lambda=1$
1	1	1	Never	Never

3.2.2. LCD codes over the field $\mathbb{F}_3$

Over the field $\mathbb{F}_3$, we cannot simplify the equations from Theorems 3.1.1 and 3.1.6. The equalities in Tables 3.3, 3.4, 3.5 and 3.6 are given in the field $\mathbb{F}_3$. Necessary conditions are given for v, k, λ, μ and γ. For the bordered construction, two more conditions are given for α and β in Theorem 3.1.6. Also, the parameters of DRTs satisfy the conditions from Lemma 1.4.20.

Table 3.3: LCD codes from SRGs over $\mathbb{F}_3$

r	s	t	Pure construction
$\neq 0$	$=0$	$=0$	Always $(x=2)$
$\neq 0$	$\neq 0$	$=0$	$k=x+1,\ \lambda=rs,\ \mu=0$
$\neq 0$	$=0$	$\neq 0$	$v=k+x+2,\ k=\lambda+x+2=2rt+\mu+x$
$\neq 0$	$\neq 0$	$\neq 0$	$v=x-1,\ rs=x+(1-k+\lambda)(st-1),\ rt=x+(\mu-k)(st-1)$
$=0$	$\neq 0$	$=0$	$k=x+2,\ \lambda=\mu=0$
$=0$	$=0$	$\neq 0$	$v=k+x,\ \lambda=k+2x=\mu+1$
$=0$	$\neq 0$	$\neq 0$	$v=x,\ \lambda=k+x+st(1-k+\lambda),\ \mu=k+x+1+st(\mu-k)$

Table 3.4: LCD codes from SRGs over $\mathbb{F}_3$

r	s	t	Bordered construction
$\neq 0$	$=0$	$=0$	$\gamma = 0$ $(x=2)$
$\neq 0$	$\neq 0$	$=0$	$2k+x+1=\gamma^2,\ \mu = k+2x+2 = 2rs+\lambda$
$\neq 0$	$=0$	$\neq 0$	$k+2v+x+2=\gamma^2,\ \lambda = k+2x+1 = 2rt+\mu+1$
$\neq 0$	$\neq 0$	$\neq 0$	$v = x-1-\gamma^2,\ rs = x+(1-k+\lambda)(st-1),\ rt = x+(\mu-k)(st-1)$
$=0$	$\neq 0$	$=0$	$2k+x+2=\gamma^2,\ \lambda = k+2x+1 = \mu$
$=0$	$=0$	$\neq 0$	$k+2v+x=\gamma^2,\ \lambda = k+2x = \mu+1$
$=0$	$\neq 0$	$\neq 0$	$v = x-\gamma^2,\ \lambda = k+x+st(1-k+\lambda),\ \mu = k+x+1+st(\mu-k)$

Table 3.5: LCD codes from DRTs over $\mathbb{F}_3$

r	s	t	Pure construction
$\neq 0$	$=0$	$=0$	Always $(x=2)$
$\neq 0$	$\neq 0$	$=0$	$k = x+1,\ \lambda = rs+x$
$\neq 0$	$=0$	$\neq 0$	$k = x+1,\ \lambda = rt+x$
$\neq 0$	$\neq 0$	$\neq 0$	$k = 2x+2,\ rs+rt+st(\lambda+\mu) = k-1-\lambda$
$=0$	$\neq 0$	$=0$	$k = x+2,\ \lambda = x+1$
$=0$	$=0$	$\neq 0$	$k = x+2,\ \lambda = x+1$
$=0$	$\neq 0$	$\neq 0$	$k = 2x+1,\ k-1-\lambda = st(\lambda+\mu)$

Table 3.6: LCD codes from DRTs over $\mathbb{F}_3$

r	s	t	Bordered construction
$\neq 0$	$=0$	$=0$	$\gamma = 0$ $(x=2)$
$\neq 0$	$\neq 0$	$=0$	$2k+x+1=\gamma^2,\ \lambda = rs+x$
$\neq 0$	$=0$	$\neq 0$	$2k+x+1=\gamma^2,\ \lambda = rt+x$
$\neq 0$	$\neq 0$	$\neq 0$	$k+x+1=\gamma^2,\ rs+rt+st(\lambda+\mu) = 2\lambda+2x+1$
$=0$	$\neq 0$	$=0$	$2k+x+2=\gamma^2,\ \lambda = x+1$
$=0$	$=0$	$\neq 0$	$2k+x+2=\gamma^2,\ \lambda = x+1$
$=0$	$\neq 0$	$\neq 0$	$k+x+2=\gamma^2,\ st(\lambda+\mu) = 2\lambda+2x$

3.2.3. LCD codes over the field $\mathbb{F}_4$

In the field $\mathbb{F}_4$, we use the Hermitian inner product. Using Lemma 2.5.1 and conditions stated in Theorems 3.1.1 and 3.1.6, we determine when the pure and the bordered constructions give LCD codes obtained from SRGs and DRTs. The equalities in Tables 3.7, 3.8, 3.9 and 3.10 are given in the field $\mathbb{F}_4$. Necessary conditions are given for v, k, λ, μ and γ. For the bordered construction, two more conditions are given for α and β in Theorem 3.1.6. Also, the parameters of DRTs satisfy the conditions from Lemma 1.4.20.

Table 3.7: LCD codes from SRGs over $\mathbb{F}_4$

r	s	t	Pure construction ($x=1$)
$\neq 0$	$=0$	$=0$	Never
$\neq 0$	$\neq 0$	$=0$	$k=1,\ rs^2+sr^2=\lambda,\ \mu=0$
$\neq 0$	$=0$	$\neq 0$	$k=v=\lambda,\ rt^2+tr^2=\mu+v$
$\neq 0$	$\neq 0$	$\neq 0$	$v=0,\ rs^2+sr^2=(ts^2+st^2)(1+k+\lambda),\ rt^2+tr^2=(ts^2+st^2)(\mu+k)$
$=0$	$\neq 0$	$=0$	$k=\lambda=\mu=0$
$=0$	$=0$	$\neq 0$	$k+1=v=\lambda=\mu$
$=0$	$\neq 0$	$\neq 0$	$v=1,\ (ts^2+st^2)(1+k+\lambda)=1,\ (ts^2+st^2)(\mu+k)=1$

Table 3.8: LCD codes from SRGs over $\mathbb{F}_4$

r	s	t	Bordered construction
$\neq 0$	$=0$	$=0$	Never
$\neq 0$	$\neq 0$	$=0$	$k+x=\mu=\gamma^2,\ rs^2+sr^2+\lambda=\gamma^2$
$\neq 0$	$=0$	$\neq 0$	$k+v+x+1=\gamma^2,\ \lambda+v=\gamma^2,\ rt^2+tr^2+\mu+v=\gamma^2$
$\neq 0$	$\neq 0$	$\neq 0$	$v+x+1=\gamma^2,\ rs^2+sr^2=(ts^2+st^2)(1+k+\lambda)+v+\gamma^2,$ $rt^2+tr^2=(ts^2+st^2)(\mu+k)+v+\gamma^2$
$=0$	$\neq 0$	$=0$	$k+x+1=\lambda=\mu=\gamma^2$
$=0$	$=0$	$\neq 0$	$k+x=\lambda=\mu=v+\gamma^2$
$=0$	$\neq 0$	$\neq 0$	$v+x=\gamma^2,\ (ts^2+st^2)(1+k+\lambda)+v=\gamma^2,\ (ts^2+st^2)(\mu+k)+v=\gamma^2$

Table 3.9: LCD codes from DRTs over $\mathbb{F}_4$

r	s	t	Pure construction $(x = 1)$
$\neq 0$	$= 0$	$= 0$	Never
$\neq 0$	$\neq 0$	$= 0$	$k = \lambda = 1,\ r = s$
$\neq 0$	$= 0$	$\neq 0$	$k = \lambda = 1,\ r = t$
$\neq 0$	$\neq 0$	$\neq 0$	Never
$= 0$	$\neq 0$	$= 0$	$k = 0,\ \lambda = 1$
$= 0$	$= 0$	$\neq 0$	$k = 0,\ \lambda = 1$
$= 0$	$\neq 0$	$\neq 0$	$st^2\lambda + ts^2\mu = 0,\ st^2\mu + ts^2\lambda = 0$

Table 3.10: LCD codes from DRTs over $\mathbb{F}_4$

r	s	t	Bordered construction
$\neq 0$	$= 0$	$= 0$	Never
$\neq 0$	$\neq 0$	$= 0$	$k + x = \gamma^2,\ \lambda = x,\ r = s$
$\neq 0$	$= 0$	$\neq 0$	$k + x = \gamma^2,\ \lambda = x,\ r = t$
$\neq 0$	$\neq 0$	$\neq 0$	$x = \gamma^2,\ rt^2 + sr^2 + st^2\lambda + ts^2\mu = \gamma^2,\ rs^2 + tr^2 + st^2\mu + ts^2\lambda = \gamma^2$
$= 0$	$\neq 0$	$= 0$	$k + x + 1 = \gamma^2,\ \lambda = x$
$= 0$	$= 0$	$\neq 0$	$k + x + 1 = \gamma^2,\ \lambda = x$
$= 0$	$\neq 0$	$\neq 0$	$x + 1 = \gamma^2,\ st^2\lambda + ts^2\mu = \gamma^2,\ st^2\mu + ts^2\lambda = \gamma^2$

4. LCD codes from some families of SRGs

In this section, we construct LCD codes from some families of strongly regular graphs. For this purpose, we consider line graphs of complete graphs and bipartite complete graphs, some notable graphs such as the Petersen, Shrikhande, Clebsch, Hoffman-Singleton and Gewirtz graph and the Chang graphs, block graphs of Steiner triple systems, graphs obtained from orthogonal arrays and rank three permutation groups. In order to provide conditions under which certain LCD codes can be constructed from these graphs over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$, we use Tables 3.1, 3.3, 3.4, 3.7, 3.8 and parameters of the corresponding SRG.

We used the software package Magma [10] to construct LCD codes from the adjacency matrices of given strongly regular graphs and then to determine their minimum distances.

4.1. LCD codes from the line graph of complete graph

Definition 4.1.1. The **line graph** $L(G)$ of a graph G is the graph with the edge set of G as vertex set, where two vertices are adjacent if the corresponding edges of G have a vertex in common.
If B is the incidence matrix of G, then $B^T B - 2I$ is the adjacency matrix of $L(G)$.

Definition 4.1.2. The **Johnson graph** $J(n,k)$ is a simple graph that has k-element subsets of an n-element set as the vertex set. Two vertices are adjacent when the intersection

of the two corresponding subsets contains $k-1$ elements.
In other words, the Johnson graph has an edge for every pair of sets at distance one in the Johnson association scheme (*i.e.* for every pair of sets which are 1st associates as defined in Section 1.6).

For more information about line graphs, we refer to [5], and for Johnson graphs to [13,14].

The line graph of a complete graph $L(K_n)$ is also known as the *triangular graph* $T(n)$, and it is equivalent to the Johnson graph $J(n,2)$. For $n \geq 4$, these graphs form a simple family of strongly regular graphs and by [29] the parameters are

$$\left(\frac{1}{2}n(n-1), 2n-4, n-2, 4\right).$$

$L(K_n)$ are uniquely determined by their parameters as strongly regular graph, the only exception is $L(K_8)$. There are up to isomorphism precisely four strongly regular graphs with parameters $(28,12,6,4)$, one of them is $L(K_8)$, and the other three are the Chang graphs (LCD codes from these graphs are obtained in Subsection 4.3.4).

4.1.1. LCD codes over the field $\mathbb{F}_2$

Below, we give conditions to obtain LCD codes from $L(K_n)$ in the pure and the bordered constructions over the field $\mathbb{F}_2$.

Pure case

- If $n \equiv 0 \pmod 2$, then $P_{\mathbb{F}_2}(0,1,0)$ is an LCD code.
- If $n \equiv 0 \pmod 4$, then $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.
- If $n \equiv 1 \pmod 4$, then $P_{\mathbb{F}_2}(1,1,1)$ is an LCD code.

Bordered case

- If $n \equiv 0 \pmod 4$, then $B_{\mathbb{F}_2}(0,0,1)$, with $\alpha = \beta = 0$, $\gamma = 1$, is an LCD code. $B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, with $\alpha = \gamma = 0$, $\beta \in \mathbb{F}_2$, are LCD codes.
- If $n \equiv 1 \pmod 4$, then $B_{\mathbb{F}_2}(1,1,1)$, with $\alpha = \gamma = 0$, $\beta \in \mathbb{F}_2$, is an LCD code.

- If $n \equiv 2 \pmod 4$, then $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$, with $\gamma = 0,\ \alpha = \beta,\ \alpha,\beta \in \mathbb{F}_2$, are LCD codes. $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, with $\gamma = 1,\ \alpha = \beta,\ \alpha,\beta \in \mathbb{F}_2$, are LCD codes.

- If $n \equiv 3 \pmod 4$, then $B_{\mathbb{F}_2}(1,1,1)$, with $\gamma = 1,\ \alpha = \beta,\ \alpha,\beta \in \mathbb{F}_2$, is an LCD code.

4.1.2. LCD codes over the field $\mathbb{F}_3$

Below, we give conditions to obtain LCD codes from $L(K_n)$ in the pure and the bordered constructions over the field $\mathbb{F}_3$.

Pure case $(a,b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b)$

- $P_{\mathbb{F}_3}(a,0,0)$ are LCD codes.

- If $n \equiv 0 \pmod 3$, then $P_{\mathbb{F}_3}(a,0,a)$ and $P_{\mathbb{F}_3}(a,a,a)$ are LCD codes.

- If $n \equiv 1 \pmod 3$, then $P_{\mathbb{F}_3}(a,a,a)$ and $P_{\mathbb{F}_3}(0,0,a)$ are LCD codes.

- If $n \equiv 2 \pmod 3$, then $P_{\mathbb{F}_3}(a,b,b)$ and $P_{\mathbb{F}_3}(0,a,b)$ are LCD codes.

Bordered case $(a,b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b)$

- $B_{\mathbb{F}_3}(a,0,0)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.

- If $n \equiv 0 \pmod 3$, then $B_{\mathbb{F}_3}(a,0,a)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, with $\gamma = 0,\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, with $\alpha\gamma + b\beta = 0,\ \alpha,\beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3 \backslash \{0\}$, are LCD codes.

- If $n \equiv 1 \pmod 3$, then $B_{\mathbb{F}_3}(a,a,a)$, with $\gamma = 0,\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(a,a,b)$, with $\alpha\gamma + a\beta = 0,\ \alpha,\beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3 \backslash \{0\}$, are LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, with $\alpha\gamma + b\beta = 0,\ \alpha,\beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3 \backslash \{0\}$, are LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.

- If $n \equiv 2 \pmod 3$, then $B_{\mathbb{F}_3}(a,b,b)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3 \backslash \{0\}$, are LCD codes. $B_{\mathbb{F}_3}(0,a,b)$, with $\gamma = 0,\ \alpha,\beta \in \mathbb{F}_3,\ 1 + \alpha^2 + \beta^2 \in \mathbb{F}_3 \backslash \{0\}$, are LCD codes.

4.1.3. LCD codes over the field $\mathbb{F}_4$

Below, we give conditions to obtain LCD codes from $L(K_n)$ in the pure and the bordered constructions over the field $\mathbb{F}_4$.

Pure case $(a, b \in \mathbb{F}_4 \setminus \{0\},\ a \neq b)$

- If $n \equiv 0 \pmod 4$, then $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$ are LCD codes.
- If $n \equiv 1 \pmod 4$, then $P_{\mathbb{F}_4}(a,a,a)$ are LCD codes.
- If $n \equiv 2 \pmod 4$, then $P_{\mathbb{F}_4}(0,a,0)$ are LCD codes.

Bordered case $(a, b \in \mathbb{F}_4 \setminus \{0\},\ a \neq b)$

- If $n \equiv 0 \pmod 4$, then $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, with $\gamma = 0,\ \alpha \in \mathbb{F}_4 \setminus \{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.
- If $n \equiv 1 \pmod 4$, then $B_{\mathbb{F}_4}(a,a,a)$, with $\gamma = 0,\ \alpha \in \mathbb{F}_4 \setminus \{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.
- If $n \equiv 2 \pmod 4$, then $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, with $\gamma = 1,\ \alpha + a\beta = 0,\ 1 + \alpha^2 + \beta^2 \in \mathbb{F}_4 \setminus \{0\},\ \alpha, \beta \in \mathbb{F}_4$, are LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, with $\gamma = 0,\ 1 + \alpha^2 + \beta^2 \in \mathbb{F}_4 \setminus \{0\},\ \alpha, \beta \in \mathbb{F}_4$, are LCD codes.
- If $n \equiv 3 \pmod 4$, then $B_{\mathbb{F}_4}(a,a,a)$, with $\gamma = 1,\ \alpha + a\beta = 0,\ 1 + \alpha^2 + \beta^2 \in \mathbb{F}_4 \setminus \{0\}$, $\alpha, \beta \in \mathbb{F}_4$, are LCD codes.

4.1.4. Examples computed in Magma

The adjacency matrices of $L(K_n)$ were obtained using the built-in functions in Magma. By applying the pure and the bordered constructions, we constructed LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$ for all $n = 4, \ldots, 10$.

1. The graf $L(K_4)$ is the unique SRG$(6,4,2,4)$.
   $P_{\mathbb{F}_2}(0,1,0)$, $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,1)$ are $[12,6,2]$ LCD codes.

$B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[14,7,1]$ LCD code.

$B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, are $[14,7,1]$ LCD codes. $B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, are $[14,7,2]$ LCD codes.

$P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,a,a)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[12,6,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[14,7,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[14,7,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[14,7,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[14,7,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[14,7,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[14,7,4]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[14,7,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[14,7,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[12,6,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\}$, are $[14,7,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,b,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[14,7,2]$ LCD codes.

2. The graf $L(K_5)$ is the unique $\mathrm{SRG}(10,6,3,4)$.

   $P_{\mathbb{F}_2}(1,1,1)$ is an $[20,10,2]$ LCD code.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[22,11,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[22,11,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[20,10,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,b)$ and $P_{\mathbb{F}_3}(0,a,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[20,10,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[22,11,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[22,11,2]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[22,11,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[22,11,4]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[22,11,1]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[22,11,2]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\gamma=0,\ a,b,\beta\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[22,11,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[20,10,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[22,11,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[22,11,2]$ LCD codes.

3. The graf $L(K_6)$ is the unique SRG$(15,8,4,4)$.

$P_{\mathbb{F}_2}(0,1,0)$ is an $[30,15,3]$ LCD code.

$B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=\gamma=0$, is an $[32,16,1]$ LCD code. $B_{\mathbb{F}_2}(0,0,1)$, where $\gamma=0,\ \alpha=\beta=1$, is an $[32,16,7]$ LCD code, and this code is near-optimal.

$B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=\gamma=0$, is an $[32,16,1]$ LCD code. $B_{\mathbb{F}_2}(0,1,0)$, where $\gamma=0,\ \alpha=\beta=1$, is an $[32,16,3]$ LCD code.

$B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[32,16,1]$ LCD code. $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=1$, is an $[32,16,4]$ LCD code.

$B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[32,16,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[32,16,2]$ LCD code.

$P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[30,15,2]$ LCD codes.

$P_{\mathbb{F}_3}(a,0,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[30,15,8]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[32,16,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,8]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,4]$ LCD codes.

$P_{\mathbb{F}_4}(0,a,0)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[30,15,3]$ LCD codes.

$B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha=\beta=0,\ \gamma=1$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\gamma=1,\ \alpha+a\beta=0$, $a,b,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[32,16,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\},\ \alpha=\beta=0,\ \gamma=1$, are $[32,16,1]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[32,16,2]$ LCD codes.

$B_{\mathbb{F}_4}(0,a,0)$, where $a\in\mathbb{F}_4\backslash\{0\},\ \alpha=\beta=\gamma=0$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[32,16,2]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4$, are $[32,16,3]$ LCD codes.

4. The graf $L(K_7)$ is the unique SRG$(21,10,5,4)$.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[44,22,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[44,22,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[42,21,2]$ LCD codes.

   $P_{\mathbb{F}_3}(0,0,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[42,21,7]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[44,22,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[44,22,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[44,22,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[44,22,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[44,22,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[44,22,8]$ LCD codes.

   $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[44,22,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[44,22,4]$ LCD codes.

   $B_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\},\ \alpha=\beta=0,\ \gamma=1$, are $[44,22,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[44,22,2]$ LCD codes.

5. The graf $L(K_8)$ is an SRG$(28,12,6,4)$.

   $P_{\mathbb{F}_2}(0,1,0)$ is an $[56,28,3]$ LCD code.

   $P_{\mathbb{F}_2}(1,0,1)$ is an $[56,28,4]$ LCD code.

   $P_{\mathbb{F}_2}(1,1,1)$ is an $[56,28,2]$ LCD code.

   $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, are $[58,29,1]$ LCD codes.

   $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\gamma=0,\ \beta=1$, are $[58,29,4]$ LCD codes.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_2}(1,1,1)$, where

$\alpha = \gamma = 0,\ \beta = 1$, are $[58,29,2]$ LCD codes.

$P_{\mathbb{F}_3}(a,0,0)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes.

$P_{\mathbb{F}_3}(a,b,b)$ and $P_{\mathbb{F}_3}(0,a,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[56,28,4]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha = \beta = \gamma = 0,\ a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta = \gamma = 0,\ a,b,\alpha \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[58,29,2]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,a)$, where $\alpha = \beta = 0,\ a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha = 0,\ a,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[58,29,4]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = \beta = \gamma = 0,\ a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[58,29,1]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,b)$, where $\beta = \gamma = 0,\ a,b,\alpha \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[58,29,2]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = \gamma = 0,\ a,b,\beta \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[58,29,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,0,a)$ and $P_{\mathbb{F}_4}(a,b,a)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[56,28,4]$ LCD codes.

$P_{\mathbb{F}_4}(0,a,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[56,28,3]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[56,28,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\alpha = \beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\}$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[58,29,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\gamma = 0$, $a,b,\beta \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[58,29,4]$ LCD codes.

$B_{\mathbb{F}_4}(0,a,0)$, where $\alpha = \beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\}$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[58,29,2]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma = 0,\ a,b,\beta \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[58,29,3]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[58,29,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma = 0,\ a,\beta \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[58,29,2]$ LCD codes.

6. The graf $L(K_9)$ is the unique SRG$(36,14,7,4)$.

   $P_{\mathbb{F}_2}(1,1,1)$ is an $[72,36,2]$ LCD code.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \beta = \gamma = 0$, is an $[74,37,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta = 1$, is an $[74,37,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,0,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,8]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[74,37,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,8]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[72,36,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[74,37,2]$ LCD codes.

7. The graf $L(K_{10})$ is the unique SRG$(45,16,8,4)$.

   $P_{\mathbb{F}_2}(0,1,0)$ is an $[90,45,3]$ LCD code.

   $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=\gamma=0$, is an $[92,46,1]$ LCD code. $B_{\mathbb{F}_2}(0,0,1)$, where $\gamma=0,\ \alpha=\beta=1$, is an $[92,46,8]$ LCD code.

   $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=\gamma=0$, is an $[92,46,1]$ LCD code. $B_{\mathbb{F}_2}(0,1,0)$, where $\gamma=0,\ \alpha=\beta=1$, is an $[92,46,3]$ LCD code.

   $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[92,46,1]$ LCD code. $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=1$, is an $[92,46,4]$ LCD code.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[92,46,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[92,46,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[90,45,2]$ LCD codes.

   $P_{\mathbb{F}_3}(0,0,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[90,45,8]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[92,46,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[92,46,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[92,46,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[92,46,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[92,46,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[92,46,8]$ LCD codes.

   $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[92,46,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[92,46,4]$ LCD codes.

$P_{\mathbb{F}_4}(0,a,0)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[90,45,3]$ LCD codes.

$B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha=\beta=0,\ \gamma=1$, are $[92,46,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\gamma=1,\ \alpha+a\beta=0$, $a,b,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[92,46,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\},\ \alpha=\beta=0,\ \gamma=1$, are $[92,46,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[92,46,2]$ LCD codes.

$B_{\mathbb{F}_4}(0,a,0)$, where $a\in\mathbb{F}_4\backslash\{0\},\ \alpha=\beta=\gamma=0$, are $[92,46,1]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[92,46,2]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4$, are $[92,46,3]$ LCD codes.

4.2. LCD CODES FROM THE LINE GRAPH OF BIPARTITE COMPLETE GRAPH

Definition 4.2.1. The **Hamming graph** $H(n,q)$ is a simple graph that has the set F_q^n of all words of length n (ordered n-tuples) over an alphabet F of q symbols as the vertex set. Two vertices are adjacent if they differ in precisely one coordinate, *i.e.* if their Hamming distance is one.
In other words, the Hamming graph has an edge for every pair of ordered n-tuples which are 1st associates in the Hamming association scheme (as defined in Section 1.6).

For more information about Hamming graphs, we refer to [13, 14].

The line graph of a complete bipartite graph $L(K_{n,n})$ is also known as the *square lattice graph*, and it is equivalent to the Hamming graph $H(2,n)$. For $n \geq 2$, these graphs form a simple family of strongly regular graphs, and by [29] the parameters are

$$(n^2, 2n-2, n-2, 2).$$

$L(K_{n,n})$ are uniquely determined by their parameters as strongly regular graphs, the only exception is $L(K_{4,4})$. There are up to isomorphism precisely two strongly regular graphs with parameters $(16,6,2,2)$, one of them is $L(K_{4,4})$ and the other one is the Shrikhande graph (LCD codes from this graph are obtained in Subsection 4.3.2).

4.2.1. LCD codes over the field $\mathbb{F}_2$

Below, we give conditions to obtain LCD codes from $L(K_{n,n})$ in the pure and the bordered constructions over the field $\mathbb{F}_2$.

Pure case

- If $n \equiv 0 \pmod 2$, then $P_{\mathbb{F}_2}(0,1,0)$, $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.

Bordered case

- If $n \equiv 0 \pmod 2$, then $B_{\mathbb{F}_2}(0,0,1)$, with $\alpha = \beta = 0$, $\gamma = 1$, is an LCD code. $B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, with $\alpha = \gamma = 0$, $\beta \in \mathbb{F}_2$, are LCD codes.

- If $n \equiv 1 \pmod 2$, then $B_{\mathbb{F}_2}(1,1,1)$, with $\alpha = \beta,\ \gamma = 1$, is an LCD code.

4.2.2. LCD codes over the field $\mathbb{F}_3$

Below, we give conditions to obtain LCD codes from $L(K_{n,n})$ in the pure and the bordered constructions over the field $\mathbb{F}_3$.

Pure case $(a, b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b)$

- $P_{\mathbb{F}_3}(a,0,0)$ are LCD codes.
- If $n \equiv 0 \pmod 3$, then $P_{\mathbb{F}_3}(a,a,a)$ are LCD codes.
- If $n \equiv 1 \pmod 3$, then $P_{\mathbb{F}_3}(a,0,a)$, $P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(a,b,b)$ are LCD codes.
- If $n \equiv 2 \pmod 3$, then $P_{\mathbb{F}_3}(a,a,b)$, $P_{\mathbb{F}_3}(a,b,b)$ and $P_{\mathbb{F}_3}(0,0,a)$ are LCD codes.

Bordered case $(a, b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b)$

- $B_{\mathbb{F}_3}(a,0,0)$ with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.
- If $n \equiv 0 \pmod 3$, then $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3 \backslash \{0\}$, are LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, with $\gamma = 0,\ \alpha, \beta \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, with $\alpha\gamma + b\beta = 0,\ \alpha, \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3 \backslash \{0\}$, are LCD codes.
- If $n \equiv 1 \pmod 3$, then $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$ with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(0,a,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3 \backslash \{0\}$, are LCD codes.
- If $n \equiv 2 \pmod 3$, then $B_{\mathbb{F}_3}(a,a,0)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3 \backslash \{0\}$, are LCD codes. $B_{\mathbb{F}_3}(a,a,b)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.

4.2.3. LCD codes over the field $\mathbb{F}_4$

Below, we give conditions to obtain LCD codes from $L(K_{n,n})$ in the pure and the bordered constructions over the field $\mathbb{F}_4$.

Pure case $(a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b)$

- If $n \equiv 0 \pmod 2$, then $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$ are LCD codes.

Bordered case $(a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b)$

- If $n \equiv 0 \pmod 2$, then $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, with $\gamma = 0,\ \alpha \in \mathbb{F}_4 \backslash \{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.
- If $n \equiv 1 \pmod 2$, then $B_{\mathbb{F}_4}(a,a,a)$, with $\gamma = 1,\ \alpha + a\beta = 0,\ 1 + \alpha^2 + \beta^2 \in \mathbb{F}_4 \backslash \{0\}$, $\alpha, \beta \in \mathbb{F}_4$, are LCD codes.

4.2.4. Examples computed in Magma

The adjacency matrices of $L(K_{n,n})$ were obtained using the built-in functions in Magma. By applying the pure and the bordered constructions, we constructed LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$ for all $n = 2, \ldots, 8$.

1. The graf $L(K_{2,2})$ is the unique SRG$(4,2,0,2)$.
   $P_{\mathbb{F}_2}(0,1,0)$, $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,1)$ are $[8,4,2]$ LCD codes.
   $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha = \beta = 0,\ \gamma = 1$, is an $[10,5,1]$ LCD code.
   $B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \beta = \gamma = 0$, are $[10,5,1]$ LCD codes. $B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta = 1$, are $[10,5,2]$ LCD codes.
   $P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a \in \mathbb{F}_3 \backslash \{0\}$, are $[8,4,2]$ LCD codes.
   $P_{\mathbb{F}_3}(a,a,b)$ and $P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[8,4,4]$ LCD codes, and these codes are optimal.
   $B_{\mathbb{F}_3}(a,a,0)$, where $\alpha = \beta = 0,\ a,\gamma \in \mathbb{F}_3 \backslash \{0\}$, are $[10,5,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,0)$, where $\alpha = 0,\ a,\beta,\gamma \in \mathbb{F}_3 \backslash \{0\}$, are $[10,5,4]$ LCD codes, and these codes are near-optimal.
   $B_{\mathbb{F}_3}(a,a,b)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha = \beta = \gamma = 0,\ a,b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[10,5,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\beta = \gamma = 0,\ a,b,\alpha \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[10,5,2]$ LCD codes.

$P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[8,4,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[10,5,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[10,5,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0$, $a,b,\beta \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[10,5,2]$ LCD codes.

2. The graf $L(K_{3,3})$ is the unique SRG$(9,4,1,2)$.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[20,10,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[20,10,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[18,9,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[20,10,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=0,\ a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[20,10,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[20,10,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\beta=\gamma=0,\ a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[20,10,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\gamma=0,\ a,\beta \in \mathbb{F}_3\backslash\{0\}$, are $[20,10,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[20,10,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[20,10,4]$ LCD codes.

   $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[20,10,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[20,10,2]$ LCD codes.

3. The graf $L(K_{4,4})$ is an SRG$(16,6,2,2)$.

   $P_{\mathbb{F}_2}(0,1,0)$ and $P_{\mathbb{F}_2}(1,0,1)$ are $[32,16,4]$ LCD codes.

   $P_{\mathbb{F}_2}(1,1,1)$ is an $[32,16,2]$ LCD code.

   $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[34,17,1]$ LCD code.

   $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\gamma=0,\ \beta=1$, are $[34,17,4]$ LCD codes.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[34,17,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[34,17,2]$ LCD code.

$P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,2]$ LCD codes.

$P_{\mathbb{F}_3}(a,0,a)$ and $P_{\mathbb{F}_3}(a,b,a)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[32,16,8]$ LCD codes.

$P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[32,16,4]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,2]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,8]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[32,16,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[32,16,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[34,17,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,b,\beta\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[34,17,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[34,17,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[34,17,2]$ LCD codes.

4. The graf $L(K_{5,5})$ is the unique SRG$(25,8,3,2)$.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[52,26,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[52,26,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[50,25,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,a,b)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[50,25,8]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[50,25,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,0)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[52,26,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,0)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[52,26,8]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,b)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[52,26,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=$

$\gamma = 0,\ a,b,\alpha \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[52,26,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha = \beta = 0,\ \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[52,26,1]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\gamma = 1,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[52,26,2]$ LCD codes.

5. The graf $L(K_{6,6})$ is the unique SRG$(36,10,4,2)$.

$P_{\mathbb{F}_2}(0,1,0)$ and $P_{\mathbb{F}_2}(1,0,1)$ are $[72,36,4]$ LCD codes.

$P_{\mathbb{F}_2}(1,1,1)$ is an $[72,36,2]$ LCD code.

$B_{\mathbb{F}_2}(0,0,1)$, where $\alpha = \beta = 0,\ \gamma = 1$, is an $[74,37,1]$ LCD code.

$B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha = \beta = \gamma = 0$, are $[74,37,1]$ LCD codes.

$B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha = \gamma = 0,\ \beta = 1$, are $[74,37,4]$ LCD codes.

$B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \beta = \gamma = 0$, is an $[74,37,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta = 1$, is an $[74,37,2]$ LCD code.

$P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = \beta = 0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = 0,\ a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[74,37,4]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\beta = \gamma = 0,\ a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[74,37,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha = \gamma = 0,\ a,\beta \in \mathbb{F}_3\backslash\{0\}$, are $[74,37,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha = \beta = 0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[74,37,1]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma + b\beta = 0,\ a,b,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[74,37,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[72,36,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[72,36,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha = \beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[74,37,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma = 0,\ a,b,\beta \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[74,37,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[74,37,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma = 0,\ a,\beta \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[74,37,2]$ LCD codes.

6. The graf $L(K_{7,7})$ is the unique SRG$(49,12,5,2)$.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \beta = 0,\ \gamma = 1$, is an $[100,50,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \beta = \gamma = 1$, is an $[100,50,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[98,49,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,0,a)$ and $P_{\mathbb{F}_3}(a,b,a)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[98,49,8]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[98,49,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha = \beta = \gamma = 0,\ a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta = \gamma = 0,\ a,b,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[100,50,2]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha = \beta = 0,\ a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha = 0,\ a,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[100,50,8]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha = \beta = 0,\ a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha = 0,\ a,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[100,50,4]$ LCD codes.

   $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha = \beta = 0,\ \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma = 1,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[100,50,2]$ LCD codes.

7. The graf $L(K_{8,8})$ is the unique SRG$(64,14,6,2)$.

   $P_{\mathbb{F}_2}(0,1,0)$ and $P_{\mathbb{F}_2}(1,0,1)$ are $[128,64,4]$ LCD codes.

   $P_{\mathbb{F}_2}(1,1,1)$ is an $[128,64,2]$ LCD code.

   $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha = \beta = 0,\ \gamma = 1$, is an $[130,65,1]$ LCD code.

   $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha = \beta = \gamma = 0$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha = \gamma = 0,\ \beta = 1$, are $[130,65,4]$ LCD codes.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \beta = \gamma = 0$, is an $[130,65,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta = 1$, is an $[130,65,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[128,64,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,a,b)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[128,64,8]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[128,64,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,0)$, where $\alpha = \beta = 0,\ a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,0)$, where $\alpha = 0,\ a,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[130,65,8]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,b)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha = \beta = \gamma = 0,\ a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq$

b, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=\gamma=0$, $a,b,\alpha\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[130,65,2]$ LCD codes.

$P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, are $[128,64,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[128,64,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0$, $a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0$, $a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, $\alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[130,65,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0$, $a,b,\beta\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, $\alpha\in\mathbb{F}_4\backslash\{1\}$, are $[130,65,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[130,65,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0$, $a,\beta\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{1\}$, are $[130,65,2]$ LCD codes.

4.3. LCD CODES FROM SOME SPECIAL GRAPHS

In this section, we give LCD codes constructed from the adjacency matrices of some special graphs, such as the Petersen, Shrikhande, Clebsch, Hoffman-Singleton and Gewirtz graph and the Chang graphs. The adjacency matrices of these graphs were obtained using the built-in functions in Magma. More about the listed graphs can be found in [14].

Below, we give the list of all LCD codes constructed over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$ from the pure and the bordered constructions.

4.3.1. The Petersen graph

The Petersen graph is the complement of the triangular graph $T(5)$. It is the unique $\mathrm{SRG}(10,3,0,1)$.

- $P_{\mathbb{F}_2}(1,1,1)$ is an $[20,10,2]$ LCD code.
- $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[22,11,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[22,11,2]$ LCD code.
- $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[20,10,2]$ LCD codes.
  $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[20,10,4]$ LCD codes.
  $P_{\mathbb{F}_3}(0,a,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[20,10,4]$ LCD codes.
- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[22,11,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[22,11,2]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[22,11,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[22,11,4]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[22,11,1]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[22,11,2]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\gamma=0,\ a,b,\beta\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[22,11,4]$ LCD codes.
- $P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[20,10,2]$ LCD codes.

- $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[22,11,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[22,11,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0$, $a,\beta\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{1\}$, are $[22,11,2]$ LCD codes.

4.3.2. The Shrikhande graph

The Shrikhande graph is an $\mathrm{SRG}(16,6,2,2)$. This graph shares strong regularity parameters with $L(K_{4,4})$ but they are not isomorphic.

- $P_{\mathbb{F}_2}(0,1,0)$ and $P_{\mathbb{F}_2}(1,0,1)$ are $[32,16,4]$ LCD codes.
  $P_{\mathbb{F}_2}(1,1,1)$ is an $[32,16,2]$ LCD code.

- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0$, $\gamma=1$, is an $[34,17,1]$ LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, are $[34,17,1]$ LCD codes.
  $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\gamma=0$, $\beta=1$, are $[34,17,4]$ LCD codes.
  $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[34,17,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0$, $\beta=1$, is an $[34,17,2]$ LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,2]$ LCD codes.
  $P_{\mathbb{F}_3}(a,0,a)$ and $P_{\mathbb{F}_3}(a,b,a)$, where $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[32,16,8]$ LCD codes.
  $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[32,16,4]$ LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\beta=\gamma=0$, $a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,2]$ LCD codes.
  $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0$, $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0$, $a,b,\alpha\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[34,17,2]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=0$, $a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=0$, $a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,8]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0$, $a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0$, $a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,4]$ LCD codes.

- $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, are $[32,16,4]$

LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a \in \mathbb{F}_4 \backslash \{0\}$, are $[32,16,2]$ LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0,\ a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4 \backslash \{0,1\}$, are $[34,17,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,b,\beta \in \mathbb{F}_4 \backslash \{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4 \backslash \{1\}$, are $[34,17,4]$ LCD codes.

  $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_4 \backslash \{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ a \in \mathbb{F}_4 \backslash \{0\},\ \alpha \in \mathbb{F}_4 \backslash \{0,1\}$, are $[34,17,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta \in \mathbb{F}_4 \backslash \{0\},\ \alpha \in \mathbb{F}_4 \backslash \{1\}$, are $[34,17,2]$ LCD codes.

4.3.3. The Clebsch graph

The Clebsch graph is either of two complementary graphs on 16 vertices, a 5-regular graph and a 10-regular graph. Hence, the Clebsch graph is the unique $\mathrm{SRG}(16,10,6,6)$ or the unique $\mathrm{SRG}(16,5,0,2)$. Below, we give the list of all LCD codes constructed only from $\mathrm{SRG}(16,10,6,6)$ because $\mathrm{SRG}(16,5,0,2)$ gives the same list of LCD codes.

- $P_{\mathbb{F}_2}(0,1,0)$ is an $[32,16,6]$ LCD code.

  $P_{\mathbb{F}_2}(1,0,1)$ is an $[32,16,8]$ LCD code, and this code is optimal.

  $P_{\mathbb{F}_2}(1,1,1)$ is an $[32,16,2]$ LCD code.

- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[34,17,1]$ LCD code.

  $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=\gamma=0$, is an $[34,17,1]$ LCD code. $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[34,17,6]$ LCD code.

  $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, is an $[34,17,1]$ LCD code. $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[34,17,7]$ LCD code, and this code is near-optimal.

  $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[34,17,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[34,17,2]$ LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, where $a \in \mathbb{F}_3 \backslash \{0\}$, are $[32,16,2]$ LCD codes.

  $P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(0,a,0)$, where $a,b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[32,16,6]$ LCD codes.

  $P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[32,16,4]$ LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[34,17,2]$ LCD codes.
  $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,8]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,0)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,2]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,4]$ LCD codes.

- $P_{\mathbb{F}_4}(0,a,0)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[32,16,6]$ LCD codes.
  $P_{\mathbb{F}_4}(a,0,a)$ and $P_{\mathbb{F}_4}(a,b,a)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[32,16,8]$ LCD codes.
  $P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[32,16,2]$ LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[34,17,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\alpha=\gamma=0,\ a,b,\beta\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[34,17,7]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\gamma=0,\ a,b,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[34,17,8]$ LCD codes.
  $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[34,17,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[34,17,2]$ LCD codes.
  $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[34,17,2]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[34,17,6]$ LCD codes.

4.3.4. The Chang graphs

The Chang graphs are a set of three SRGs with the same parameters $(28,12,6,4)$. These graphs share strong regularity parameters with $L(K_8)$ but they are not isomorphic. All three graphs give the same list of LCD codes.

- $P_{\mathbb{F}_2}(0,1,0)$ and $P_{\mathbb{F}_2}(1,0,1)$ are $[56,28,4]$ LCD codes.
  $P_{\mathbb{F}_2}(1,1,1)$ is an $[56,28,2]$ LCD code.

- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[58,29,1]$ LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, are $[58,29,1]$ LCD codes.
  $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\gamma=0,\ \beta=1$, are $[58,29,4]$ LCD codes.
  $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[58,29,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[59,29,2]$ LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes. $P_{\mathbb{F}_3}(a,b,b)$ and $P_{\mathbb{F}_3}(0,a,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[56,28,4]$ LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[58,29,2]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[58,29,4]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[58,29,1]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\gamma=0,\ a,b,\beta\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[58,29,4]$ LCD codes.

- $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[56,28,4]$ LCD codes.
  $P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[56,28,2]$ LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[58,29,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,b,\beta\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[58,29,4]$ LCD codes.
  $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[58,29,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[58,29,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[58,29,2]$ LCD codes.

4.3.5. The Hoffman-Singleton graph

The Hoffman-Singleton graph is the unique $\mathrm{SRG}(50,7,0,1)$.

- $P_{\mathbb{F}_2}(1,1,1)$ is an $[100,50,2]$ LCD code.

- $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[102,51,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[102,51,2]$ LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,2]$ LCD codes.
  $P_{\mathbb{F}_3}(a,0,b)$ and $P_{\mathbb{F}_3}(0,a,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[100,50,14]$ LCD codes.
  $P_{\mathbb{F}_3}(0,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,4]$ LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,0,b)$, $B_{\mathbb{F}_3}(0,a,a)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[102,51,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,0,b)$, $B_{\mathbb{F}_3}(0,a,a)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[102,51,2]$ LCD codes.
  $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[102,51,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha\gamma+2a\beta=0,\ a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[102,51,2]$ LCD codes.

- $P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[100,50,2]$ LCD codes.

- $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[102,51,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[102,51,2]$ LCD codes.

4.3.6. The Gewirtz graph

The Gewirtz graph is the unique $\mathrm{SRG}(56,10,0,2)$.

- $P_{\mathbb{F}_2}(0,1,0)$ and $P_{\mathbb{F}_2}(1,0,1)$ are $[112,56,6]$ LCD codes.
  $P_{\mathbb{F}_2}(1,1,1)$ is an $[112,56,2]$ LCD code.

- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[114,57,1]$ LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, are $[114,57,1]$ LCD codes.
  $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\gamma=0,\ \beta=1$, are $[114,57,6]$ LCD codes.
  $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[114,57,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[114,57,2]$ LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[112,56,2]$ LCD codes.
  $P_{\mathbb{F}_3}(0,0,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[112,56,7]$ LCD codes.
  $P_{\mathbb{F}_3}(0,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[112,56,4]$ LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_3\backslash\{0\}$, are $[114,57,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\beta=\gamma=0$, $a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[114,57,2]$ LCD codes.

  $B_{\mathbb{F}_3}(a,a,0)$ and $B_{\mathbb{F}_3}(a,a,b)$, where $\alpha=\beta=0$, $a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[114,57,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,0)$ and $B_{\mathbb{F}_3}(a,a,b)$, where $\alpha\gamma+2a\beta=0$, $a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[114,57,8]$ LCD codes.

  $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=0$, $a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[114,57,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha\gamma+2a\beta=0$, $a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[114,57,2]$ LCD codes.

  $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_3\backslash\{0\}$, are $[114,57,1]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=\gamma=0$, $a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[114,57,2]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\gamma=0$, $a,\alpha,\beta\in\mathbb{F}_3\backslash\{0\}$, are $[114,57,7]$ LCD codes.

- $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, are $[112,56,6]$ LCD codes.

  $P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[112,56,2]$ LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0$, $a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, are $[114,57,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0$, $a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, $\alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[114,57,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0$, $a,b,\beta\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, $\alpha\in\mathbb{F}_4\backslash\{1\}$, are $[114,57,6]$ LCD codes.

  $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[114,57,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[114,57,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0$, $a,\beta\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{1\}$, are $[114,57,2]$ LCD codes.

4.4. LCD CODES FROM THE BLOCK GRAPH OF STEINER TRIPLE SYSTEM

Steiner triple systems are a simple family of block designs, but also the most studied type of designs. We have already seen the Steiner triple system of order 7 in Example 1.2.14, and here we define these designs.

Definition 4.4.1. A **Steiner triple system** $\mathrm{STS}(n)$ of order n is a 2-$(n,3,1)$ design. It is a collection of 3-element subsets of an n-element set such that any pair of elements of the n-element set is contained in a unique one among these 3-element sets.

Theorem 4.4.2. A Steiner triple system of order n exists if and only if $n \equiv 1,3 \pmod 6$, $n \geq 7$.

More information on Steiner triple systems, as well as the proof of the previous theorem, can be found in [59].

Definition 4.4.3. The **block graph** (or line graph) of a Steiner triple system is the graph with 3-element sets as vertices, where two 3-element sets are adjacent when they have non-empty intersection.

The following result can be found in [37].

Theorem 4.4.4. The block graph of an $\mathrm{STS}(n)$, $n \geq 9$, is strongly regular with parameters

$$\Big(\frac{1}{6}n(n-1), \frac{3}{2}(n-3), \frac{1}{2}(n+3), 9\Big).$$

4.4.1. LCD codes over the field $\mathbb{F}_2$

Below, we give conditions to obtain LCD codes from the block graph of $\mathrm{STS}(n)$ in the pure and the bordered constructions over the field $\mathbb{F}_2$.

Pure case

- If $n \equiv 1,3 \pmod 6$ and $n \equiv 3 \pmod 4$, then $P_{\mathbb{F}_2}(0,0,1)$ is an LCD code.

Bordered case

- If $n \equiv 1,3 \pmod 6$ and $n \equiv 3 \pmod 4$, then $B_{\mathbb{F}_2}(0,0,1)$, with $\gamma = 0,\ \alpha = \beta,\ \alpha,\beta \in \mathbb{F}_2$, is an LCD code. $B_{\mathbb{F}_2}(1,1,0)$ and $B_{\mathbb{F}_2}(1,1,1)$, with $\gamma = 1,\ \alpha = \beta,\ \alpha,\beta \in \mathbb{F}_2$, are LCD codes.

4.4.2. LCD codes over the field $\mathbb{F}_3$

Below, we give conditions to obtain LCD codes from the block graph of STS(n) in the pure and the bordered constructions over the field $\mathbb{F}_3$.

Pure case ($a,b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$)

- $P_{\mathbb{F}_3}(a,0,0)$ are LCD codes.
- If $n \equiv 1 \pmod 6$, then $P_{\mathbb{F}_3}(a,b,0)$ are LCD codes.
- If $n \equiv 1 \pmod 6$ and $n \equiv 1 \pmod 9$, then $P_{\mathbb{F}_3}(a,a,a)$ are LCD codes.
- If $n \equiv 1 \pmod 6$ and $n \equiv 4 \pmod 9$, then $P_{\mathbb{F}_3}(0,a,b)$ are LCD codes.
- If $n \equiv 1 \pmod 6$ and $n \equiv 7 \pmod 9$, then $P_{\mathbb{F}_3}(a,0,b)$ and $P_{\mathbb{F}_3}(a,b,b)$ are LCD codes.
- If $n \equiv 3 \pmod 6$, then $P_{\mathbb{F}_3}(0,a,0)$ are LCD codes.
- If $n \equiv 3 \pmod 6$ and $n \equiv 0 \pmod 9$, then $P_{\mathbb{F}_3}(a,0,a)$, $P_{\mathbb{F}_3}(a,a,a)$ and $P_{\mathbb{F}_3}(a,b,a)$ are LCD codes.
- If $n \equiv 3 \pmod 6$ and $n \equiv 3 \pmod 9$, then $P_{\mathbb{F}_3}(a,b,b)$ are LCD codes.
- If $n \equiv 3 \pmod 6$ and $n \equiv 6 \pmod 9$, then $P_{\mathbb{F}_3}(0,a,a)$ are LCD codes.

Bordered case ($a,b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$)

- $B_{\mathbb{F}_3}(a,0,0)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.
- If $n \equiv 1 \pmod 6$, then $B_{\mathbb{F}_3}(a,b,0)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.

- If $n \equiv 1 \pmod 6$ and $n \equiv 1 \pmod 9$, then $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, with $\alpha\gamma + b\beta = 0,\ \alpha,\beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes. $B_{\mathbb{F}_3}(a,a,a)$ with $\gamma = 0,\ \alpha,\ \beta \in \mathbb{F}_3$, are LCD codes.

- If $n \equiv 1 \pmod 6$ and $n \equiv 4 \pmod 9$, then $B_{\mathbb{F}_3}(0,a,b)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, with $\alpha\gamma + 2a\beta = 0,\ \alpha,\beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- If $n \equiv 1 \pmod 6$ and $n \equiv 7 \pmod 9$, then $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(0,a,b)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- If $n \equiv 3 \pmod 6$, then $B_{\mathbb{F}_3}(0,a,0)$, with $\gamma = 0,\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes.

- If $n \equiv 3 \pmod 6$ and $n \equiv 0 \pmod 9$, then $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,a,a)$ and $B_{\mathbb{F}_3}(a,b,a)$ with $\gamma = 0,\ \alpha,\ \beta \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, with $\alpha\gamma + b\beta = 0,\ \alpha,\beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- If $n \equiv 3 \pmod 6$ and $n \equiv 3 \pmod 9$, then $B_{\mathbb{F}_3}(0,a,a)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.

- If $n \equiv 3 \pmod 6$ and $n \equiv 6 \pmod 9$, then $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,a,a)$ and $B_{\mathbb{F}_3}(a,b,a)$, with $\alpha\gamma + 2a\beta = 0,\ \alpha,\beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.

4.4.3. LCD codes over the field $\mathbb{F}_4$

Below, we give conditions to obtain LCD codes from the block graph of STS(n) in the pure and the bordered constructions over the field $\mathbb{F}_4$.

Pure case $(a \in \mathbb{F}_4\backslash\{0\})$

- If $n \equiv 1,3 \pmod 6$ and $n \equiv 1 \pmod 4$, then $P_{\mathbb{F}_4}(a,a,a)$ are LCD codes.

- If $n \equiv 1,3 \pmod 6$ and $n \equiv 3 \pmod 4$, then $P_{\mathbb{F}_4}(0,0,a)$ are LCD codes.

Bordered case $(a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b)$

- If $n \equiv 1,3 \pmod 6$ and $n \equiv 1 \pmod 4$, then $B_{\mathbb{F}_4}(a,a,a)$, with $\gamma = 0,\ \alpha \in \mathbb{F}_4 \backslash \{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.
- If $n \equiv 1,3 \pmod 6$ and $n \equiv 3 \pmod 4$, then $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,a)$ and $B_{\mathbb{F}_4}(a,a,b)$, with $\gamma = 1,\ \alpha + a\beta = 0,\ \alpha,\beta \in \mathbb{F}_4,\ 1+\alpha^2+\beta^2 \in \mathbb{F}_4 \backslash \{0\}$, are LCD codes. $B_{\mathbb{F}_4}(0,0,a)$, with $\gamma = 0,\ \alpha,\beta \in \mathbb{F}_4,\ 1+\alpha^2+\beta^2 \in \mathbb{F}_4 \backslash \{0\}$, are LCD codes.

4.4.4. Examples computed in Magma

We determined the parameters of SRGs obtained from certain STS(n). Then, by applying the pure and the bordered constructions, we constructed LCD codes from non-isomorphic strongly regular graphs with these parameters. For STS(n), $n = 9, 13, 15$, collections of 3-element subsets of an n-element set were obtained from [49]. For STS(19), one collection of 3-element subsets of a 19-element set was obtained from [59] and this collection can be found at http://www.math.uniri.hr/ abaric/matrice.zip. Using these collections, we constructed the adjacency matrices of corresponding SRGs, with the help of programmes written in Magma. LCD codes were constructed over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$.

1. STS(9) gives the unique SRG$(12,9,6,9)$.
   $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,0,a)$, $P_{\mathbb{F}_3}(a,a,a)$, $P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(0,a,0)$, where $a,b \in \mathbb{F}_3 \backslash \{0\}$, $a \neq b$, are $[24,12,2]$ LCD codes.
   $B_{\mathbb{F}_3}(a,0,0)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_3 \backslash \{0\}$, are $[26,13,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, where $\beta = \gamma = 0,\ a,\alpha \in \mathbb{F}_3 \backslash \{0\}$, are $[26,13,2]$ LCD codes.
   $B_{\mathbb{F}_3}(a,0,a)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_3 \backslash \{0\}$, are $[26,13,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\beta = \gamma = 0,\ a,\alpha \in \mathbb{F}_3 \backslash \{0\}$, are $[26,13,2]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\gamma = 0,\ a,\beta \in \mathbb{F}_3 \backslash \{0\},\ \alpha \in \mathbb{F}_3$, are $[26,13,2]$ LCD codes.
   $B_{\mathbb{F}_3}(a,a,a)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha = \beta = \gamma = 0,\ a,b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[26,13,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\beta = \gamma = 0,\ a,b,\alpha \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[26,13,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\gamma = 0,\ a,b,\beta \in \mathbb{F}_3 \backslash \{0\},\ a \neq b,\ \alpha \in \mathbb{F}_3$, are $[26,13,2]$ LCD codes.
   $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha = \beta = 0,\ a,b,\gamma \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$, are $[26,13,1]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[26,13,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[24,12,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[26,13,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[26,13,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[26,13,2]$ LCD codes.

2. STS(13) gives an SRG$(26,15,8,9)$. There are two non-isomorphic strongly regular graphs with these parameters, which are obtained with the use of [49]. They give LCD codes with the same parameters, except for one code in the field $\mathbb{F}_3$.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[52,26,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,0)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[52,26,10]$ LCD codes for the 1st graph, and $[52,26,11]$ LCD codes for the 2nd graph constructed with the use of [49].

   $P_{\mathbb{F}_3}(0,a,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[52,26,8]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[54,27,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[54,27,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[54,27,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha\gamma+2a\beta=0,\ a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[54,27,2]$ LCD codes.

   $P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[52,26,2]$ LCD codes.

   $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[54,27,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[54,27,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[54,27,2]$ LCD codes.

3. STS(15) gives an SRG$(35,18,9,9)$. There are 80 pairwise non-isomorphic strongly regular graphs with these parameters, which are obtained with the use of [49].

   $P_{\mathbb{F}_2}(0,0,1)$ is an $[70,35,3]$ LCD code for graphs No. 1 - No. 24, No. 61, an $[70,35,4]$ LCD code for graphs No. 25 - No. 60, No. 62 - No. 79, and an $[70,35,5]$ LCD code for graph No. 80.

   $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=\gamma=0$, is an $[72,36,1]$ LCD code. $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=1,\ \gamma=0$, is an $[72,36,3]$ LCD code for graphs No. 1 - No. 24, No. 61, an $[72,36,4]$ LCD code for graphs No. 25 - No. 60, No. 62 - No. 79, and an $[72,36,5]$

LCD code for graph No. 80.

$B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[72,36,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=\gamma=1$, is an $[72,36,4]$ LCD code for graphs No. 1 - No. 79, and an $[72,36,6]$ LCD code for graph No. 80.

$B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[72,36,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[72,36,2]$ LCD code.

$P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[70,35,2]$ LCD codes.

$P_{\mathbb{F}_3}(0,a,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[70,35,5]$ LCD codes for graphs No. 1 - No. 15, No. 17 - No. 80, and $[70,35,6]$ LCD codes for graph No. 16.

$P_{\mathbb{F}_3}(0,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[70,35,4]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[72,36,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,a)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+2a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[72,36,6]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha\gamma+2a\beta=0,\ a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[72,36,2]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,0)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[72,36,2]$ LCD codes. $B_{\mathbb{F}_3}(0,a,0)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[72,36,5]$ LCD codes for graphs No. 1 - No. 15, No. 17 - No. 80, and $[72,36,6]$ LCD codes for graph No. 16.

$P_{\mathbb{F}_4}(0,0,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[70,35,3]$ LCD codes for graphs No. 1 - No. 24, No. 61, $[70,35,4]$ LCD codes for graphs No. 25 - No. 60, No. 62 - No. 79, and $[70,35,5]$ LCD codes for graph No. 80.

$B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\alpha=\beta=0,\ \gamma=1,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,b,\alpha,$ $\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[72,36,4]$ LCD codes for graphs No. 1 - No. 79, and $[72,36,6]$ LCD codes for graph No. 80.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[72,36,2]$

LCD codes.

$B_{\mathbb{F}_4}(0,0,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_4}(0,0,a)$, where $\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[72,36,2]$ LCD codes. $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma=0$, $a,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4$, are $[72,36,3]$ LCD codes for graphs No. 1 - No. 24, No. 61, $[72,36,4]$ LCD codes for graphs No. 25 - No. 60, No. 62 - No. 79, and $[72,36,5]$ LCD codes for graph No. 80.

4. STS(19) gives an SRG$(57,24,11,9)$. The full automorphism group of STS(19) is $\mathbb{Z}_3$.

   $P_{\mathbb{F}_2}(0,0,1)$ is an $[114,57,4]$ LCD code.

   $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=\gamma=0$, is an $[116,58,1]$ LCD code. $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=1$, $\gamma=0$, is an $[116,58,4]$ LCD code.

   $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=0$, $\gamma=1$, is an $[116,58,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=\gamma=1$, is an $[116,58,4]$ LCD code.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0$, $\gamma=1$, is an $[116,58,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[116,58,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[114,57,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,0)$, where $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[114,57,10]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,0)$, where $\alpha=\beta=\gamma=0$, $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[116,58,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,0)$, where $\beta=\gamma=0$, $a,b,\alpha\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[116,58,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,b)$, where $\alpha=\beta=0$, $a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[116,58,1]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,b)$, where $\alpha\gamma+b\beta=0$, $a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[116,58,12]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_3\backslash\{0\}$, are $[116,58,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\beta=\gamma=0$, $a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[116,58,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0$, $a,\beta\in\mathbb{F}_3\backslash\{0\}$, $\alpha\in\mathbb{F}_3$, are $[116,58,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0$, $a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[116,58,1]$ LCD codes.

   $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0$, $a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[116,58,4]$ LCD codes.

   $P_{\mathbb{F}_4}(0,0,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[114,57,4]$ LCD codes.

   $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\alpha=\beta=0$, $\gamma=1$, $a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, are

$[116,58,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,b,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[116,58,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[116,58,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[116,58,2]$ LCD codes.

$B_{\mathbb{F}_4}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[116,58,1]$ LCD codes. $B_{\mathbb{F}_4}(0,0,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[116,58,2]$ LCD codes. $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma=0,\ a,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4$, are $[116,58,4]$ LCD codes.

4.5. LCD CODES FROM AN ORTHOGONAL ARRAY

Strongly regular graphs can be obtained from orthogonal arrays, which are considered as a generalization of mutually orthogonal Latin squares.

Definition 4.5.1. A **Latin square** of order n is an $n \times n$ matrix with entries from an alphabet of size n such that each row and column contains each symbol exactly once.

Definition 4.5.2. Two Latin squares $L = [l_{i,j}]$ and $M = [m_{i,j}]$ of order n are said to be **orthogonal** if the n^2 ordered pairs $(l_{i,j}, m_{i,j})$, $1 \leq i, j \leq n$, are all distinct. A matrix obtained by the superposition of L on M is called a **Graeco-Latin square** or **Euler square** and its elements are the ordered pairs $(l_{i,j}, m_{i,j})$.
A set of Latin squares is called **mutually orthogonal** if every pair of Latin squares from that set is orthogonal.

Example 4.5.3. Two orthogonal Latin squares L and M of order 3 and their Euler square S are given by

$$L = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} (1,1) & (2,2) & (3,3) \\ (2,3) & (3,1) & (1,2) \\ (3,2) & (1,3) & (2,1) \end{bmatrix}.$$

Definition 4.5.4. An **orthogonal array** $\mathrm{OA}(h,n)$ is an $h \times n^2$ array (or matrix) with entries from an alphabet of size n such that the n^2 ordered pairs defined by any two rows of the matrix are all distinct.

Theorem 4.5.5. An $\mathrm{OA}(h,n)$ is equivalent to a set of $h-2$ mutually orthogonal Latin squares.

Example 4.5.6. From the Theorem 4.5.5 follows that a Latin square of order n gives rise to an $\mathrm{OA}(3,n)$ by taking the three rows of the orthogonal array to be the row number i, column number j and the (i,j)-entry of the Latin square.

The Latin square L in example 4.5.3 is equivalent to $\mathrm{OA}(3,3)$ given by

$$\begin{array}{ccccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \end{array}.$$

Remark 4.5.7. A graph defined by an $\mathrm{OA}(2,n)$ is isomorphic to $L(K_{n,n})$.

For more information on Latin squares and orthogonal arrays we refer the reader to [38, 59].

As defined in [29], let G be a graph where vertices are the n^2 columns of the orthogonal array (viewed as column vectors of length h) and two vertices are adjacent if they have the same entries in one coordinate position.

Theorem 4.5.8. A graph defined by an $\mathrm{OA}(h,n)$, $n \geq 2$, is strongly regular with parameters

$$(n^2,(n-1)h,n-2+(h-1)(h-2),h(h-1)).$$

The proofs of the two preceding theorems and more about graphs from orthogonal arrays can be found in [29, 57].

4.5.1. LCD codes over the field $\mathbb{F}_2$

Below, we give conditions to obtain LCD codes from $\mathrm{OA}(h,n)$ in the pure and the bordered constructions over the field $\mathbb{F}_2$.

Pure case

- If $h \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 2$, then $P_{\mathbb{F}_2}(0,1,0)$, $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.
- If $h \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 2$, then $P_{\mathbb{F}_2}(0,0,1)$, $P_{\mathbb{F}_2}(1,1,0)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.

Bordered case

- If $h \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 2$, then $B_{\mathbb{F}_2}(0,0,1)$, with $\alpha = \beta = 0,\ \gamma = 1$, is an LCD code. $B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, with $\alpha = \gamma = 0,\ \beta \in \mathbb{F}_2$, are LCD codes.

- If $h \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 2$, then $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(1,1,0)$ and $B_{\mathbb{F}_2}(1,1,1)$, with $\alpha = \gamma = 0,\ \beta \in \mathbb{F}_2$, are LCD codes.

- If $h \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, then $B_{\mathbb{F}_2}(1,1,1)$, with $\gamma = 1,\ \alpha = \beta,\ \alpha, \beta \in \mathbb{F}_2$, is an LCD code.

- If $h \equiv 1 \pmod 2$ and $n \equiv 1 \pmod 2$, then $B_{\mathbb{F}_2}(1,1,1)$, with $\gamma = 1,\ \alpha = \beta,\ \alpha, \beta \in \mathbb{F}_2$, is an LCD code.

4.5.2. LCD codes over the field $\mathbb{F}_3$

Below, we give conditions to obtain LCD codes from $\mathrm{OA}(h,n)$ in the pure and the bordered constructions over the field $\mathbb{F}_3$.

Pure case ($a, b \in \mathbb{F}_3 \backslash \{0\},\ a \neq b$)

- $P_{\mathbb{F}_3}(a,0,0)$ are LCD codes.

- If $h \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 3$, then $P_{\mathbb{F}_3}(a,0,a)$, $P_{\mathbb{F}_3}(a,a,a)$, $P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(0,a,0)$ are LCD codes.

- If $h \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 3$, then $P_{\mathbb{F}_3}(a,a,0)$, $P_{\mathbb{F}_3}(a,a,b)$ and $P_{\mathbb{F}_3}(a,b,b)$ are LCD codes.

- If $h \equiv 0 \pmod 3$ and $n \equiv 2 \pmod 3$, then $P_{\mathbb{F}_3}(a,b,0)$, $P_{\mathbb{F}_3}(a,0,b)$ and $P_{\mathbb{F}_3}(a,b,b)$ are LCD codes.

- If $h \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 3$, then $P_{\mathbb{F}_3}(a,a,0)$, $P_{\mathbb{F}_3}(a,a,a)$, $P_{\mathbb{F}_3}(a,a,b)$ and $P_{\mathbb{F}_3}(0,0,a)$ are LCD codes.

- If $h \equiv 1 \pmod 3$ and $n \equiv 1 \pmod 3$, then $P_{\mathbb{F}_3}(a,b,0)$, $P_{\mathbb{F}_3}(a,0,b)$ and $P_{\mathbb{F}_3}(a,b,b)$ are LCD codes.

- If $h \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$, then $P_{\mathbb{F}_3}(a,b,a)$, $P_{\mathbb{F}_3}(a,b,b)$ and $P_{\mathbb{F}_3}(0,a,0)$ are LCD codes.

- If $h \equiv 2 \pmod 3$ and $n \equiv 0 \pmod 3$, then $P_{\mathbb{F}_3}(a,a,a)$ are LCD codes.

- If $h \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 3$, then $P_{\mathbb{F}_3}(a,0,a)$, $P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(a,b,b)$ are LCD codes.

- If $h \equiv 2 \pmod 3$ and $n \equiv 2 \pmod 3$, then $P_{\mathbb{F}_3}(a,a,b)$, $P_{\mathbb{F}_3}(a,b,b)$ and $P_{\mathbb{F}_3}(0,0,a)$ are LCD codes.

Bordered case $(a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b)$

- $B_{\mathbb{F}_3}(a,0,0)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.

- If $h \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 3$, then $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,a,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(0,a,0)$, with $\gamma = 0,\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, with $\alpha\gamma + b\beta = 0,\ \alpha,\beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- If $h \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 3$, then $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(0,0,a)$ and $B_{\mathbb{F}_3}(0,a,a)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- If $h \equiv 0 \pmod 3$ and $n \equiv 2 \pmod 3$, then $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(0,a,b)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- If $h \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 3$, then $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,a,a)$, $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, with $\gamma = 0,\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, with $\alpha\gamma + b\beta = 0,\ \alpha,\beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- If $h \equiv 1 \pmod 3$ and $n \equiv 1 \pmod 3$, then $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(0,a,b)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- If $h \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$, then $B_{\mathbb{F}_3}(a,0,a)$ and $B_{\mathbb{F}_3}(0,a,a)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes. $B_{\mathbb{F}_3}(a,b,a)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,a,0)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.

- If $h \equiv 2 \pmod 3$ and $n \equiv 0 \pmod 3$, then $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, with $\gamma = 0,\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, with $\alpha\gamma + b\beta = 0,\ \alpha,\beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- If $h \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 3$, then $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes. $B_{\mathbb{F}_3}(0,a,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- If $h \equiv 2 \pmod 3$ and $n \equiv 2 \pmod 3$, then $B_{\mathbb{F}_3}(a,a,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, with $\alpha = 0,\ \beta \in \mathbb{F}_3,\ \gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes. $B_{\mathbb{F}_3}(a,a,b)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, with $\beta = \gamma = 0,\ \alpha \in \mathbb{F}_3$, are LCD codes.

4.5.3. LCD codes over the field $\mathbb{F}_4$

Below, we give conditions to obtain LCD codes from $\mathrm{OA}(h,n)$ in the pure and the bordered constructions over the field $\mathbb{F}_4$.

Pure case $(a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b)$

- If $h \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 2$, then $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$ are LCD codes.

- If $h \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 2$, then $P_{\mathbb{F}_4}(a,a,0)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(0,0,a)$ are LCD codes.

Bordered case $(a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b)$

- If $h \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 2$, then $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, with $\gamma = 0,\ \alpha \in \mathbb{F}_4\backslash\{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.

- If $h \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 2$, then $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, with $\gamma = 0, \alpha \in \mathbb{F}_4\backslash\{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.

- If $h \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, then $B_{\mathbb{F}_4}(a,a,a)$, with $\gamma = 1,\ \alpha + a\beta = 0,\ \alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are LCD codes.

- If $h \equiv 1 \pmod 2$ and $n \equiv 1 \pmod 2$, then $B_{\mathbb{F}_4}(a,a,a)$, with $\gamma = 1$, $\alpha + a\beta = 0$, $\alpha, \beta, 1+\alpha^2+\beta^2 \in \mathbb{F}_4 \backslash \{0\}$, are LCD codes.

4.5.4. Examples computed in Magma

We determined the parameters of SRGs obtained from certain $\mathrm{OA}(h,n)$. Then, by applying the pure and the bordered constructions, we constructed LCD codes from non-isomorphic strongly regular graphs with these parameters. The adjacency matrices of two non-isomorphic SRGs, which are constructed from $\mathrm{OA}(3,4)$, were obtained from [4]. $\mathrm{OA}(4,5)$ gives the unique SRG, and the adjacency matrix of its complement was taken from [56]. Orthogonal arrays $\mathrm{OA}(3,5)$, $\mathrm{OA}(3,6)$, $\mathrm{OA}(3,7)$, $\mathrm{OA}(4,7)$, $\mathrm{OA}(4,8)$, $\mathrm{OA}(5,7)$, $\mathrm{OA}(5,8)$ and $\mathrm{OA}(5,9)$ were obtained from [58] and their matrices can be found at http://www.math.uniri.hr/ abaric/matrice.zip. Using these orthogonal arrays, we constructed the adjacency matrices of corresponding SRGs, with the help of programmes written in Magma. LCD codes were constructed over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$.

1. $\mathrm{OA}(3,4)$ gives an $\mathrm{SRG}(16,9,4,6)$. Both non-isomorphic strongly regular graphs with these parameters give LCD codes with the same parameters.
   $P_{\mathbb{F}_2}(0,0,1)$ and $P_{\mathbb{F}_2}(1,1,0)$ are $[32,16,4]$ LCD codes.
   $P_{\mathbb{F}_2}(1,1,1)$ is an $[32,16,2]$ LCD code.
   $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha = \beta = \gamma = 0$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha = \gamma = 0$, $\beta = 1$, are $[34,17,4]$ LCD codes.
   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \beta = \gamma = 0$, is an $[34,17,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0$, $\beta = 1$, is an $[34,17,2]$ LCD code.
   $P_{\mathbb{F}_3}(a,0,0)$, where $a \in \mathbb{F}_3 \backslash \{0\}$, are $[32,16,2]$ LCD codes.
   $P_{\mathbb{F}_3}(a,a,0)$ and $P_{\mathbb{F}_3}(a,a,b)$, where $a,b \in \mathbb{F}_3 \backslash \{0\}$, $a \neq b$, are $[32,16,8]$ LCD codes.
   $P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3 \backslash \{0\}$, $a \neq b$, are $[32,16,4]$ LCD codes.
   $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha = \beta = \gamma = 0$, $a,b \in \mathbb{F}_3 \backslash \{0\}$, $a \neq b$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta = \gamma = 0$, $a,b,\alpha \in \mathbb{F}_3 \backslash \{0\}$, $a \neq b$, are $[34,17,2]$ LCD codes.
   $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha = \beta = 0$, $a, \gamma \in \mathbb{F}_3 \backslash \{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha = 0$, $a,\beta,\gamma \in \mathbb{F}_3 \backslash \{0\}$, are $[34,17,8]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[34,17,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,0)$, $P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[32,16,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[32,16,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[34,17,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma=0,\ a,b,\beta\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[34,17,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[34,17,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[34,17,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[34,17,2]$ LCD codes.

2. $\mathrm{OA}(3,5)$ gives an $\mathrm{SRG}(25,12,5,6)$.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[52,26,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[52,26,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[50,25,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,0)$ and $P_{\mathbb{F}_3}(a,0,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[50,25,10]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[50,25,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[52,26,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[52,26,2]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[52,26,1]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=0,\ a,b,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[52,26,12]$ LCD codes.

   $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[52,26,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[52,26,2]$ LCD codes.

3. $\mathrm{OA}(3,6)$ gives an $\mathrm{SRG}(36,15,6,6)$.

   $P_{\mathbb{F}_2}(0,0,1)$ is an $[72,36,3]$ LCD code.

   $P_{\mathbb{F}_2}(1,1,0)$ is an $[72,36,4]$ LCD code.

$P_{\mathbb{F}_2}(1,1,1)$ is an $[72,36,2]$ LCD code.

$B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=\gamma=0$, is an $[74,37,1]$ LCD code. $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[74,37,3]$ LCD code.

$B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=\gamma=0$, is an $[74,37,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[74,37,4]$ LCD code.

$B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[74,37,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[74,37,2]$ LCD code.

$P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[72,36,2]$ LCD codes.

$P_{\mathbb{F}_3}(a,0,a)$, $P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(0,a,0)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[72,36,6]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,a)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[74,37,6]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[74,37,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,2]$ LCD codes. $B_{\mathbb{F}_3}(a,b,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[74,37,6]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[74,37,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,0)$ and $P_{\mathbb{F}_4}(a,a,b)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[72,36,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[72,36,2]$ LCD codes.

$P_{\mathbb{F}_4}(0,0,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[72,36,3]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[74,37,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\gamma=0,\ a,b,\beta\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[74,37,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[74,37,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[74,37,2]$ LCD codes.

$B_{\mathbb{F}_4}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[74,37,1]$ LCD codes. $B_{\mathbb{F}_4}(0,0,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[74,37,2]$ LCD codes. $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[74,37,3]$ LCD codes.

4. OA$(3,7)$ gives an SRG$(49,18,7,6)$.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[100,50,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[100,50,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[98,49,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,a,0)$ and $P_{\mathbb{F}_3}(a,a,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[98,49,12]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[98,49,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[100,50,2]$ LCD codes.

   $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,12]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,4]$ LCD codes.

   $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[100,50,2]$ LCD codes.

5. OA$(4,5)$ gives the unique SRG$(25,16,9,12)$.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[52,26,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[52,26,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[50,25,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(0,a,0)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[50,25,8]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[50,25,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,a)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[52,26,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,a)$, $B_{\mathbb{F}_3}(a,b,b)$ and

$B_{\mathbb{F}_3}(0,a,0)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[52,26,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[52,26,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[52,26,8]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[52,26,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[52,26,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[52,26,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[52,26,2]$ LCD codes.

6. OA$(4,7)$ gives an SRG$(49,24,11,12)$.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[100,50,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[100,50,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[98,49,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,0)$ and $P_{\mathbb{F}_3}(a,0,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[98,49,14]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[98,49,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[100,50,1]$. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[100,50,2]$.

   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[100,50,1]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=0,\ a,b,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[100,50,16]$ LCD codes.

   $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[100,50,2]$ LCD codes.

7. OA$(4,8)$ gives an SRG$(64,28,12,12)$.

   $P_{\mathbb{F}_2}(0,1,0)$ and $P_{\mathbb{F}_2}(1,0,1)$ are $[128,64,4]$ LCD codes.

   $P_{\mathbb{F}_2}(1,1,1)$ is an $[128,64,2]$ LCD code.

   $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[130,65,1]$ LCD code.

   $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, are $[130,65,1]$ LCD codes.

   $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\gamma=0,\ \beta=1$, are $[130,65,4]$ LCD codes.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[130,65,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[130,65,2]$ LCD code.

$P_{\mathbb{F}_3}(a,0,0)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[128,64,2]$ LCD codes.

$P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(0,a,0)$, where $a,b \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[128,64,16]$ LCD codes.

$P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[128,64,4]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,a)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=\gamma=0$, $a,b \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,a)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\beta=\gamma=0$, $a,b,\alpha \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[130,65,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=\beta=0$, $a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=0$, $a,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[130,65,16]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0$, $a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0$, $a,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[130,65,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, are $[128,64,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[128,64,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0$, $a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0$, $a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, $\alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[130,65,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0$, $a,b,\beta \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, $\alpha \in \mathbb{F}_4\backslash\{1\}$, are $[130,65,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0$, $a \in \mathbb{F}_4\backslash\{0\}$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0$, $a \in \mathbb{F}_4\backslash\{0\}$, $\alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[130,65,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0$, $a,\beta \in \mathbb{F}_4\backslash\{0\}$, $\alpha \in \mathbb{F}_4\backslash\{1\}$, are $[130,65,2]$ LCD codes.

8. OA$(5,7)$ gives an SRG$(49,30,17,20)$.

   $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0$, $\gamma=1$, is an $[100,50,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[100,50,2]$ LCD code.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[98,49,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,0,a)$ and $P_{\mathbb{F}_3}(a,b,a)$, where $a,b \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[98,49,12]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[98,49,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0$, $a \in \mathbb{F}_3\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and

$B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,2]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,12]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[100,50,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[100,50,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[100,50,2]$ LCD codes.

9. OA$(5,8)$ gives an SRG$(64,35,18,20)$.

$P_{\mathbb{F}_2}(0,0,1)$ and $P_{\mathbb{F}_2}(1,1,0)$ are $[128,64,4]$ LCD codes.

$P_{\mathbb{F}_2}(1,1,1)$ is an $[128,64,2]$ LCD code.

$B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=\gamma=0$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\gamma=0,\ \beta=1$, are $[130,65,4]$ LCD codes. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[130,65,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[130,65,2]$ LCD code.

$P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[128,64,2]$ LCD codes.

$P_{\mathbb{F}_3}(a,a,b)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[128,64,16]$ LCD codes.

$P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[128,64,4]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,a,b)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,a,b)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[130,65,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,0)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,0)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[130,65,16]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[130,65,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,0)$, $P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[128,64,4]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[128,64,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where

$\beta = \gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[130,65,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma = 0,\ a,b,\beta \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[130,65,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[130,65,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[130,65,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma = 0,\ a,\beta \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[130,65,2]$ LCD codes.

10. OA$(5,9)$ gives an SRG$(81,40,19,20)$.

$B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \beta = 0,\ \gamma = 1$, is an $[164,82,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \beta = \gamma = 1$, is an $[164,82,2]$ LCD code.

$P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$ are $[162,81,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[164,82,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, where $\beta = \gamma = 0,\ a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[164,82,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,0)$ and $B_{\mathbb{F}_3}(a,0,b)$, where $\alpha = \beta = 0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[164,82,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,0)$ and $B_{\mathbb{F}_3}(a,0,b)$, where $\alpha = 0,\ a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[164,82,6]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = \beta = 0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[164,82,1]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = 0,\ a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[164,82,16]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[164,82,1]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\beta = \gamma = 0,\ a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[164,82,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma = 0,\ a,\beta \in \mathbb{F}_3\backslash\{0\},\ \alpha \in \mathbb{F}_3$, are $[164,82,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha = \beta = 0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[164,82,1]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma + b\beta = 0,\ a,b,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[164,82,4]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha = \beta = 0,\ \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[164,82,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma = 1,\ \alpha + a\beta = 0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[164,82,2]$ LCD codes.

4.6. LCD CODES FROM RANK THREE GROUPS

Any rank three permutation group gives rise to a strongly regular graph. The study of these groups was started by D. G. Higman, which can be found in [34]. We gave the basic theory concerning permutation groups in Section 1.1, and here we define rank three permutation groups.

Definition 4.6.1. Let G be a permutation group on a set Ω. We define a group action of G on the set $\Omega \times \Omega$ by

$$g \cdot (x,y) = (g \cdot x, g \cdot y) \text{ for all } x,y \in \Omega,\ g \in G.$$

An orbit of G in this action is called an **orbital**.

Remark 4.6.2. To each orbital Δ_i we associate a matrix A_i which has rows and columns labelled by the elements of Ω. The entry in row x and column y of A_i is 1 if (x,y) is in the ith orbital, and 0 otherwise.

Definition 4.6.3. For each orbital Δ there is a **paired orbital** defined as

$$\Delta^* = \{(y,x) \mid (x,y) \in \Delta\}.$$

If $\Delta = \Delta^*$, then Δ is said to be a **self-paired orbital**.

Definition 4.6.4. Let G be a transitive permutation group on Ω. Then the diagonal set

$$\Delta_0 = \{(x,x) \mid x \in \Omega\}$$

is always an orbital, and it is called the **trivial orbital**.

Lemma 4.6.5. *G has a non-trivial self-paired orbital if and only if $|G|$ is even.*

Lemma 4.6.6. *If G is transitive on Ω and $x \in \Omega$, then there is a bijection between the orbitals of G and the orbits of a point stabilizer G_x.*

Remark 4.6.7. Let $\Delta_0, \Delta_1, \ldots, \Delta_d$ be the orbitals of a permutation group G on Ω. Then the sets $\Delta_i(x) = \{y \in \Omega \mid (x,y) \in \Delta_i\}$ are the orbits of G_x on Ω. It holds that $g \cdot \Delta_i(x) = \Delta_i(g \cdot x)$ for all $g \in G,\ i = 1, \ldots, d$.

Example 4.6.8. Let $G = D_8$ be the dihedral group of order 8, which is defined as the group of all symmetries of the square. This group is a subgroup of the symmetric group S_4, and is given by

$$G = \{(), (1,2,3,4), (1,3)(2,4), (1,4,3,2), (1,3), (2,4), (1,4)(2,3), (1,2)(3,4)\}.$$

Therefore, G acts on the set $\Omega = \{1,2,3,4\}$ and the orbitals in this action are

$$\begin{aligned}
\Delta_0 &= \{(1,1), (2,2), (3,3), (4,4)\},\\
\Delta_1 &= \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\},\\
\Delta_2 &= \{(1,3), (2,4), (3,1), (4,2)\}.
\end{aligned}$$

The stabilizer of the point $1 \in \Omega$ in G is the subgroup $G_1 = \{(), (2,4)\}$, and it has three orbits: $\{1\}$, $\{2,4\}$ and $\{3\}$. The orbital corresponding to the orbit $\{1\}$ is the closure of the set $\{(1,1)\}$ under the action of G, and this is the diagonal orbital Δ_0. The orbital corresponding to the orbit $\{2,4\}$ is the closure of the set $\{(1,2),(1,4)\}$ under the action of G, and this is Δ_1. The orbital corresponding to the orbit $\{3\}$ is the closure of the set $\{(1,3)\}$ under the action of G, and this is Δ_2.

Definition 4.6.9. The **rank** of a transitive permutation group G on a set Ω is the number of orbitals of G, *i.e.* the number of orbits of the stabilizer of a point G_x on Ω.

Remark 4.6.10. An orbital is self-paired if and only if the corresponding matrix is symmetric.
In the case that G is a rank three group, and G has a non-trivial self-paired orbital, then both of the non-trivial orbitals of G are self-paired. Hence, all matrices A_i are symmetric.

Definition 4.6.11. Let G be a permutation group on a set Ω. The **orbital graph** associated with an orbital $\Delta \subseteq \Omega \times \Omega$ is a graph with the vertex set Ω and the edge set Δ, *i.e.* there is an edge from x to y for each $(x,y) \in \Delta$. If Δ is self-paired, then its orbital graph (Ω, Δ) is an undirected graph.

The following theorem shows how strongly regular graphs can be constructed from rank three permutation groups.

Theorem 4.6.12. Let G be a rank three permutation group of degree v and A a $v \times v$ incidence matrix of a non-trivial self-paired orbital. Then A is the incidence matrix of a strongly regular graph.

Remark 4.6.13. Let G be a rank three permutation group on a set Ω. Then G has three orbitals $\Delta_0, \Delta_1, \Delta_2$ on $\Omega \times \Omega$. It follows that Δ_1 and Δ_2 are inverse relations, *i.e.* $\Delta_2 = \{(y,x) \mid (x,y) \in \Delta_1\}$, or Δ_1 and Δ_2 are symmetric relations. In the first case, (Ω, Δ_1) is a tournament, and (Ω, Δ_2) is the opposite tournament. In the second case, (Ω, Δ_1) and (Ω, Δ_2) are a complementary pair of graphs. When Ω is finite, they are a complementary pair of strongly regular graphs. The group G acts as a group of automorphisms on the graphs (Ω, Δ_1) and (Ω, Δ_2).

More about rank 3 groups and all the above results can be found in [6, 60].

Definition 4.6.14. A permutation group G on a set Ω is ***k*-transitive** ($k \geq 1$) if given any two ordered k-tuples $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ of distinct elements of Ω, there is a $g \in G$ such that $g(x_i) = y_i$, $i = 1, \ldots, k$.
G is called **sharply *k*-transitive** if the element $g \in G$ is unique.

Remark 4.6.15. A k-transitive group is also l-transitive, for $l = 1, \ldots, k$.

Definition 4.6.16. A **simple group** is a group G that does not have any normal subgroups except for the trivial group and G itself.

The *sporadic groups* are the 26 finite simple groups that do not belong to any of the three infinite families of finite simple groups (the cyclic groups of prime order, the alternating groups of degree at least five, and the groups of Lie type). For more information, we refer to [62]. Several of the sporadic simple groups were discovered as rank 3 permutation groups. Therefore, we searched for parameters of strongly regular graphs obtained from these rank 3 groups in order to construct LCD codes. Higman provided parameters for some groups in Table 10A.1 in [31] and others can be found in [14].

We used all sporadic simple groups which are rank 3 permutation groups to obtain LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$ from the pure and the bordered constructions. Certain adjacency matrices of SRGs used in the construction of LCD codes were obtained using the built-in functions in Magma or in the way described in [11] with the use of Magma.

4.6.1. Mathieu groups

The Mathieu groups form a set of five sporadic simple groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24} introduced by É. L. Mathieu. They were the first sporadic groups to be discovered.

Mathieu group $\mathbf{M_{11}}$

M_{11} is the smallest sporadic simple group, and it is a sharply 4-transitive permutation group on 11 points. It has a rank 3 action on 55 points, which gives the unique SRG$(55,18,9,4)$. These parameters correspond to the parameters of a triangular graph $T(11)$. Below, we give the list of all LCD codes constructed with the use of Magma.

- $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[112,56,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[112,56,2]$ LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[110,55,2]$ LCD codes.
  $P_{\mathbb{F}_3}(0,a,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[110,55,4]$ LCD codes.
  $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[110,55,4]$ LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[112,56,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[112,56,2]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[112,56,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,b)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[112,56,2]$ LCD codes. $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\gamma=0,\ a,b,\beta\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[112,56,4]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[112,56,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[112,56,4]$ LCD codes.

- $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[112,56,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[112,56,2]$ LCD codes.

Mathieu group $\mathbf{M_{12}}$

M_{12} is a sharply 5-transitive permutation group on 12 points. It has a rank 3 action on 66 points, which gives the unique SRG$(66,20,10,4)$. These parameters correspond to the parameters of a triangular graph $T(12)$. Below, we give the list of all LCD codes constructed with the use of Magma.

- $P_{\mathbb{F}_2}(0,1,0)$ is an $[132,66,3]$ LCD code.
  $P_{\mathbb{F}_2}(1,0,1)$ is an $[132,66,4]$ LCD code.
  $P_{\mathbb{F}_2}(1,1,1)$ is an $[132,66,2]$ LCD code.

- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[134,67,1]$ LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=\gamma=0$, is an $[134,67,1]$ LCD code. $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[134,67,3]$ LCD code.
  $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, is an $[134,67,1]$ LCD code. $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[134,67,4]$ LCD code.
  $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[134,67,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[134,67,2]$ LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[132,66,2]$ LCD codes.
  $P_{\mathbb{F}_3}(a,0,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[132,66,8]$ LCD codes.
  $P_{\mathbb{F}_3}(a,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[132,66,2]$ LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[134,67,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[134,67,2]$ LCD codes.
  $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[134,67,1]$ LCD codes.
  $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0,\ a,\alpha,\beta\in\mathbb{F}_3\backslash\{0\}$, are $[134,67,2]$ LCD codes.
  $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[134,67,1]$ LCD codes.
  $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+2a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[134,67,8]$ LCD codes.
  $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[134,67,1]$ LCD codes.
  $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+2a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[134,67,4]$ LCD codes.

- $P_{\mathbb{F}_4}(a,0,a)$ and $P_{\mathbb{F}_4}(a,b,a)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[132,66,4]$ LCD codes.
  $P_{\mathbb{F}_4}(0,a,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[132,66,3]$ LCD codes.
  $P_{\mathbb{F}_4}(a,a,a)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[132,66,2]$ LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\alpha=\beta=\gamma=0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[134,67,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\beta=\gamma=0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[134,67,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\gamma=0,\ a,b,\beta \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[134,67,4]$ LCD codes.
  $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[134,67,1]$ LCD codes.
  $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0,\ a \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[134,67,2]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,\beta \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[134,67,3]$ LCD codes.
  $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[134,67,1]$ LCD codes.
  $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ \alpha \in \mathbb{F}_4\backslash\{0,1\},\ a \in \mathbb{F}_4\backslash\{0\}$, are $[134,67,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[134,67,2]$ LCD codes.

Mathieu group $\mathbf{M_{22}}$

M_{22} is a 3-transitive permutation group on 22 points. It has a rank 3 action on 77 points, which gives the unique SRG$(77,16,0,4)$. Below, we give the list of all LCD codes constructed with the use of Magma.

- $P_{\mathbb{F}_2}(0,1,0)$ is an $[154,77,5]$ LCD code.

- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=\gamma=0$, is an $[156,78,1]$ LCD code. $B_{\mathbb{F}_2}(0,0,1)$, where $\gamma=0,\ \alpha=\beta=1$, is an $[156,78,12]$ LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=\gamma=0$, is an $[156,78,1]$ LCD code. $B_{\mathbb{F}_2}(0,1,0)$, where $\gamma=0,\ \alpha=\beta=1$, is an $[156,78,5]$ LCD code.
  $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[156,78,1]$ LCD code. $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=1$, is an $[156,78,6]$ LCD code.
  $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[156,78,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=1$, is an $[156,78,2]$ LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[154,77,2]$ LCD codes.
  $P_{\mathbb{F}_3}(a,0,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[154,77,14]$ LCD codes.
  $P_{\mathbb{F}_3}(0,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[154,77,4]$ LCD codes.
  $P_{\mathbb{F}_3}(0,a,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[154,77,16]$ LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,0,b)$, $B_{\mathbb{F}_3}(0,a,a)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=\gamma=0,\ a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[156,78,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,0,b)$, $B_{\mathbb{F}_3}(0,a,a)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\beta=\gamma=0,\ a,b,\alpha \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[156,78,2]$ LCD codes.
  $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[156,78,1]$ LCD codes.
  $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha\gamma+2a\beta=0,\ a,b,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[156,78,2]$ LCD codes.

- $P_{\mathbb{F}_4}(0,a,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[154,77,5]$ LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[156,78,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\gamma=1,\ \alpha+a\beta=0$, $a,b,\alpha,\beta \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[156,78,6]$ LCD codes.
  $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[156,78,1]$ LCD codes.
  $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha+a\beta=0,\ \gamma=1,\ a,\alpha,\beta \in \mathbb{F}_4\backslash\{0\}$, are $[156,78,2]$ LCD codes.
  $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[156,78,1]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[156,78,5]$ LCD codes.

M_{22} has also a rank 3 action on 176 points, which gives the unique $\mathrm{SRG}(176,70,18,34)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,1,0)$, $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.

- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta \in \mathbb{F}_2$, are LCD codes.

- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,0,b)$, $P_{\mathbb{F}_3}(0,a,a)$ and $P_{\mathbb{F}_3}(0,a,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,0,b)$, $B_{\mathbb{F}_3}(0,a,a)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\beta = \gamma = 0$, $a,b \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, $\alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha\gamma + 2a\beta = 0$, $a,\gamma \in \mathbb{F}_3\backslash\{0\}$, $\alpha,\beta \in \mathbb{F}_3$, are LCD codes.

- $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, are LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma = 0$, $a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b$, $\alpha \in \mathbb{F}_4\backslash\{1\}$, $\beta \in \mathbb{F}_4$, are LCD codes.

Mathieu group $\mathbf{M_{23}}$

M_{23} is a 4-transitive permutation group on 23 points. It has two different rank 3 actions on 253 points. One rank 3 action gives the unique SRG$(253,42,21,4)$. These parameters correspond to the parameters of a triangular graph $T(23)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $B_{\mathbb{F}_2}(1,1,1)$, where $\gamma = 1$, $\alpha = \beta$, $\alpha,\beta \in \mathbb{F}_2$, is an LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(0,a,b)$ and $P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta = \gamma = 0$, $a,b \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, $\alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(0,a,b)$, where $\gamma = 0$, $a,b,1+\alpha^2+\beta^2 \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, $\alpha,\beta \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha = 0$, $a,\gamma \in \mathbb{F}_3\backslash\{0\}$, $\beta \in \mathbb{F}_3$, are LCD codes.

- $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma = 1$, $\alpha + a\beta = 0$, $a,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, $\alpha,\beta \in \mathbb{F}_4$, are LCD codes.

The second rank 3 action of M_{23} gives an SRG$(253,112,36,60)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,1,0)$ is an LCD code.

- $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$, where $\gamma = 0,\ \alpha = \beta,\ \alpha,\beta \in \mathbb{F}_2$, are LCD codes.
  $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\gamma = 1,\ \alpha = \beta,\ \alpha,\beta \in \mathbb{F}_2$, are LCD codes.

- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,b,a)$, $P_{\mathbb{F}_3}(a,b,b)$ and $P_{\mathbb{F}_3}(0,a,0)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,a)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\beta = \gamma = 0,\ a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(a,0,a)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha = 0,\ a,\gamma \in \mathbb{F}_3\backslash\{0\},\ \beta \in \mathbb{F}_3$, are LCD codes.

- $P_{\mathbb{F}_4}(0,a,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\gamma = 1,\ \alpha + a\beta = 0,\ a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b,\ \alpha,\beta \in \mathbb{F}_4$, are LCD codes.
  $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma = 0,\ a, 1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\},\ \alpha,\beta \in \mathbb{F}_4$, are LCD codes.

Mathieu group $\mathbf{M_{24}}$

M_{24} is a 5-transitive permutation group on 24 points. It has a rank 3 action on 276 points, which gives the unique SRG$(276,44,22,4)$. These parameters correspond to the parameters of a triangular graph $T(24)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,1,0)$, $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.

- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha = \beta = 0,\ \gamma = 1$, is an LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta \in \mathbb{F}_2$, are LCD codes.

- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,0,a)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\},\ \alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\},\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma + 2a\beta = 0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b,\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes.

- $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{1\},\ \beta\in\mathbb{F}_4$, are LCD codes.

M_{24} has also a rank 3 action on 1288 points, which gives an SRG$(1288,495,206,180)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,0,1)$, $P_{\mathbb{F}_2}(1,1,0)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.

- $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(1,1,0)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta\in\mathbb{F}_2$, are LCD codes.

- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,b,0)$, $P_{\mathbb{F}_3}(a,0,b)$ and $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b,\ \alpha\in\mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b,\ \beta\in\mathbb{F}_3$, are LCD codes.

- $P_{\mathbb{F}_4}(a,a,0)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are LCD codes.

- $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{1\},\ \beta\in\mathbb{F}_4$, are LCD codes.

4.6.2. Janko group J_2

J_2 is a rank 3 permutation group on 100 points, which gives an SRG$(100,36,14,12)$. Below, we give the list of all LCD codes constructed with the use of Magma.

- $P_{\mathbb{F}_2}(0,1,0)$ and $P_{\mathbb{F}_2}(1,0,1)$ are $[200,100,8]$ LCD codes.
  $P_{\mathbb{F}_2}(1,1,1)$ is an $[200,100,2]$ LCD code.

- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[202,101,1]$ LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, are $[202,101,1]$ LCD codes.
  $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\gamma=0,\ \beta=1$, are $[202,101,8]$ LCD codes.
  $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[202,101,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[202,101,2]$ LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[200,100,2]$ LCD codes.
  $P_{\mathbb{F}_3}(a,b,0)$ and $P_{\mathbb{F}_3}(a,0,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[200,100,16]$ LCD codes.
  $P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[200,100,4]$ LCD codes.
- $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$ and $B_{\mathbb{F}_3}(a,0,b)$, where $\alpha=\beta=\gamma=0,\ a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[202,101,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$ and $B_{\mathbb{F}_3}(a,0,b)$, where $\beta=\gamma=0,\ a,b,\alpha \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[202,101,2]$ LCD codes.
  $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[202,101,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[202,101,2]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[202,101,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[202,101,4]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[202,101,1]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=0,\ a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[202,101,16]$ LCD codes.
- $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[200,100,8]$ LCD codes.
  $P_{\mathbb{F}_4}(a,a,a)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[200,100,2]$ LCD codes.
- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[202,101,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[202,101,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,b,\beta \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[202,101,8]$ LCD codes.
  $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[202,101,1]$ LCD codes.
  $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ \alpha \in \mathbb{F}_4\backslash\{0,1\},\ a \in \mathbb{F}_4\backslash\{0\}$, are $[202,101,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=0,\ a,\beta \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[202,101,2]$ LCD codes.

4.6.3. Higman Sims group *HS*

HS is a rank 3 permutation group on 100 points, which gives the unique SRG$(100,22,0,6)$. Below, we give the list of all LCD codes constructed with the use of Magma.

- $P_{\mathbb{F}_2}(0,1,0)$ and $P_{\mathbb{F}_2}(1,0,1)$ are $[200,100,6]$ LCD codes.
  $P_{\mathbb{F}_2}(1,1,1)$ is an $[200,100,2]$ LCD code.

- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[202,101,1]$ LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, are $[202,101,1]$ LCD codes.
  $B_{\mathbb{F}_2}(0,1,0)$ and $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\gamma=0,\ \beta=1$, are $[202,101,6]$ LCD codes.
  $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\beta=\gamma=0$, is an $[202,101,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha=\gamma=0,\ \beta=1$, is an $[202,101,2]$ LCD code.

- $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[200,100,2]$ LCD codes.
  $P_{\mathbb{F}_3}(a,b,a)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[200,100,16]$ LCD codes.
  $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[200,100,4]$ LCD codes.
  $P_{\mathbb{F}_3}(0,a,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[200,100,16]$ LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,a)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[202,101,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,a)$, $B_{\mathbb{F}_3}(a,b,b)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\beta=\gamma=0,\ a,b,\alpha\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[202,101,2]$ LCD codes.
  $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[202,101,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[202,101,16]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[202,101,1]$ LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[202,101,4]$ LCD codes.

- $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[200,100,6]$ LCD codes.
  $P_{\mathbb{F}_4}(a,a,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[200,100,2]$ LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[202,101,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\beta=\gamma=0,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[202,101,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma=0,\ a,b,\beta\in\mathbb{F}_4\backslash\{0\},\ a\neq b,\ \alpha\in\mathbb{F}_4\backslash\{1\}$, are $[202,101,6]$ LCD codes.
  $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[202,101,1]$ LCD codes.
  $B_{\mathbb{F}_4}(a,a,a)$, where $\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\},\ \alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[202,101,2]$ LCD

codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma = 0,\ a,\beta \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{1\}$, are $[202,101,2]$ LCD codes.

4.6.4. McLaughlin group *McL*

McL is a rank 3 permutation group on 275 points, which gives the unique SRG$(275,112,30,56)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,1,0)$ is an LCD code.
- $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$, where $\gamma = 0,\ \alpha = \beta,\ \alpha,\beta \in \mathbb{F}_2$, are LCD codes.
- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,0,a)$, $P_{\mathbb{F}_3}(0,0,a)$ and $P_{\mathbb{F}_3}(0,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.
- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\},\ \alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(a,a,0)$, $B_{\mathbb{F}_3}(a,a,a)$ and $B_{\mathbb{F}_3}(a,a,b)$, where $\alpha\gamma + 2a\beta = 0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b,\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(0,0,a)$, where $\gamma = 0,\ a, 1+\alpha^2+2\beta^2 \in \mathbb{F}_3\backslash\{0\},\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes.
- $P_{\mathbb{F}_4}(0,a,0)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are LCD codes.
- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\gamma = 1,\ \alpha + a\beta = 0,\ a,b,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\},\ a \neq b,\ \alpha,\beta \in \mathbb{F}_4$, are LCD codes.
  $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma = 0,\ a, 1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\},\ \alpha,\beta \in \mathbb{F}_4$, are LCD codes.

4.6.5. Suzuki group *Suz*

Suz is a rank 3 permutation group on 1782 points, which gives an SRG$(1782,416,100,96)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,1,0)$, $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.
- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha = \beta = 0,\ \gamma = 1$, is an LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta \in \mathbb{F}_2$, are LCD codes.

- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,a,0)$, $P_{\mathbb{F}_3}(a,a,a)$, $P_{\mathbb{F}_3}(a,a,b)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a,b \in \mathbb{F}_3 \backslash \{0\}$, $a \neq b$, are LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\},\ \alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(a,a,0),\ B_{\mathbb{F}_3}(a,a,a),\ B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\gamma = 0,\ a,b,\alpha,\beta \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are LCD codes.

- $P_{\mathbb{F}_4}(a,0,a),\ P_{\mathbb{F}_4}(a,a,a),\ P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are LCD codes.

- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.

4.6.6. Conway group Co_2

Co_2 is a rank 3 permutation group on 2300 points, which gives an SRG$(2300,891,378,324)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,0,1)$, $P_{\mathbb{F}_2}(1,1,0)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.

- $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(1,1,0)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta \in \mathbb{F}_2$, are LCD codes.

- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(0,a,0)$ and $P_{\mathbb{F}_3}(0,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\},\ \alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(a,0,a),\ B_{\mathbb{F}_3}(a,a,a)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma + 2a\beta = 0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are LCD codes.
  $B_{\mathbb{F}_3}(0,a,0)$, where $\gamma = 0,\ a, 1+\alpha^2+2\beta^2 \in \mathbb{F}_3\backslash\{0\},\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes.

- $P_{\mathbb{F}_4}(a,a,0),\ P_{\mathbb{F}_4}(a,a,a),\ P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are LCD codes.

- $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.

4.6.7. Rudvalis group *Ru*

Ru is a rank 3 permutation group on 4060 points, which gives an SRG$(4060, 2304, 1328, 1208)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,1,0)$, $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.
- $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha = \beta = 0,\ \gamma = 1$, is an LCD code.
  $B_{\mathbb{F}_2}(0,1,0)$, $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta \in \mathbb{F}_2$, are LCD codes.
- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,0,a)$, $P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are LCD codes.
- $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\}$, $\alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(0,a,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha = 0,\ \beta \in \mathbb{F}_3,\ a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are LCD codes.
- $P_{\mathbb{F}_4}(a,0,a)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,b,a)$ and $P_{\mathbb{F}_4}(0,a,0)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are LCD codes.
- $B_{\mathbb{F}_4}(a,0,a)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.

4.6.8. Fischer groups

The Fischer sporadic groups are the three sporadic groups Fi_{22}, Fi_{23} and Fi'_{24} introduced by B. Fischer.

Fischer group $\mathbf{Fi_{22}}$

Fi_{22} is a rank 3 permutation group on 3510 points, which gives an SRG$(3510, 693, 180, 126)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,0,1)$, $P_{\mathbb{F}_2}(1,1,0)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.
- $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(1,1,0)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta \in \mathbb{F}_2$, are LCD codes.

- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,0,a)$, $P_{\mathbb{F}_3}(a,a,a)$, $P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(0,a,0)$, where $a,b \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\},\ \alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,a,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\gamma = 0,\ a,b \in \mathbb{F}_3\backslash\{0\}$, $a \neq b,\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes.

- $P_{\mathbb{F}_4}(a,a,0)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are LCD codes.

- $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.

Fi_{22} has also a rank 3 action on 14080 points, which gives an SRG$(14080,3159,918,648)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,0,1)$, $P_{\mathbb{F}_2}(1,1,0)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.

- $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(1,1,0)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta \in \mathbb{F}_2$, are LCD codes.

- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,b,b)$ and $P_{\mathbb{F}_3}(0,a,0)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta = \gamma = 0,\ a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b,\ \alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(0,a,0)$, where $\gamma = 0,\ a, 1+\alpha^2+\beta^2 \in \mathbb{F}_3\backslash\{0\},\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha = 0,\ a,\gamma \in \mathbb{F}_3\backslash\{0\},\ \beta \in \mathbb{F}_3$, are LCD codes.

- $P_{\mathbb{F}_4}(a,a,0)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are LCD codes.

- $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma = 0,\ a,b \in \mathbb{F}_4\backslash\{0\}$, $a \neq b,\ \alpha \in \mathbb{F}_4\backslash\{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.

Fischer group $\mathbf{Fi_{23}}$

Fi_{23} is a rank 3 permutation group on 31671 points, which gives an SRG$(31671, 3510, 693, 351)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,0,1)$ is an LCD code.
- $B_{\mathbb{F}_2}(0,0,1)$, where $\gamma = 0,\ \alpha = \beta,\ \alpha,\beta \in \mathbb{F}_2$, is an LCD code.
  $B_{\mathbb{F}_2}(1,1,0)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\gamma = 1,\ \alpha = \beta,\ \alpha,\beta \in \mathbb{F}_2$, are LCD codes.
- $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,0,a)$, $P_{\mathbb{F}_3}(a,a,a)$, $P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(0,a,0)$, where $a,b \in \mathbb{F}_3 \backslash \{0\}$, $a \neq b$, are LCD codes.
- $B_{\mathbb{F}_3}(a,0,0)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_3 \backslash \{0\},\ \alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,a,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\gamma = 0,\ a,b \in \mathbb{F}_3 \backslash \{0\}$, $a \neq b,\ \alpha,\beta \in \mathbb{F}_3$, are LCD codes.
- $P_{\mathbb{F}_4}(0,0,a)$, where $a \in \mathbb{F}_4 \backslash \{0\}$, are LCD codes.
- $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,a)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\gamma = 1,\ \alpha + a\beta = 0,\ a,b,1+\alpha^2+\beta^2 \in \mathbb{F}_4 \backslash \{0\},\ a \neq b,\ \alpha,\beta \in \mathbb{F}_4$, are LCD codes.
  $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma = 0,\ a, 1+\alpha^2+\beta^2 \in \mathbb{F}_4 \backslash \{0\},\ \alpha,\beta \in \mathbb{F}_4$, are LCD codes.

Fi_{23} has also a rank 3 action on 137632 points, which gives an SRG$(137632, 28431, 6030, 5832)$. This graph gives the same list of LCD codes as SRG$(14080, 3159, 918, 648)$, which is obtained from Fi_{22}.

Fischer group $\mathbf{Fi_{24}}'$

Fi_{24}' is a rank 3 permutation group on 306936 points, which gives an SRG$(306936, 31671, 3510, 3240)$. Below, we give the list of all obtained LCD codes, but without construction in Magma.

- $P_{\mathbb{F}_2}(0,0,1)$, $P_{\mathbb{F}_2}(1,1,0)$ and $P_{\mathbb{F}_2}(1,1,1)$ are LCD codes.
- $B_{\mathbb{F}_2}(0,0,1)$, $B_{\mathbb{F}_2}(1,1,0)$ and $B_{\mathbb{F}_2}(1,1,1)$, where $\alpha = \gamma = 0,\ \beta \in \mathbb{F}_2$, are LCD codes.

- $P_{\mathbb{F}_3}(a,0,0), P_{\mathbb{F}_3}(a,0,a), P_{\mathbb{F}_3}(a,a,a), P_{\mathbb{F}_3}(a,b,a)$ and $P_{\mathbb{F}_3}(0,a,0)$, where $a,b \in \mathbb{F}_3 \backslash \{0\}$, $a \neq b$, are LCD codes.

- $B_{\mathbb{F}_3}(a,0,0)$, where $\beta = \gamma = 0,\ a \in \mathbb{F}_3 \backslash \{0\},\ \alpha \in \mathbb{F}_3$, are LCD codes.
  $B_{\mathbb{F}_3}(a,0,a)$, $B_{\mathbb{F}_3}(a,a,a)$, $B_{\mathbb{F}_3}(a,b,a)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\gamma = 0,\ a,b \in \mathbb{F}_3 \backslash \{0\}$, $a \neq b,\ \alpha, \beta \in \mathbb{F}_3$, are LCD codes.

- $P_{\mathbb{F}_4}(a,a,0)$, $P_{\mathbb{F}_4}(a,a,a)$, $P_{\mathbb{F}_4}(a,a,b)$ and $P_{\mathbb{F}_4}(0,0,a)$, where $a,b \in \mathbb{F}_4 \backslash \{0\},\ a \neq b$, are LCD codes.

- $B_{\mathbb{F}_4}(a,a,0)$, $B_{\mathbb{F}_4}(a,a,a)$, $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma = 0,\ a,b \in \mathbb{F}_4 \backslash \{0\}$, $a \neq b,\ \alpha \in \mathbb{F}_4 \backslash \{1\},\ \beta \in \mathbb{F}_4$, are LCD codes.

5. LCD CODES FROM DRTS

In this section, we construct LCD codes from some doubly regular tournaments, which are defined in Section 1.4. In order to obtain LCD codes that can be constructed from a certain DRT over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$, we use Tables 3.2, 3.5, 3.6, 3.9, 3.10 and parameters of the corresponding DRT. The adjacency matrices of most DRTs are taken from [47]. Besides, we applied the following result, which can be found in [50], to obtain adjacency matrices of some DRTs of higher orders.

Theorem 5.0.1. Let $n = 4\lambda + 3$, where λ is a non-negative integer. Then there exists a DRT of order n if and only if there exists a skew Hadamard matrix of order $n+1$.

The proposition that follows can help us to obtain LCD codes constructed over the field $\mathbb{F}_2$ from the pure and the bordered constructions.

Proposition 5.0.2. Let (X, R_1) be a DRT with parameters (v, k, λ, μ). In the case of the pure construction, $v \equiv 7 \pmod 8$ if and only if $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,0)$ are LCD codes. Further, in the case of the bordered construction, suppose that $\alpha = \beta = \gamma = 1$. Then $v \equiv 7 \pmod 8$ if and only if $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$ are LCD codes. Now suppose that $\alpha = \beta = 1,\ \gamma = 0$. Then $v \equiv 7 \pmod 8$ if and only if $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$ are LCD codes.

Proof. As mentioned in Lemma 1.6.3, if A is the adjacency matrix of a DRT, then $AA^\top = kI + (k-1-\lambda)A + (k-\mu)\overline{A}$. Now using Lemma 1.4.20, we have $AA^\top = kI + \lambda A + \lambda\overline{A}$ and $v \equiv 7 \pmod 8$ if and only if $\lambda \equiv 1 \pmod 2$. First we consider the pure construction.

Suppose that $v \equiv 7 \pmod 8$. Then

$$\begin{aligned} Q_{\mathbb{F}_2}(1,1,0)Q_{\mathbb{F}_2}(1,1,0)^\top &= (I+A)(I+A)^\top = I + A^\top + A + A\overline{A} \\ &= J + kI + \lambda A + \lambda \overline{A} = J + (2\lambda+1)I + \lambda A + \lambda \overline{A} \\ &\equiv (J + I + A + \overline{A}) \pmod 2 \quad (\text{as } \lambda \equiv 1 \pmod 2) \\ &\equiv \mathbf{0} \pmod 2. \end{aligned}$$

From this follows $\det(P_{\mathbb{F}_2}(1,1,0) \cdot P_{\mathbb{F}_2}(1,1,0)^\top) = \det(I) = 1 \neq 0$. Therefore, $P_{\mathbb{F}_2}(1,1,0)$ is an LCD code. Similarly, we can show that $P_{\mathbb{F}_2}(1,0,1)$ is an LCD code.

Conversely, suppose that $P_{\mathbb{F}_2}(1,1,0)$ is an LCD code. Then

$$\begin{aligned} Q_{\mathbb{F}_2}(1,1,0)Q_{\mathbb{F}_2}(1,1,0)^\top &= J + kI + \lambda A + \lambda \overline{A} = J + (2\lambda+1)I + \lambda A + \lambda(J - I - A) \\ &= (\lambda+1)J + (\lambda+1)I. \end{aligned}$$

Therefore, to have $\det(P_{\mathbb{F}_2}(1,1,0) \cdot P_{\mathbb{F}_2}(1,1,0)^\top) \neq 0$, it must be $\lambda \equiv 1 \pmod 2$ and hence $v \equiv 7 \pmod 8$.

Now we consider the bordered construction, for the case $\alpha = \beta = \gamma = 1$. Suppose that $v \equiv 7 \pmod 8$. Then $\lambda \equiv 1 \pmod 2$ and we have

$$\begin{aligned} Q_{\mathbb{F}_2}(0,1,0)Q_{\mathbb{F}_2}(0,1,0)^\top &= AA^\top = kI + \lambda A + \lambda \overline{A} = (2\lambda+1)I + \lambda A + \lambda \overline{A} \\ &\equiv (I + A + \overline{A}) \pmod 2 \equiv J \pmod 2. \end{aligned}$$

Next, we need to determine $B_{\mathbb{F}_2}(0,1,0) \cdot B_{\mathbb{F}_2}(0,1,0)^\top$, and so we obtain these equations

$$\begin{aligned} &1 + \alpha^2 + v\beta^2 = 1 + 1 + 4\lambda + 3 \equiv 1 \pmod 2, \\ &\alpha\gamma + \beta(r + sk + t(v-k-1)) = 1 + 2\lambda + 1 \equiv 0 \pmod 2, \\ &I + \gamma^2 J + Q_{\mathbb{F}_2}(0,1,0)Q_{\mathbb{F}_2}(0,1,0)^\top \equiv (I + J + J) \pmod 2 \equiv I \pmod 2. \end{aligned}$$

From this follows $\det(B_{\mathbb{F}_2}(0,1,0) \cdot B_{\mathbb{F}_2}(0,1,0)^\top) = \det(I) = 1 \neq 0$. Therefore, $B_{\mathbb{F}_2}(0,1,0)$ is an LCD code. Similarly, we can show that $B_{\mathbb{F}_2}(0,0,1)$ is an LCD code.

Conversely, suppose that $B_{\mathbb{F}_2}(0,1,0)$ is an LCD code. Then from the above calculation follows

$$Q_{\mathbb{F}_2}(0,1,0)Q_{\mathbb{F}_2}(0,1,0)^\top = (2\lambda+1)I + \lambda A + \lambda \overline{A} = \lambda J + (\lambda+1)I$$

and

$$I + \gamma^2 J + Q_{\mathbb{F}_2}(0,1,0)Q_{\mathbb{F}_2}(0,1,0)^\top = (\lambda+2)I + (\lambda+1)J.$$

Therefore, to have $\det(B_{\mathbb{F}_2}(0,1,0)\cdot B_{\mathbb{F}_2}(0,1,0)^\top)\neq 0$, it must be $\lambda\equiv 1 \pmod 2$ and hence $v\equiv 7 \pmod 8$.

Analogously, we can prove the second case for $\alpha=\beta=1,\ \gamma=0$ that $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$ are LCD codes if and only if $v\equiv 7 \pmod 8$.

□

We used the software package Magma [10] to construct LCD codes from the adjacency matrices of given doubly regular tournaments and then to determine their minimum distances. By applying the pure and the bordered constructions, we constructed LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$ for DRTs of order $n=4\lambda+3$ where $\lambda=0,1,\dots,8$.

Examples computed in Magma

1. Let G be a DRT with parameters $(3,1,0,1)$. There is the unique DRT of order 3 which is of a Paley type.

   $P_{\mathbb{F}_3}(a,0,0)$, $P_{\mathbb{F}_3}(a,a,a)$, $P_{\mathbb{F}_3}(0,a,0)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[6,3,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[8,4,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[8,4,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,b,0)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[8,4,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,0)$, where $\alpha=0,\ a,b,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[8,4,3]$ LCD codes and these codes are near-optimal.

   $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[8,4,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[8,4,3]$ LCD codes and these codes are near-optimal.

   $B_{\mathbb{F}_3}(a,a,a)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[8,4,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[8,4,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\gamma=0,\ a,\alpha\in\mathbb{F}_3,\ \beta\in\mathbb{F}_3\backslash\{0\}$, are $[8,4,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[8,4,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[8,4,4]$ LCD codes and these codes are optimal.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0$, $a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[8,4,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0$, $a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[8,4,4]$ LCD codes and these codes are optimal.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0$, $\gamma=1$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[8,4,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1$, $\alpha\gamma+a\beta=0$, $a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[8,4,2]$ LCD codes.

2. Let G be a DRT with parameters $(7,3,1,2)$. There is the unique DRT of order 7 (No. 2 of [47]).

   $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,0)$ are $[14,7,3]$ LCD codes and these codes are near-optimal. $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=0$, $\gamma=1$, are $[16,8,1]$ LCD codes. $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=\gamma=1$, are $[16,8,4]$ LCD codes and these codes are near-optimal.

   $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=\gamma=0$, are $[16,8,1]$ LCD codes. $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=1$, $\gamma=0$, are $[16,8,3]$ LCD codes.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[14,7,2]$ LCD codes. $P_{\mathbb{F}_3}(a,b,0)$ and $P_{\mathbb{F}_3}(a,0,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[14,7,5]$ LCD codes and these codes are near-optimal.

   $P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[14,7,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0$, $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[16,8,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0$, $a,b,\alpha\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[16,8,2]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0$, $a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[16,8,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0$, $a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[16,8,4]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0$, $a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[16,8,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=0$, $a,b,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[16,8,6]$ LCD codes and these codes are optimal.

   $P_{\mathbb{F}_4}(a,a,0)$ and $P_{\mathbb{F}_4}(a,0,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[14,7,3]$ LCD codes.

   $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[16,8,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[16,8,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\beta\in\mathbb{F}_4\backslash\{0,1\}$, are $[16,8,3]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\gamma=0$, $a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[16,8,3]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[16,8,1]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[16,8,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\alpha=\beta=0,\ \gamma=1,\ a,b\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[16,8,1]$ LCD codes. $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\gamma=1,\ \alpha+b\beta=0,\ a,b,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\},\ a\neq b$, are $[16,8,4]$ LCD codes.

$B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[16,8,1]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[16,8,4]$ LCD codes.

3. Let G be a DRT with parameters $(11,5,2,3)$. There is the unique DRT of order 11 (No. 2 of [47]).

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[22,11,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,a,0)$ and $P_{\mathbb{F}_3}(a,0,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[22,11,5]$ LCD codes.

   $P_{\mathbb{F}_3}(0,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[22,11,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\gamma=0,\ a,\alpha,\beta\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,5]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha\gamma+2a\beta=0,\ a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,6]$ LCD codes.

   $B_{\mathbb{F}_3}(0,a,0)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=0,\ a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,0)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha\gamma+2a\beta=0,\ a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[24,12,6]$ LCD codes.

   $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1$, are $[24,12,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha\gamma+a\beta=0,\ \gamma=1,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[24,12,2]$ LCD codes.

4. Let G be a DRT with parameters $(15,7,3,4)$. There is the unique DRT of order 15 (No. 5 of [47]).
   $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,0)$ are $[30,15,3]$ LCD codes.
   $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=0,\ \gamma=1$, are $[32,16,1]$ LCD codes.
   $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=\gamma=1$, are $[32,16,4]$ LCD codes.
   $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=\gamma=0$, are $[32,16,1]$ LCD codes.
   $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=1,\ \gamma=0$, are $[32,16,3]$ LCD codes.
   $P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[30,15,2]$ LCD codes.
   $P_{\mathbb{F}_3}(0,a,0)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[30,15,8]$ LCD codes.
   $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,2]$ LCD codes.
   $B_{\mathbb{F}_3}(a,b,0)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[32,16,1]$ LCD codes.
   $B_{\mathbb{F}_3}(a,b,0)$, where $\alpha=0,\ a,b,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[32,16,6]$ LCD codes.
   $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[32,16,6]$ LCD codes.
   $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[32,16,2]$ LCD codes.
   $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[32,16,8]$ LCD codes.
   $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[32,16,1]$ LCD codes.
   $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[32,16,4]$ LCD codes.
   $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,2]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\gamma=0,\ a,\alpha,\beta\in\mathbb{F}_3\backslash\{0\}$, are $[32,16,8]$ LCD codes.
   $P_{\mathbb{F}_4}(a,a,0)$ and $P_{\mathbb{F}_4}(a,0,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[30,15,3]$ LCD codes.
   $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[16,8,1]$ LCD

codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[32,16,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\beta\in\mathbb{F}_4\backslash\{0,1\}$, are $[32,16,3]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\gamma=0$, $a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[32,16,3]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0$, $\gamma=1$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1$, $\alpha+a\beta=0$, $a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[32,16,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\alpha=\beta=0$, $\gamma=1$, $a,b\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_4}(a,b,a)$ and $B_{\mathbb{F}_4}(a,a,b)$, where $\gamma=1$, $\alpha+b\beta=0$, $a,b,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, $a\neq b$, are $[32,16,4]$ LCD codes.

$B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\alpha=\beta=0$, $\gamma=1$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[32,16,1]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma=1$, $\alpha+a\beta=0$, $a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[32,16,4]$ LCD codes.

5. Let G be a DRT with parameters $(19,9,4,5)$. There are two DRTs of order 19 (No. 2 and No. 3 of [47]). They only give LCD codes with different parameters for the pure case in the field $\mathbb{F}_3$.

$P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[38,19,2]$ LCD codes.

$P_{\mathbb{F}_3}(a,b,0)$ and $P_{\mathbb{F}_3}(a,0,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[38,19,9]$ LCD codes for DRT No. 2, and $[38,19,10]$ LCD codes for DRT No. 3.

$P_{\mathbb{F}_3}(a,b,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[38,19,4]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0$, $a,b\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[40,20,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0$, $a,b,\alpha\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[40,20,2]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0$, $a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[40,20,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0$, $a,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[40,20,4]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0$, $a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[40,20,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=0$, $a,b,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, $a\neq b$, are $[40,20,10]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0$, $\gamma=1$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[40,20,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1$, $\alpha\gamma+a\beta=0$, $a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[40,20,2]$ LCD codes.

6. Let G be a DRT with parameters $(23,11,5,6)$. There are 19 DRTs of order 23 (No. 2 - No. 20 of [47]). They all give LCD codes with the same parameters except for DRTs No. 19 and No. 20.

   $P_{\mathbb{F}_2}(1,0,1)$ is an $[46,23,3]$ LCD code for DRTs No. 2 - No. 19, and an $[46,23,7]$ LCD code for DRT No. 20.

   $P_{\mathbb{F}_2}(1,1,0)$ is an $[46,23,3]$ LCD code for DRTs No. 2 - No. 18, and an $[46,23,7]$ LCD code for DRTs No. 19 and No. 20.

   $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[48,24,1]$ LCD code. $B_{\mathbb{F}_2}(0,0,1)$, where $\alpha=\beta=\gamma=1$, is an $[48,24,4]$ LCD code for DRTs No. 2 - No. 18, an $[48,24,7]$ LCD code for DRT No. 19, and an $[48,24,8]$ LCD code for DRT No. 20.

   $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=0,\ \gamma=1$, is an $[48,24,1]$ LCD code. $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=\gamma=1$, is an $[48,24,4]$ LCD code for DRTs No. 2 - No. 19, and an $[48,24,8]$ LCD code for DRT No. 20.

   $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=\gamma=0$, is an $[48,24,1]$ LCD code. $B_{\mathbb{F}_2}(1,0,1)$, where $\alpha=\beta=1,\ \gamma=0$, is an $[48,24,3]$ LCD code for DRTs No. 2 - No. 19, and an $[48,24,7]$ LCD code for DRT No. 20.

   $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=\gamma=0$, is an $[48,24,1]$ LCD code. $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=1,\ \gamma=0$, is an $[48,24,3]$ LCD code for DRTs No. 2 - No. 18, and an $[48,24,7]$ LCD code for DRTs No. 19 and No. 20.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[46,23,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,a,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[46,23,5]$ LCD codes for DRTs No. 2 - No. 19, and $[46,23,8]$ LCD codes for DRT No. 20.

   $P_{\mathbb{F}_3}(a,0,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[46,23,5]$ LCD codes for DRTs No. 2 - No. 18, and $[46,23,8]$ LCD codes for DRTs No. 19 and No. 20.

   $P_{\mathbb{F}_3}(0,a,a)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[46,23,4]$ LCD codes.

   $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,2]$ LCD codes.

   $B_{\mathbb{F}_3}(a,a,0)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,0)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,0)$, where $\gamma=0,\ a,\alpha,\beta\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,5]$ LCD codes for DRTs No. 2 - No. 19, and

$[48,24,8]$ LCD codes for DRT No. 20.

$B_{\mathbb{F}_3}(a,0,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$, where $\beta=\gamma=0$, $a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,2]$ LCD codes. $B_{\mathbb{F}_3}(a,0,a)$, where $\gamma=0$, $a,\alpha,\beta\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,5]$ LCD codes for DRTs No. 2 - No. 18, and $[48,24,8]$ LCD codes for DRTs No. 19 and No. 20.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=0$, $a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha\gamma+2a\beta=0$, $a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,b)$, where $\alpha=\beta=0$, $a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$, where $\alpha\gamma+a\beta=0$, $a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,6]$ LCD codes for DRTs No. 2 - No. 19, and $[48,24,9]$ LCD codes for DRT No. 20.

$B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=0$, $a,b,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+a\beta=0$, $a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,6]$ LCD codes for DRTs No. 2 - No. 18, and $[48,24,9]$ LCD codes for DRTs No. 19 and No. 20.

$B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=0$, $a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha\gamma+2a\beta=0$, $a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,6]$ LCD codes for DRTs No. 2 - No. 18, and $[48,24,9]$ LCD codes for DRTs No. 19 and No. 20.

$B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=0$, $a,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha\gamma+2a\beta=0$, $a,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\}$, are $[48,24,6]$ LCD codes for DRTs No. 2 - No. 19, and $[48,24,9]$ LCD codes for DRT No. 20.

$P_{\mathbb{F}_4}(a,a,0)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[46,23,3]$ LCD codes for DRTs No. 2 - No. 18, and $[46,23,7]$ LCD codes for DRTs No. 19 and No. 20.

$P_{\mathbb{F}_4}(a,0,a)$, where $a\in\mathbb{F}_4\backslash\{0\}$, are $[46,23,3]$ LCD codes for DRTs No. 2 - No. 19, and $[46,23,7]$ LCD codes for DRT No. 20.

$B_{\mathbb{F}_4}(a,a,0)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$, where $\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[48,24,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$, where $\alpha=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\beta\in\mathbb{F}_4\backslash\{0,1\}$, are $[48,24,3]$ LCD codes for DRTs No. 2 - No. 18, and $[48,24,7]$ LCD codes for DRTs No. 19 and No. 20. $B_{\mathbb{F}_4}(a,a,0)$, where $\gamma=0$, $a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[48,24,3]$ LCD codes for DRTs No. 2 - No. 18, and $[48,24,7]$ LCD codes for DRTs No. 19 and No. 20.

$B_{\mathbb{F}_4}(a,0,a)$, where $\alpha=\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, where $\beta=\gamma=0$, $a\in\mathbb{F}_4\backslash\{0\}$, $\alpha\in\mathbb{F}_4\backslash\{0,1\}$, are $[48,24,2]$ LCD codes. $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha = \gamma = 0,\ a \in \mathbb{F}_4\backslash\{0\},\ \beta \in \mathbb{F}_4\backslash\{0,1\}$, are $[48,24,3]$ LCD codes for DRTs No. 2 - No. 19, and $[48,24,7]$ LCD codes for DRT No. 20. $B_{\mathbb{F}_4}(a,0,a)$, where $\gamma = 0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[48,24,3]$ LCD codes for DRTs No. 2 - No. 19, and $[48,24,7]$ LCD codes for DRT No. 20.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha = \beta = 0,\ \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma = 1,\ \alpha + a\beta = 0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[48,24,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,b)$, where $\alpha = \beta = 0,\ \gamma = 1,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,b)$, where $\gamma = 1,\ \alpha + b\beta = 0,\ a,b,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[48,24,4]$ LCD codes for DRTs No. 2 - No. 18, and $[48,24,8]$ LCD codes for DRTs No. 19 and No. 20.

$B_{\mathbb{F}_4}(a,b,a)$, where $\alpha = \beta = 0,\ \gamma = 1,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_4}(a,b,a)$, where $\gamma = 1,\ \alpha + b\beta = 0,\ a,b,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[48,24,4]$ LCD codes for DRTs No. 2 - No. 19, and $[48,24,8]$ LCD codes for DRT No. 20.

$B_{\mathbb{F}_4}(0,a,0)$, where $\alpha = \beta = 0,\ \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$, where $\gamma = 1,\ \alpha + a\beta = 0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[48,24,4]$ LCD codes for DRTs No. 2 - No. 19, and $[48,24,8]$ LCD codes for DRT No. 20.

$B_{\mathbb{F}_4}(0,0,a)$, where $\alpha = \beta = 0,\ \gamma = 1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[48,24,1]$ LCD codes. $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma = 1,\ \alpha + a\beta = 0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[48,24,4]$ LCD codes for DRTs No. 2 - No. 18, $[48,24,7]$ LCD codes for DRT No. 19, and $[48,24,8]$ LCD codes for DRT No. 20.

7. Let G be a DRT with parameters $(27,13,6,7)$. There are 374 DRTs of order 27 (No. 5 - No. 378 of [47]).

   (a) DRTs No. 5 - No. 54, No. 57 - No. 66, No. 68 - No. 119, No. 121, No. 122, No. 124, No. 127 - No. 155, No. 158 - No. 169, No. 171 - No. 191, No. 194 - No. 266, No. 268 - No. 292, No. 294 - No. 297, No. 299 - No. 320, No. 323 - No. 334, No. 336, No. 337, No. 339 - No. 346, No. 348 - No. 350, No. 352, No. 353, No. 355, No. 357, No. 359 - No. 363, No. 365, No. 367, No. 369, No. 371, No. 373, No. 374, No. 376:

$P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[54,27,2]$ LCD codes.

$P_{\mathbb{F}_3}(0,a,0)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[54,27,10]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha = \beta = \gamma = 0$, $a \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\beta = \gamma = 0$, $a, \alpha \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,0)$, where $\alpha = \beta = 0$, $a,b,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,0)$, where $\alpha = 0$, $a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[56,28,6]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha = \beta = 0$, $a,b,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha\gamma + a\beta = 0$, $a,b,\alpha,$ $\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[56,28,6]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha = \beta = \gamma = 0$, $a \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\beta = \gamma = 0$, $a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma = 0$, $\alpha \in \mathbb{F}_3$, $a,\beta \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha = \beta = 0$, $a,b,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma + a\beta = 0$, $a,b,\alpha,$ $\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[56,28,12]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha = \beta = 0$, $a,b,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma + b\beta = 0$, $a,b,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, $a \neq b$, are $[56,28,4]$ LCD codes.

$B_{\mathbb{F}_3}(0,0,a)$, where $\alpha = \beta = \gamma = 0$, $a \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\beta = \gamma = 0$, $a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\gamma = 0$, $a,\alpha,\beta \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,10]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha = \beta = 0$, $\gamma = 1$, $a \in \mathbb{F}_4\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma = 1$, $\alpha\gamma + a\beta = 0$, $a,\alpha,\beta, 1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[56,28,2]$ LCD codes.

(b) DRTs No. 55, No. 56, No. 120, No. 123, No. 125, No. 126, No. 156, No. 157, No. 170, No. 192, No. 193, No. 267, No. 293, No. 298, No. 321, No. 322, No. 335, No. 351, No. 354, No. 356, No. 358, No. 366, No. 368, No. 372, No. 377:

$P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[54,27,2]$ LCD codes.

$P_{\mathbb{F}_3}(0,a,0)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[54,27,10]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\beta=\gamma=0,\ a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,0)$, where $\alpha=\beta=0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,0)$, where $\alpha=0,\ a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[56,28,9]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha\gamma+\beta a=0,\ a,b,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[56,28,9]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\beta=\gamma=0,\ a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0,\ a,\beta \in \mathbb{F}_3\backslash\{0\},\ \alpha \in \mathbb{F}_3$, are $[56,28,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[56,28,12]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[56,28,4]$ LCD codes.

$B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=\gamma=0,\ a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\gamma=0,\ a,\alpha,\beta \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,10]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha\gamma+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[56,28,2]$ LCD codes.

(c) DRTs No. 67, No. 338, No. 347, No. 364, No. 370, No. 375, No. 378:

$P_{\mathbb{F}_3}(a,0,0)$ and $P_{\mathbb{F}_3}(a,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[54,27,2]$ LCD codes.

$P_{\mathbb{F}_3}(0,a,0)$ and $P_{\mathbb{F}_3}(0,0,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[54,27,12]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$ and $B_{\mathbb{F}_3}(0,a,0)$, where $\beta=\gamma=0,\ a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,0)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,0)$, where $\alpha=0,\ a,b,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[56,28,6]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(0,a,b)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[56,28,6]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\gamma=0,\ a,\beta\in\mathbb{F}_3\backslash\{0\},\ \alpha\in\mathbb{F}_3$, are $[56,28,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma+a\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[56,28,12]$ LCD codes.

$B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=0,\ a,b,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha\gamma+b\beta=0,\ a,b,\alpha,\beta,\gamma\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[56,28,4]$ LCD codes.

$B_{\mathbb{F}_3}(0,0,a)$, where $\alpha=\beta=\gamma=0,\ a\in\mathbb{F}_3\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\beta=\gamma=0,\ a,\alpha\in\mathbb{F}_3\backslash\{0\}$, are $[56,28,2]$ LCD codes. $B_{\mathbb{F}_3}(0,0,a)$, where $\gamma=0,\ a,\alpha,\beta\in\mathbb{F}_3\backslash\{0\}$, are $[56,28,12]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a\in\mathbb{F}_4\backslash\{0\}$, are $[56,28,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha\gamma+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2\in\mathbb{F}_4\backslash\{0\}$, are $[56,28,2]$ LCD codes.

8. Let G be a DRT with parameters $(31,15,7,8)$. There are at least 6 skew Hadamard matrices of order 32 [41], so according to Theorem 5.0.1, there exist DRTs of order 31. Here we only consider the Paley type matrix.

   $P_{\mathbb{F}_2}(1,0,1)$ and $P_{\mathbb{F}_2}(1,1,0)$ are $[62,31,7]$ LCD codes.

   $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=0,\ \gamma=1$, are $[64,32,1]$ LCD codes. $B_{\mathbb{F}_2}(0,0,1)$ and $B_{\mathbb{F}_2}(0,1,0)$, where $\alpha=\beta=\gamma=1$, are $[64,32,8]$ LCD codes. $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=\gamma=0$, are $[64,32,1]$ LCD codes. $B_{\mathbb{F}_2}(1,0,1)$ and $B_{\mathbb{F}_2}(1,1,0)$, where $\alpha=\beta=1,\ \gamma=0$, are $[64,32,7]$ LCD codes.

   $P_{\mathbb{F}_3}(a,0,0)$, where $a\in\mathbb{F}_3\backslash\{0\}$, are $[62,31,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,b,0)$ and $P_{\mathbb{F}_3}(a,0,b)$, where $a,b\in\mathbb{F}_3\backslash\{0\},\ a\neq b$, are $[62,31,12]$ LCD codes.

$P_{\mathbb{F}_3}(a,b,b)$, where $a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[62,31,4]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\alpha=\beta=\gamma=0,\ a,b \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[64,32,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,b,0)$, $B_{\mathbb{F}_3}(a,0,b)$ and $B_{\mathbb{F}_3}(a,b,b)$, where $\beta=\gamma=0,\ a,b,\alpha \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[64,32,2]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=\beta=0,\ a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[64,32,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\alpha=0,\ a,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[64,32,4]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=\beta=0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[64,32,1]$ LCD codes.

$B_{\mathbb{F}_3}(0,a,b)$, where $\alpha=0,\ a,b,\beta,\gamma \in \mathbb{F}_3\backslash\{0\},\ a \neq b$, are $[64,32,14]$ LCD codes.

$P_{\mathbb{F}_4}(a,a,0)$ and $P_{\mathbb{F}_4}(a,0,a)$, where $a \in \mathbb{F}_4\backslash\{0\}$, are $[62,31,7]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha=\beta=\gamma=0,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[64,32,1]$ LCD codes. $B_{\mathbb{F}_2}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\beta=\gamma=0,\ a \in \mathbb{F}_4\backslash\{0\},\ \alpha \in \mathbb{F}_4\backslash\{0,1\}$, are $[64,32,2]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\alpha=\gamma=0,\ a \in \mathbb{F}_4\backslash\{0\},\ \beta \in \mathbb{F}_4\backslash\{0,1\}$, are $[64,32,7]$ LCD codes. $B_{\mathbb{F}_4}(a,a,0)$ and $B_{\mathbb{F}_4}(a,0,a)$, where $\gamma=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[64,32,7]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[64,32,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[64,32,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a,b \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[64,32,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,b)$ and $B_{\mathbb{F}_4}(a,b,a)$, where $\gamma=1,\ \alpha+b\beta=0,\ a,b,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\},\ a \neq b$, are $[64,32,8]$ LCD codes.

$B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\alpha=\beta=0,\ \gamma=1,\ a \in \mathbb{F}_4\backslash\{0\}$, are $[64,32,1]$ LCD codes. $B_{\mathbb{F}_4}(0,a,0)$ and $B_{\mathbb{F}_4}(0,0,a)$, where $\gamma=1,\ \alpha+a\beta=0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\}$, are $[64,32,8]$ LCD codes.

9. Let G be a DRT with parameters $(35,17,8,9)$. There are at least 18 skew Hadamard matrices of order 36 [41], so there exist DRTs of order 35. Here we consider all of them in the order as skew Hadamard matrices are listed on [41].

   $P_{\mathbb{F}_3}(a,0,0)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[70,35,2]$ LCD codes.

   $P_{\mathbb{F}_3}(a,a,0)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[70,35,5]$ LCD codes for DRTs No. 4, No. 5, No. 6, No. 7, No. 8, No. 9, No. 10, No. 12, No. 13, No. 14, No. 17 and No. 18, $[70,35,6]$ LCD codes for DRTs No. 1, No. 2, No. 3 and No. 11, and $[70,35,8]$ LCD codes for DRTs No. 15 and No. 16.

$P_{\mathbb{F}_3}(a,0,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[70,35,5]$ LCD codes for DRTs No. 1, No. 6, No. 8, No. 10, No. 12, No. 13, No. 17 and No. 18, $[70,35,6]$ LCD codes for DRTs No. 2, No. 3, No. 4, No. 5, No. 7, No. 9, No. 11 and No. 14, and $[70,35,8]$ LCD codes for DRTs No. 15 and No. 16.

$P_{\mathbb{F}_3}(0,a,a)$, where $a \in \mathbb{F}_3\backslash\{0\}$, are $[70,35,4]$ LCD codes.

$B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,a,0)$ and $B_{\mathbb{F}_3}(a,0,a)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_3}(a,0,0)$, $B_{\mathbb{F}_3}(a,a,0)$ and $B_{\mathbb{F}_2}(a,0,a)$, where $\beta = \gamma = 0,\ a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,0)$, where $\gamma = 0,\ a,\alpha,\beta \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,5]$ LCD codes for DRTs No. 4, No. 5, No. 6, No. 7, No. 8, No. 9, No. 10, No. 12, No. 13, No. 14, No. 17 and No. 18, $[72,36,6]$ LCD codes for DRTs No. 1, No. 2, No. 3 and No. 11, and $[72,36,8]$ LCD codes for DRTs No. 15 and No. 16.

$B_{\mathbb{F}_3}(a,0,a)$, where $\gamma = 0,\ a,\alpha,\beta \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,5]$ LCD codes for DRTs No. 1, No. 6, No. 8, No. 10, No. 12, No. 13, No. 17 and No. 18, $[72,36,6]$ LCD codes for DRTs No. 2, No. 3, No. 4, No. 5, No. 7, No. 9, No. 11 and No. 14, and $[72,36,8]$ LCD codes for DRTs No. 15 and No. 16.

$B_{\mathbb{F}_3}(a,a,a)$, where $\alpha = \beta = 0,\ a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,a)$, where $\alpha\gamma + 2a\beta = 0,\ a,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,2]$ LCD codes.

$B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha = \beta = 0,\ a,b,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_3}(a,a,b)$ and $B_{\mathbb{F}_3}(a,b,a)$, where $\alpha\gamma + 2a\beta = 0,\ a,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,9]$ LCD codes for DRTs No. 15 and No. 16, $[72,36,6]$ LCD codes for all other DRTs.

$B_{\mathbb{F}_3}(0,a,0)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha = \beta = 0,\ a,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,0)$ and $B_{\mathbb{F}_3}(0,0,a)$, where $\alpha\gamma + 2\beta a = 0,\ a,\alpha,\beta,\gamma \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,9]$ LCD codes for DRTs No. 15 and No. 16, $[72,36,6]$ LCD codes for all other DRTs.

$B_{\mathbb{F}_3}(0,a,a)$, where $\alpha = \beta = \gamma = 0,\ a \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_3}(0,a,a)$, where $\beta = \gamma = 0,\ a,\alpha \in \mathbb{F}_3\backslash\{0\}$, are $[72,36,2]$ LCD codes.

$B_{\mathbb{F}_4}(a,a,a)$, where $\alpha = \beta = 0,\ \gamma = 1$, are $[72,36,1]$ LCD codes. $B_{\mathbb{F}_4}(a,a,a)$, where $\alpha\gamma + a\beta = 0,\ a,\alpha,\beta,1+\alpha^2+\beta^2 \in \mathbb{F}_4\backslash\{0\},\ \gamma = 1$, are $[72,36,2]$ LCD codes.

Conclusion

The aim of this dissertation was to develop a method for constructing LCD codes from two class association schemes. For this purpose, we showed that it is possible to construct LCD codes using the adjacency matrices of strongly regular graphs and doubly regular tournaments. The proposed method consists of pure and bordered construction, and it is theoretically related to previously known methods of constructing self-dual codes. Due to this, we described a method given by P. Gaborit and another given by S. T. Dougherty, J.-L. Kim and P. Solé. We also applied their methods to construct self-dual codes from Paley designs and Paley graphs over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$. As a result, we obtained many self-dual codes with large minimum distance. In the case of Paley designs, self-dual codes with the largest minimum distance with respect to a given length and dimension were obtained from the bordered constuction over the fields $\mathbb{F}_2$ and $\mathbb{F}_3$, and in the case of Paley graphs, self-dual codes with the largest minimum distance with respect to a given length and dimension were obtained from the bordered constuction over the field $\mathbb{F}_3$. Many obtained self-dual codes are best known, and some of them are even optimal. Regarding LCD codes, we gave the conditions under which the developed construction method gives LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$. In addition to theoretical results, we constructed examples of LCD codes from some strongly regular graphs and doubly regular tournaments. We also obtained many LCD codes with large minimum distance. The graphs $L(K_6)$, $L(K_{2,2})$ and the Clebsch graph are examples of strongly regular graphs which gave some near-optimal and optimal codes. In the case of doubly regular tournaments, LCD codes with the largest minimum distance with respect to a given length and dimension were constructed over the field $\mathbb{F}_3$. For example, doubly regular tournaments with parameters $(3,1,0,1)$ and $(7,3,1,2)$ provided several near-optimal and optimal codes.

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Curriculum Vitae

Ana Grbac was born on May 4, 1985 in Rijeka where she finished ”Prva riječka hrvatska gimnazija”, high school. She graduated from the Faculty of Arts and Sciences, University of Rijeka, where she acquired the title MSc in Mathematics and Computer Science in 2008. The same year she enrolled in a doctoral program in mathematics which is coordinated by the Department of Mathematics of the Faculty of Science, University of Zagreb.

She worked as a teaching assistant at the Department of Mathematics, Faculty of Arts and Sciences, University of Rijeka from September 2008 to April 2009. Since then, she has been working as a teaching assistant at the Department of Mathematics, University of Rijeka, where she is a member of the Division of Discrete Mathematics. She is a member of the Society of Mathematicians and Physicists Rijeka, a member of the Alumni Club, Department of Mathematics, University of Rijeka, and a member of the Seminar on Finite Mathematics in Rijeka, as part of which she held a series of seminars.

She improved her skills by participating in the international scientific conference ”2016 PhD Summer School in Discrete Mathematics” in June 2016, Rogla, Slovenia.

She is married and has three children.