Multilinear singular integrals associated with hypergraphs

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Multilinearni singularni integrali pridruženi hipergrafovima

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SUMMARY

This work contributes to the theory of entangled multilinear singular integral forms by giving the first characterizations of L^p boundedness of dyadic versions of these forms associated with hypergraphs. Moreover, it establishes the first weighted estimates and sparse domination results for such forms. The work proceeds by applying the obtained characterizations to an open problem in probability theory. It introduces the notion of ergodic-martingale paraproducts and establishes their boundedness and convergence in a certain range of the L^p norms. This gives a possible answer to a classical question by Kakutani. Finally, the work discusses connections with the removal lemmas from arithmetic combinatorics and graph theory.

Keywords: hypergraph, singular integral forms, entangled multilinear singular integral forms, T(1)-type conditions, ergodic-martingale paraproduct, Kakutani's question, L^p estimates, removal lemmas

SAŽETAK

Ovaj rad doprinosi teoriji zapetljanih multilinearnih singularnih integralnih formi tako što daje prve karakterizacije L^p omeđenosti dijadskih verzija tih formi pridruženih hipergrafovima. Nadalje, dokazuje prve težinske ocjene i dominaciju rijetkim operatorima za takve forme. Rad potom primjenjuje dobivene karakterizacije na jedan otvoreni problem u teoriji vjerojatnosti. Uvodi se pojam ergodičko-martingalnih paraprodukata i pokazuje njihovu omeđenost i konvergenciju u izvjesnom rasponu L^p normi. To daje jedan mogući odgovor na klasično Kakutanijevo pitanje. Konačno, rad diskutira veze s lemama o uklanjanju iz aritmetičke kombinatorike i teorije grafova.

Ključne riječi: hipergraf, singularne integralne forme, zapetljane multilinearne singularne integralne forme, T(1) uvjeti, ergodičko-martingalni paraprodukt, Kakutanijevo pitanje, L^p ocjene, leme o uklanjanju

Contents

Introduction

Entangled multilinear singular integral forms have been studied by several authors over the last ten years; see the papers by Kovač [34], [33], Kovač and Thiele [37], Durcik [10], [12], and Durcik and Thiele [18]. They recently found applications in ergodic theory [35], [16], in arithmetic combinatorics [13], [14], to stochastic integration [38], and within the harmonic analysis itself [15], [17]. Therefore, it would be useful to have a reasonably general theory establishing (or characterizing) L^p bounds for these objects. As a step in this program we take results from the papers [34] and [37], where the forms are dyadic and indexed by bipartite graphs, and generalize them to r-partite r-uniform hypergraphs. Some higher-dimensional instances of dyadic entangled forms were already discussed by Kovač [33] and Durcik [11], but our hypergraph generalization prefers a combinatorial description of the structure over a geometric one. Consequently, we can study less symmetric entangled forms and show their estimates in an open range of L^p spaces.

Working in a dyadic model certainly limits the applicability of our results, but this choice is justified in several ways. First, quite often dyadic models help in developing the techniques that are used later to approach the original, continuous-type problems. The reader can compare the result from Chapter 2 with the work of Durcik and Thiele [18], which is the current state-of-the-art on the continuous singular entangled forms. Second, in some applications it is possible to transfer an estimate easily from dyadic to continuous setting; see [33] and [17]. Third, below we formulate an entangled T(1) theorem for dyadic forms associated with hypergraphs. Even its particular case dealing with graphs, which was discussed in [37], has not yet been formulated in the continuous setting and leaves an interesting open problem.

The general theory that we develop has one interesting application on the verge of both probability theory and ergodic theory. There are many similarities in the behaviors of ergodic averages and (forward or backward) martingales. Back in 1950 they inspired Kakutani [32] to formulate an open-ended problem of finding a single concept that generalizes both of these notions. He was primarily looking for “a general theorem which contains both the maximal ergodic theorem and the martingale theorem” (a quote from [32]), and both of these are results on convergence almost surely. However, one can understand his question in a broader sense, by also considering other modes of convergence. Kakutani’s question was answered in versatile ways by many different authors over the course of the last 70 years. The most notable unifying theories were developed by Jerison [27], Rota [48], A. and C. Ionescu Tulcea [26], Petz [45], Kachurovskii [30], and Kachurovskii and Vershik [61]; see the survey by Kachurovskii [31]. It is also interesting to mention a largely forgotten paper of Neveu [44], who deduced almost sure convergence of backward martingales from the pointwise ergodic theorem for contractions. The question of unifying ergodic averages and martingales still attracts some attention of the mathematical community; see the more recent papers by Podvigin [46,47], Ganiev and Shahidi [52], and Shahidi [51].

This work attempts to approach the aforementioned question of Kakutani via bilinear operators and in the spirit of classical harmonic analysis. Quite surprisingly, already the simplest nontrivial bilinear objects formed by ergodic averages and discrete martingales turns out to be somewhat involved.

In Chapter 1 we will present definitions and results needed for this dissertation. Most of them are about very common objects and fundamental results in analysis and probability theory, which we are going to cite from [21] and [19], while other statements are results from recent scientific articles. Along with that, we will introduce a hypergraph setting by giving the definition of a hypergraph, which will turn out to be a direct generalization of a standard object from the graph theory, and certain assumptions on hypergraphs that we consider, which will be our requirements needed for the proofs to work. We will also shortly discuss the connections of results in this dissertation with removal lemmas, which are another interesting topic, this time in combinatorics. These lemmas inspired the proofs of the aforementioned results on multilinear singular integrals.

In Chapter 2 we will introduce a singular integral form corresponding to a perfect dyadic Calderón-Zygmund kernel K and associated with a hypergraph $H=(V,E)$ in a way that its formula will be expressed in terms of the set of edges E, along with corresponding vertices that each edge $e \in E$ contains. At the level of a precise formula, it will be defined as

$$\Lambda_E(\mathbf{F}) := \int_{\mathbb{R}^n} \Big(\prod_{e\in E} F_e(\mathbb{x}_e)\Big) K(\mathbb{x})\, d\mathbb{x}.$$

We will consider L^p boundedness of this form where the range of exponent p_e for each function F_e will be $\langle d_e, \infty]$, where d_e is a constant determined by the structure of the given hypergraph. In Section 2.1 we will state the main result, which was also published in the form of an article [58]. We will show that the boundedness is equivalent to several other conditions, such as: the weak boundedness property and the T(1)-type condition; L^p boundedness for only one choice of exponents in stated range; domination by the sparse form; the weighted L^p boundedness for certain tuples of Muckenhoupt weights. In order to obtain such result, in Section 2.2 we are proving L^p boundedness of a similarly defined form $\Lambda_{E,S}$ where the kernel K will be replaced with a much simpler one; more precisely, with

$$\sum_{Q=I_1\times\cdots\times I_r\in\mathscr{C}_r} |Q| \prod_{i=1}^{r} \Big(\prod_{v^{(i)}\in S^{(i)}} \mathbb{h}^1_{I_i}(x_{v^{(i)}}) \prod_{v^{(i)}\in (S^{(i)})^c} \mathbb{h}^0_{I_i}(x_{v^{(i)}}) \Big),$$

where $\mathbb{h}^0$ and $\mathbb{h}^1$ are L^1-normalized Haar functions; see (1.16). We are going to show estimates using an induction on a certain collection of hypergraphs, a telescoping formula for certain expressions and estimates of the form that is localized on a finite convex tree, where we will also apply a stopping time argument. In Section 2.3 we are going to consider paraproducts Λ_E^S which sum over all observed tuples of vertex-sets S, decompose the form Λ_E of main interest, and lead to localized estimates for Λ_E. In Section 2.4 we will prove the required characterizations of L^p boundedness.

In Chapter 3 the main object of interest are the ergodic-martingale paraproducts, defined as the sequence $(\Pi_n^{\mathrm{em}})_{n\in\mathbb{N}_0}$ of bilinear operators given with

$$\Pi_n^{\mathrm{em}}(f,g) := \sum_{i=0}^{n-1} (A_{\lfloor a^i \rfloor} f)(\mathbb{E}(g|\mathscr{G}_{i+1}) - \mathbb{E}(g|\mathscr{G}_i)).$$

Here $A_n f$ stands for n-th Cesàro average of a function f, with a certain transformation given, and $(\mathscr{G}_i)_{i\in\mathbb{N}_0}$ is a decreasing sequence of σ-algebras, so that $\mathbb{E}(g|\mathscr{G}_n)$ stands for the

n-th term of a backward martingale $(\mathbb{E}(g|\mathscr{G}_i))_{i\in\mathbb{N}_0}$. We will give a proof (later published in the form of the paper [36]), that the sequence $(\Pi_n^{\mathrm{em}}(f,g))_{n\in\mathbb{N}_0}$ converges in L^r space for $r\in\left[1,\frac{4}{3}\right]$ and $p,q\in\left[\frac{4}{3},4\right]$, for every $f\in\mathrm{L}^p$ and $g\in\mathrm{L}^q$, and assuming $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. The question of convergence will turn out to be equivalent to the same question for a similar paraproduct $(\Pi_n^{\mathrm{me}}(f,g))_{n\in\mathbb{N}_0}$ where

$$\Pi_n^{\mathrm{me}}(f,g) := \sum_{i=0}^{n-1}(A_{\lfloor a^{i+1}\rfloor}f - A_{\lfloor a^i\rfloor}f))\mathbb{E}(g|\mathscr{G}_{i+1}),$$

which is why the main focus in the proofs will be only on the first sequence mentioned. The convergence of this sequence will follow from its L^r boundedness, which we will show by gradually performing various reductions. In Subsection 3.2.6 we will deduce the estimate for the triple of exponents $(p,q,r)=(4,2,4)$ and then, in Subsection 3.2.7, we will use additional techniques (such as multilinear interpolation) of expanding the region of exponents in order to obtain the required estimate.

1. Definitions and preliminary results

In this chapter we will list definitions of basic objects that we plan to work with in this dissertation and we will state various results from the analysis, probability theory, and ergodic theory. Most of these definitions and results are fundamental and commonly applied in various texts, while some of them are objects of interest in most recent articles. We will also introduce hypergraph setting with conventions required for the proofs to follow.

Some results in this section will be accompanied with complete proofs, while the others will come with references to the existing literature. We will prefer to prove those results that are either slightly non-standard, or will be applied in slightly nonstandard formulations.

1.1. Analysis

For a locally integrable function F and any bounded measurable set $S \subseteq \mathbb{R}^r$ of strictly positive Lebesgue measure $|S| > 0$ we introduce the notation

$$[F]_S := \frac{1}{|S|} \int_S F(x)dx.$$

In words, $[F]_S$ is simply the average of F on S.

For nonnegative quantities A and B we write $A \lesssim B$ if $A \leq CB$ holds with some unimportant finite constant C. If we wish to emphasize that the exact value of the constant C depends on some of the parameters that appear in our calculations, we are going to write those parameters as lower indices of the symbol $\lesssim$. For example, if $A \leq C(p,q,r)B$, where

$C(p,q,r)$ is a constant that depends (only) on p,q and r, we will write $A \lesssim_{p,q,r} B$.

With this notation our goal will be to write down all the parameters that influence the value of the constant, but also to indirectly imply that the constant does not depend on other factors appearing in the calculations. This will be important for some estimates where the dependance on certain parameters might affect proofs in undesireable ways.

For the rest of this section let us observe a fixed measure space $(X, \mathscr{F}, \mu)$. Recall that the measure space is finite if $\mu(X) < \infty$ and it is a probability space if $\mu(X) = 1$. If not stated otherwise, definitions and results in this section, as well as some additional details, can be found in [21].

The next theorem is a well known result in real analysis theory, which we are going to use in two of its variants. One is commonly stated as follows.

Theorem 1.1.1. *(Jensen's inequality)* Let $(X, \mathscr{F}, \mu)$ be a probability space, $g : X \to \mathbb{R}$ an $\mathscr{F}$-measurable function such that $\int_X |g(x)| d\mu(x) < \infty$, and let $F : \mathbb{R} \to \mathbb{R}$ be convex on $[a,b]$, meaning $F(\lambda s + (1-\lambda)t) \leq \lambda F(s) + (1-\lambda)F(t)$ for each $s,t \in [a,b]$ and $\lambda \in [0,1]$. Then we have

$$F\left(\int_X g d\mu\right) \leq \int_X F \circ g d\mu.$$

In completely analogous way it can be shown that if F is a concave on $[a,b]$, meaning the reverse inequality $F(\lambda s + (1-\lambda)t) \geq \lambda F(s) + (1-\lambda)F(t)$ is valid for each $s,t \in [a,b]$ and $\lambda \in [0,1]$, then we have the reverse integral inequality $F\left(\int_X g d\mu\right) \geq \int F \circ g d\mu$.

Definition 1.1.2. If for a certain statement exists a set $E \in \mathscr{F}$ such that the statement is valid for all $x \in E$ and $\mu(E^c) = 0$, then we will say that the statement is true *almost everywhere*.

If $\mu(X) = 1$, then we will say that the stament is true *almost surely*.

One of the most fundamental functional spaces in real analysis is defined as follows. First, assume that $p \in \langle 0, \infty \rangle$ and denote

$$\|F\|_p := \left(\int_X |F|^p(x) d\mu(x)\right)^{\frac{1}{p}}$$

and

$$\|F\|_\infty := \inf\{t \in [0,\infty] : \mu(\{x \in X : |F(x)| > t\}) = 0\}.$$

We define *the Lebesgue space* as

$$\mathrm{L}^p(X) := \{F : X \to \mathbb{C} : F \text{ is measurable and } \|F\|_p < \infty\}.$$

It can be shown that $\mathrm{L}^p(X)$ is a vector space for each $p \in \langle 0, \infty]$; moreover, in cases $p \in [1, \infty]$ it is a normed vector space with additional identification that two functions are considered the same if they are identical μ-almost everywhere. Also, the value $\|F\|_\infty$ turns out to coincide with

$$\|F\|_\infty = \text{ess sup}_{x \in X} |F(x)|$$

where $\text{ess sup}_{x \in X}$ stands for supremum on X up to sets of μ-measure 0.

For each $p \in \langle 0, \infty\rangle$ we define the additional quantity

$$\|F\|_{p,\infty} := \left(\sup_{\alpha \in \langle 0, \infty\rangle} \alpha^p \mu(\{x \in X : |f(x)| > \alpha\}) \right)^{\frac{1}{p}}.$$

Now we define *the weak Lebesgue space* as

$$\mathrm{L}^{p,\infty}(X) := \{F : X \to \mathbb{C} : F \text{ is measurable and } \|F\|_{p,\infty} < \infty\}.$$

To be more precise and to emphasize the underlying set on which we observe each of these values, we introduce additional notations $\|F\|_{\mathrm{L}^p(X)} := \|F\|_p$ and $\|F\|_{\mathrm{L}^{p,\infty}(X)} := \|F\|_{p,\infty}$.

Another variant of such spaces that we will consider is known as *the weighted* L^p space along with *the weight* w, denoted as $\mathrm{L}^p(w)$ and defined as the standard L^p space according to the measure ν such that $d\nu = wd\lambda$, w being a nonnegative function and λ being the standard Lebesgue measure.

One of the most important inequalities in the measure theory is the well known Hölder's inequality which gives an L^1 estimate of the product of two functions. For the purposes of our proofs we will state its more general version.

Theorem 1.1.3. *(Generalized Hölder's inequality)* Let $n \in \mathbb{N}$, $r \in [1, \infty]$ and suppose that $p_j \in [1, \infty]$ for each $j \in \{1, \ldots, n\}$ such that $\sum_{j=1}^n \frac{1}{p_j} = \frac{1}{r}$. If $f_j \in \mathrm{L}^{p_j}(X)$ for each $j \in \{1, \ldots, n\}$, then $\prod_{j=1}^n f_j \in \mathrm{L}^r(X)$ and $\|\prod_{j=1}^n f_j\|_{\mathrm{L}^r(X)} \leq \prod_{j=1}^n \|f_j\|_{\mathrm{L}^{p_j}(X)}$.

The most common application of this theorem will be for $n = 2, r = 1$ and $p_1 = p_2 = 2$ which, in real analysis theory, is also known as *the Cauchy-Schwarz inequality*.

A direct consequence of Hölder's inequality is that the function $p \mapsto \mathrm{L}^p(X)$ on $[1, \infty]$ is decreasing with respect to the set inclusion if the initial measure space is finite.

Proposition 1.1.4. If $\mu(X) < \infty$ and $0 < p < q \leq \infty$ then $\mathrm{L}^q(X) \subseteq \mathrm{L}^p(X)$ and $\|f\|_{\mathrm{L}^p(X)} \leq \|f\|_{\mathrm{L}^q(X)} \mu(X)^{\frac{1}{p}-\frac{1}{q}}$.

Each $\mathscr{F}$-measurable function $f : X \to \mathbb{C}$ that takes finitely many values is called *a simple function*. Equivalently, f is a finite linear combination of indicator functions of sets from $\mathscr{F}$.

Lemma 1.1.5. (a) If $f : X \to [0,\infty]$ is measurable, there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of simple functions such that $0 \leq \varphi_n \leq \varphi_{n+1} \leq f$ for each $n \in \mathbb{N}$, $\lim_{n\to\infty} \varphi_n = f$ pointwise and $\lim_{n\to\infty} \varphi_n = f$ uniformly on any set on which f is bounded.

(b) If $f : X \to \mathbb{C}$ is measurable, there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of simple functions such that $0 \leq |\varphi_n| \leq |\varphi_{n+1}| \leq |f|$, $\lim_{n\to\infty} \varphi_n = f$ pointwise and $\lim_{n\to\infty} \varphi_n = f$ uniformly on any set on which f is bounded.

As a consequence, the family of all simple functions is dense in $\mathrm{L}^p(X)$, meaning that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of simple functions such that $\lim_{n\to\infty} \|f - \varphi_n\|_{\mathrm{L}^p(X)} = 0$.

One of the most interesting topics in harmonic analysis is obtaining L^p estimates for certain operators. Usually we try to make the range of $p \in [1,\infty]$ as large as possible for which we have boundedness of the operator, which may also give us the answers to some related interesting questions. The following theorem shows that it is enough to prove boundedness for two different pairs of exponents in order to derive the conclusion for all intermediate values.

Theorem 1.1.6. *(The Marcinkiewicz interpolation theorem)* Let $(X,\mathscr{M},\mu)$, $(Y,\mathscr{N},\nu)$ be measure spaces and $p_0, p_1, q_0, q_1 \in [1,\infty]$ such that $p_0 \leq q_0$, $p_1 \leq q_1$ and $q_0 \neq q_1$ and

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1} \quad \text{where } 0 < t < 1.$$

If T is a sublinear map from $\mathrm{L}^{p_0}(X) + \mathrm{L}^{p_1}(X)$ to the space of measurable functions on Y and if there exist constants $C_0, C_1 \in \langle 0,\infty\rangle$ such that for each measurable function f we have

$$\|Tf\|_{\mathrm{L}^{q_0,\infty}(X)} \leq C_0 \|f\|_{\mathrm{L}^{p_0}(X)} \quad \text{and} \quad \|Tf\|_{\mathrm{L}^{q_1,\infty}(X)} \leq C_1 \|f\|_{\mathrm{L}^{p_1}(X)}, \tag{1.1}$$

then there exists a constant $C_p \in \langle 0,\infty\rangle$ such that for each measurable function f we have

$$\|Tf\|_{\mathrm{L}^q(X)} \leq C_p \|f\|_{\mathrm{L}^p(X)}.$$

Another variant of this theorem is stated with the assumption that T is the linear operator and that we have $\|Tf\|_{\mathrm{L}^{q_0}(X)}$ and $\|Tf\|_{\mathrm{L}^{p_0}(X)}$ in place of $\|Tf\|_{\mathrm{L}^{q_0,\infty}(X)}$ and $\|Tf\|_{\mathrm{L}^{p_0,\infty}(X)}$ in (1.1), in which case it is also known as *the Riesz-Thorin theorem*.

A dyadic BMO-seminorm is defined as

$$\|F\|_{\mathrm{BMO}(\mathbb{R}^r)} := \sup_{Q\in\mathscr{C}_r}\left(\frac{1}{|Q|}\int_Q\left|F(x)-\frac{1}{|Q|}\int_Q F(y)dy\right|^2 dx\right)^{\frac{1}{2}}. \tag{1.2}$$

Here, $\mathscr{C}_r$ stands for a collection of dyadic cubes; see (1.15) for precise definition.

Before we give a useful proposition related to these seminorms, let us state a useful estimate that can be found in [57].

Lemma 1.1.7. *(The John-Nirenberg inequality)* There exist constants $C_1, C_2 \in \langle 0,\infty\rangle$ such that for each $t\in\langle 0,\infty\rangle$ and each dyadic cube $Q\in\mathscr{C}_r$ we have

$$|\{x\in Q : |f(x)-[f]_Q|>t\}| \le C_1 e^{-\frac{C_2 t}{\|f\|_{BMO}}}|Q|.$$

The previous lemma helps us giving a valuable characterization of the BMO-seminorm (which works not only for the dyadic variant that we consider here, but also in greater generality).

Proposition 1.1.8. For each $p\in[1,\infty\rangle$ we have

$$\|F\|_{\mathrm{BMO}(\mathbb{R}^r)} \sim_p \sup_{Q\in\mathscr{C}_r}\left(\frac{1}{|Q|}\int_Q\left|F(x)-\frac{1}{|Q|}\int_Q F(y)dy\right|^p dx\right)^{\frac{1}{p}}.$$

This means that the BMO-seminorm could be defined with any exponent p that we prefer. For the proofs to follow the best choice for the definition seems to be $p=2$ while in the literature it is also common to define the BMO space with $p=1$.

Proof. Let us denote

$$\|F\|_{\mathrm{BMO},p} := \sup_{Q\in\mathscr{C}_r}\left(\frac{1}{|Q|}\int_Q\left|F(x)-\frac{1}{|Q|}\int_Q F(y)dy\right|^p dx\right)^{\frac{1}{p}}$$

and fix $Q\in\mathscr{C}_r$. By Proposition 1.1.4 applied to the measure $Q\mapsto\frac{1}{|Q|}\lambda$, where λ is the Lebesgue measure,

$$\frac{1}{|Q|}\int_Q\left|F(x)-\frac{1}{|Q|}\int_Q F(y)dy\right|dx \le \left(\frac{1}{|Q|}\int_Q\left|F(x)-\frac{1}{|Q|}\int_Q F(y)dy\right|^p dx\right)^{\frac{1}{p}}.$$

Let us try to obtain some form of inverse inequality (up to a multiplicative constant). We are going to use the well known layer cake representation (which is an easy consequence of the Fubini-Tonelli theorem) which, for a general measure space $(X,\mathscr{F},\mu)$ and an $\mathscr{F}$-measurable function f gives us:

$$\|f\|_{\mathrm{L}^p(X)}^p = \int_0^\infty pt^{p-1}\mu(\{x\in X : |f(x)|>t\})dt.$$

Let us apply this to the function $x\mapsto F(x)-[F]_Q$ on a space of Lebesgue measure $(\mathbb{R}^r,\mathscr{B}(\mathbb{R}^r),\lambda)$ restricted to the set Q. Additionally, the application of Lemma 1.1.7 gives us

$$\begin{aligned}\int_Q |F(x)-[F]_Q|^p dx &= p\int_0^\infty t^{p-1}\,|\{x\in Q : |F(x)-[F]_Q\}|>t|\,dt\\ &\le C_1 p|Q|\int_0^\infty t^{p-1}e^{-\frac{C_2 t}{\|F\|_{\mathrm{BMO},1}}}dt.\end{aligned}$$

By substituting $s:=\frac{C_2 t}{\|F\|_{\mathrm{BMO},1}}$ we obtain

$$\frac{1}{|Q|}\int_Q |F(x)-[F]_Q|^p dx \le \left(\frac{C_1 p}{C_2^{p-1}}\int_0^\infty s^{p-1}e^{-s}ds\right)\|F\|_{\mathrm{BMO},1}^p.$$

The integral appearing inside the brackets on the right side of the inequality is convergent and can be recognized as the value of the Gamma function at p.

Finally, by taking the supremum over all $Q\in\mathscr{C}_r$ we obtain

$$\|F\|_{\mathrm{BMO},1}\le\|F\|_{\mathrm{BMO},p}\le C_p\|F\|_{\mathrm{BMO},1}$$

where $C_p := \left(\frac{C_1 p}{C_2^{p-1}}\int_0^\infty s^{p-1}e^{-s}ds\right)^{\frac{1}{p}}$. The statement of this Proposition follows by recognizing $\|F\|_{\mathrm{BMO},2}=\|F\|_{\mathrm{BMO}(\mathbb{R}^r)}$ and

$$\|F\|_{\mathrm{BMO}(\mathbb{R}^r)}\le C_2\|F\|_{\mathrm{BMO},1}\le C_2\|F\|_{\mathrm{BMO},p}\le C_2C_p\|F\|_{\mathrm{BMO},1}\le C_2C_p\|F\|_{\mathrm{BMO}(\mathbb{R}^r)}.$$

■

Let us close this section with definitions of a few less familiar objects that we are going to observe in this thesis.

The notion of sparse collections of cubes and the associated sparse forms was introduced by Lerner [40]; the reader can also compare the dyadic setting of Lerner and Nazarov [41]. Since we are dealing with multilinear forms, we will need the multilinear modification of the theory developed by Culiuc, Di Plinio, and Ou [8], so several major concepts and many ideas of proofs will be adapted from that paper.

Definition 1.1.9. For a fixed $c > 0$ we say that $\mathscr{S} \subseteq \mathscr{C}_r$ is *a sparse family* if it is a collection of dyadic cubes such that, for each $Q \in \mathscr{S}$, there exists a measurable set $E_Q \subseteq Q$ with the following properties:

- for each $Q \in \mathscr{S}$ we have $|E_Q| \geq c|Q|$,
- for each $Q, Q' \in \mathscr{S}, Q \neq Q'$, sets E_Q and $E_{Q'}$ are mutually disjoint.

A sparse (multisublinear) form associated with $\mathscr{S}$ is given by

$$\Theta_{\mathscr{S}}(\mathbf{F}) := \sum_{Q \in \mathscr{S}} |Q| \prod_{e \in E} [|F_e|^{d_e}]_Q^{\frac{1}{d_e}},$$

where $d_e \in [1, \infty\rangle$, $e \in E$ are some numbers, while $\mathbf{F} = (F_e)_{e \in E}$ is a tuple of measurable functions.

Intuitively, elements of sparse family are those dyadic cubes that are mutually disjoint in certain volumes that are large enough in terms of the Lebesgue measure (i.e., that take at least $100 \cdot c$ percent of the whole cube).

Once again, we merely adapt the trilinear setting from the paper [8] by Culiuc, Di Plinio, and Ou. Given the set E and a tuple of integers $\mathbf{d} = (d_e)_{e \in E}$, let $\mathbf{p} = (p_e)_{e \in E}$ be an arbitrary tuple of exponents from $[1, \infty]$ such that $p_e > d_e$ for each $e \in E$ and $\sum_{e \in E} \frac{1}{p_e} = 1$. Also, let $\mathbf{w} = (w_e)_{e \in E}$ be a tuple of strictly positive functions satisfying

$$\prod_{e \in E} w_e^{\frac{1}{p_e}} \equiv 1. \tag{1.3}$$

We will define *the multilinear Muckenhoupt constant* of the tuple $\mathbf{w}$ to be an expression

$$[\mathbf{w}]_{\mathbf{p},\mathbf{d}} := \sup_{Q \in \mathscr{C}_r} \prod_{e \in E} \left[w_e^{\frac{-d_e}{p_e - d_e}} \right]_Q^{\frac{1}{d_e} - \frac{1}{p_e}}.$$

1.2. PROBABILITY

In this section we are going to consider a fixed probability space, denoted as $(\Omega, \mathscr{F}, \mathbb{P})$. All of the definitions and results, as well as some of the comments in this section, with the exception of the last stated Theorem 1.2.13, are from [19].

Definition 1.2.1. Let X be a random variable such that $\mathbb{E}|X| < \infty$ and let $\mathscr{G} \subseteq \mathscr{F}$ be a σ-algebra. *A conditional expectation of X with respect to* $\mathscr{G}$ is any $\mathscr{G}$-measurable random variable Y such that we have

$$\mathbb{E}(X\mathbb{1}_A) = \mathbb{E}(Y\mathbb{1}_A) \quad \text{for each } A \in \mathscr{G}.$$

Such random variable exists and is unique $\mathbb{P}|_{\mathscr{G}}$-almost everywhere; $\mathbb{P}|_{\mathscr{G}}$ stands for a probability obtained by restricting $\mathbb{P}$ to σ-algebra $\mathscr{G}$. We introduce the notation $\mathbb{E}[X|\mathscr{G}]$ for such Y.

Notice from this definition that, if X is $\mathscr{G}$-measurable, then $\mathbb{E}(X|\mathscr{G}) = X$ $\mathbb{P}$-almost surely. In fact, we have a slightly more general statement.

Theorem 1.2.2. If X is $\mathscr{G}$-measurable and Y is any random variable such that $Y, XY \in \mathrm{L}^1(\Omega)$, then

$$\mathbb{E}(XY|\mathscr{G}) = X\mathbb{E}(Y|\mathscr{G}).$$

Definition 1.2.3. A sequence $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$ is *a filtration* if it is an increasing sequence of σ-algebras, meaning $\mathscr{G}_n \subseteq \mathscr{G}_{n+1}$ for each $n \in \mathbb{N}_0$.

A sequence $X = (X_n)_{n\in\mathbb{N}_0}$ of random variables is *a martingale* if for each $n \in \mathbb{N}_0$ we have $\mathbb{E}|X_n| < \infty$, X_n is $\mathscr{G}_n$-measurable and

$$\mathbb{E}(X_{n+1}|\mathscr{G}_n) = X_n.$$

If the sign "$=$" in this equation is replaced with "$\leq$" or "$\geq$", then X is called, respectively, *supermartingale* or *submartingale*.

Intuitively, the best prediction of value $X_{n+1}(\omega)$ for each $\omega \in \Omega$ (the value of the martingale at the following moment $n+1$), considering information that we have at moment n (which is what $\mathscr{G}_n$ contains) would be to go for the value $X_n(\omega)$ (the value at the current moment n).

In a similar way we can look at martingales reversed in time, which motivates the following similar definition.

Definition 1.2.4. A sequence $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$ is *a backward filtration* if it is a decreasing sequence of σ-algebras, meaning $\mathscr{G}_n \supseteq \mathscr{G}_{n+1}$ for each $n \in \mathbb{N}_0$, and such that $\mathscr{G}_0 = \mathscr{F}$.

A sequence $X = (X_n)_{n\in\mathbb{N}_0}$ of random variables is *a backward martingale* if for each $n \in \mathbb{N}_0$ we have $\mathbb{E}|X_n| < \infty$, X_n is $\mathscr{G}_n$-measurable and

$$\mathbb{E}(X_n|\mathscr{G}_{n+1}) = X_{n+1}.$$

If the sign "$=$" in this equation is replaced with "$\leq$" or "$\geq$", then X is called, respectively, *backward supermartingale* or *backward submartingale*.

In order to emphasise which of these two objects we are observing, sometimes we will refer to the objects from Definition 1.2.3 as *the forward filtration* and *the forward martingale*. The most common example of both forward and backward martingale (depending on whether $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$ is a forward or a backward filtration) and also the one we are going to observe in this dissertation is $(\mathbb{E}(f|\mathscr{G}_n))_{n\in\mathbb{N}_0}$ for any function $f \in \mathrm{L}^p(\Omega)$, where $p \in [1,\infty\rangle$. It can be shown that every L^p-bounded backward martingale arises this way.

Proposition 1.2.5. (a) Let $(X_n)_{n\in\mathbb{N}_0}$ be a (forward) martingale. For each $i, j \in \mathbb{N}_0$, $j < i$ we have

$$\mathbb{E}\left(\mathbb{E}(f|\mathscr{G}_i)|\mathscr{G}_j\right) = \mathbb{E}\left(\mathbb{E}(f|\mathscr{G}_j)|\mathscr{G}_i\right) = \mathbb{E}(f|\mathscr{G}_j).$$

(b) Let $(X_n)_{n\in\mathbb{N}_0}$ be a backward martingale. For each $i, j \in \mathbb{N}_0$, $j < i$ we have

$$\mathbb{E}\left(\mathbb{E}(f|\mathscr{G}_i)|\mathscr{G}_j\right) = \mathbb{E}\left(\mathbb{E}(f|\mathscr{G}_j)|\mathscr{G}_i\right) = \mathbb{E}(f|\mathscr{G}_i).$$

Take any $n, m \in \mathbb{N}_0$ such that $n > m$. From Definition 1.2.3 and Proposition 1.2.5, since X_m is $\mathscr{G}_m$-measurable, we can notice that

$$\mathbb{E}(X_n|\mathscr{G}_m) = \mathbb{E}(\mathbb{E}(X_n|\mathscr{G}_{n-1})|\mathscr{G}_m) = \mathbb{E}(X_{n-1}|\mathscr{G}_m).$$

We can continue lowering the index of a random variable inside conditional expectation by repeating this procedure $n - m - 1$ times in order to get

$$\mathbb{E}(X_n|\mathscr{G}_m) = \mathbb{E}(X_m|\mathscr{G}_m) = X_m.$$

It is also well known that the conditional expectation is a contraction in $\mathrm{L}^p(\Omega)$. The proof easily follows by using Jensen's inequality for conditional expectation.

Theorem 1.2.6. Let $p \in [1,\infty]$. We have

$$\|\mathbb{E}(g|\mathscr{F})\|_{\mathrm{L}^p(\Omega)} \leq \|g\|_{\mathrm{L}^p(\Omega)}.$$

Notice that, for any $p \in [1,\infty]$, $n \in \mathbb{N}_0$ and a (forward) martingale $(X_n)_{n\in\mathbb{N}_0}$

$$\|X_n\|_{\mathrm{L}^p(\Omega)} = \|\mathbb{E}(X_{n+1}|\mathscr{G}_n)\|_{\mathrm{L}^p(\Omega)} \leq \|X_{n+1}\|_{\mathrm{L}^p(\Omega)},$$

from which follows that $(\|X_n\|_{\mathrm{L}^p(\Omega)})_{n\in\mathbb{N}_0}$ is an increasing sequence of nonnegative real numbers.

We are going to use the following two theorems on martingale convergence.

Theorem 1.2.7. *(Martingale convergence theorem)* If $(X_n)_{n\in\mathbb{N}}$ is a submartingale with $\sup_{n\in\mathbb{N}} \mathbb{E}X_n^+ < \infty$, then $(X_n)_{n\in\mathbb{N}_0}$ converges almost surely to a limit X with $\mathbb{E}|X| < \infty$.

Consequently, if $(X_n)_{n\in\mathbb{N}}$ is a nonnegative supermartingale, the sequence converges almost surely.

Theorem 1.2.8. *(Dominated convergence theorem for conditional expectations)* Suppose X_n, X and Z, $n \in \mathbb{N}$, are random variables such that $\lim_{n\to\infty} X_n = X$ $\mathbb{P}$-almost surely and $|X_n| \leq Z$ for all $n \in \mathbb{N}_0$ where $\mathbb{E}Z < \infty$. Let $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$ be a (forward) filtration and denote $\mathscr{G}_\infty := \sigma(\cup_{n=0}^\infty \mathscr{G}_n)$. We have

$$\lim_{n\to\infty} \mathbb{E}(X_n|\mathscr{G}_n) = \mathbb{E}(X|\mathscr{G}_\infty) \quad \mathbb{P}-\text{almost surely.}$$

We will need a theorem of convergence similar to the previous one, for backward filtration.

Theorem 1.2.9. Let X be a random variable and let $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$ be a backward filtration. Denote $\mathscr{G}_\infty := \cap_{n=0}^\infty \mathscr{G}_n$. We have

$$\lim_{n\to\infty} \mathbb{E}(X|\mathscr{G}_n) = \mathbb{E}(X|\mathscr{G}_\infty) \quad \mathbb{P}-\text{almost surely and in } \mathrm{L}^1(\Omega).$$

Because of Proposition 1.1.4 we can conclude that the convergence in Theorem 1.2.9 is also valid in $\mathrm{L}^p(\Omega)$ for each $p \in [1,\infty]$.

Definition 1.2.10. Let $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$ be a (forward) filtration. *A stopping time* is any random variable $N:\Omega\to\mathbb{N}_0\cup\{\infty\}$ such that $\{N=n\}\in\mathscr{F}_n$ for each $n\in\mathbb{N}_0$.

For any sequence $(X_n)_{n\in\mathbb{N}_0}$ we define X_N as the random variable such that

$$X_N = X_n \quad \text{on } \{N=n\}, \quad n\in\mathbb{N}_0.$$

Additionally, if $\mathbb{P}(N=\infty)>0$ and if $\lim_{n\to\infty} X_n$ exists almost surely on $\{N=\infty\}$, then we define $X_N=\lim_{n\to\infty} X_n$ on $\{N=\infty\}$.

It is easy to verify that, by replacing the condition $\{N=n\}\in\mathscr{F}_n$ in the definition of the stopping time with $\{N\le n\}\in\mathscr{F}_n$, we obtain the equivalent definition.

Theorem 1.2.11. If $(X_n)_{n\in\mathbb{N}_0}$ is a martingale, then for any stopping time N the sequence $(X_{\min\{N,n\}})_{n\in\mathbb{N}_0}$ is also a martingale with respect to the same filtration.

Theorem 1.2.12. If $(X_n)_{n\in\mathbb{N}_0}$ is a martingale such that $X_n\in\mathrm{L}^p(\Omega)$ for each $n\in\mathbb{N}_0$, then a martingale $(X_{\min\{N,n\}})_{n\in\mathbb{N}_0}$ also satisfies $X_{\min\{N,n\}}\in\mathrm{L}^p(\Omega)$ for each $n\in\mathbb{N}_0$ and we have

$$\|X_{\min\{N,n\}}\|_{\mathrm{L}^p(\Omega)} \le \|X_n\|_{\mathrm{L}^p(\Omega)}.$$

The next theorem is a result from [24]. We are going to use a statement and present a proof that can be found in [25].

Theorem 1.2.13. *(The Gundy decomposition)* Let $f=(f_n)_{n\in\mathbb{N}_0}$ be a martingale in $\mathrm{L}^1(\Omega)$ on a probability space $(\Omega,\mathscr{F},\mathbb{P})$. For every $\alpha>0$ there exists a decomposition

$$f = g+b+h,$$

where g, b and h are martingales with respect to the same filtration, such that

$$g_0=f_0, \quad b_0=h_0\equiv 0,$$

and the following estimates hold for all $n\in\mathbb{N}_0$:

$$\|g_n\|_{\mathrm{L}^\infty(\Omega)}\le 2\alpha, \quad \|g_n\|_{\mathrm{L}^1(\Omega)}\le 4\|f_n\|_{\mathrm{L}^1(\Omega)}, \tag{1.4}$$

$$\mathbb{P}(\max_{0\le m\le n}|b_m|>0)\le 3\alpha^{-1}\|f_n\|_{\mathrm{L}^1(\Omega)}, \tag{1.5}$$

$$\sum_{m=0}^{n-1}\|h_{m+1}-h_m\|_{\mathrm{L}^1(\Omega)}\le 4\|f_n\|_{\mathrm{L}^1(\Omega)}. \tag{1.6}$$

Proof. We will first prove the theorem in the case when $|f_0| < \alpha$ $\mathbb{P}$-almost everywhere. Let us define stopping times

$$T := \min\{n \in \mathbb{N}_0 : |f_n| > \alpha\},$$
$$S := \min\left\{n \in \mathbb{N}_0 : \sum_{k=1}^{n+1} \mathbb{E}(\mathbb{1}_{\{T=k\}}|f_k - f_{k-1}||\mathscr{G}_{k-1}) > \alpha\right\}.$$

Note that $T \geq 1$. Now, let

$$g_0 := f_0, \quad g_n := f_{\min\{n,S,T-1\}} + \sum_{k=1}^{\min\{n,S\}} \mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k - f_{k-1})|\mathscr{G}_{k-1}), \qquad n \in \mathbb{N},$$
$$b_n := f_n - f_{\min\{n,S,T\}}, \qquad n \in \mathbb{N}_0,$$
$$h_n := \mathbb{1}_{\{T\leq\min\{n,S\}\}}(f_T - f_{T-1}) - \sum_{k=1}^{\min\{n,S\}} \mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k - f_{k-1})|\mathscr{G}_{k-1}), \qquad n \in \mathbb{N}_0.$$

We can readily check that $b_0 = h_0 \equiv 0$ $\mathbb{P}$-almost everywhere. Also, for each $n \in \mathbb{N}$,

$$g_n + b_n + h_n = f_n + \mathbb{1}_{\{T\leq\min\{n,S\}\}}(f_T - f_{T-1}) - (f_{\min\{n,S,T\}} - f_{\min\{n,S,T-1\}}) = f_n.$$

We also need to confirm that these three constructed sequences are indeed martingales. We have

$$\begin{aligned}
\mathbb{E}(g_{n+1}|\mathscr{G}_n) &= \mathbb{E}\left(f_{\min\{n+1,S,T-1\}} + \sum_{k=1}^{\min\{n+1,S\}} \mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k - f_{k-1})|\mathscr{G}_{k-1})\middle|\mathscr{G}_n\right)\\
&= \mathbb{E}\Bigg(f_0 + \sum_{k=1}^{n+1} \mathbb{1}_{\{k\leq\min\{S,T-1\}\}}(f_k - f_{k-1}) + \sum_{k=1}^{\min\{n,S\}} \mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k - f_{k-1})|\mathscr{G}_{k-1})\\
&\qquad + \mathbb{1}_{\{S\geq n+1\}}\mathbb{E}(\mathbb{1}_{\{T=n+1\}}(f_{n+1} - f_n)|\mathscr{G}_n)\Bigg|\mathscr{G}_n\Bigg)\\
&= f_0 + \sum_{k=1}^{n+1} \mathbb{E}(\mathbb{1}_{\{k\leq S\}}\mathbb{1}_{\{k+1\leq T\}}(f_k - f_{k-1})|\mathscr{G}_n)\\
&\qquad + \mathbb{E}\left(\sum_{k=1}^{n} \mathbb{1}_{\{k\leq S\}}\mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k - f_{k-1})|\mathscr{G}_{k-1})\middle|\mathscr{G}_n\right)\\
&\qquad + \mathbb{E}(\mathbb{1}_{\{S\geq n+1\}}\mathbb{E}(\mathbb{1}_{\{T=n+1\}}(f_{n+1} - f_n)|\mathscr{G}_n)|\mathscr{G}_n)\\
&= f_0 + \sum_{k=1}^{n} \mathbb{1}_{\{k\leq S\}}\mathbb{1}_{\{k+1\leq T\}}\mathbb{E}(f_k - f_{k-1}|\mathscr{G}_n) + \mathbb{1}_{\{n+1\leq S\}}\mathbb{E}(\mathbb{1}_{\{n+2\leq T\}}(f_{n+1} - f_n)|\mathscr{G}_n)\\
&\qquad + \sum_{k=1}^{n} \mathbb{1}_{\{k\leq S\}}\mathbb{E}(\mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k - f_{k-1})|\mathscr{G}_{k-1})|\mathscr{G}_n)\\
&\qquad + \mathbb{1}_{\{S\geq n+1\}}\mathbb{E}(\mathbb{1}_{\{T\geq n+1\}}(f_{n+1} - f_n)|\mathscr{G}_n) - \mathbb{1}_{\{S\geq n+1\}}\mathbb{E}(\mathbb{1}_{\{T>n+1\}}(f_{n+1} - f_n)|\mathscr{G}_n)
\end{aligned}$$

$$
\begin{aligned}
=&f_0+\sum_{k=1}^{n}\mathbb{1}_{\{k\le\min\{S,T-1\}\}}(f_k-f_{k-1})+\mathbb{1}_{\{n+1\le S\}}\mathbb{E}(\mathbb{1}_{\{n+2\le T\}}(f_{n+1}-f_n)|\mathscr{G}_n)\\
&+\sum_{k=1}^{n}\mathbb{1}_{\{k\le S\}}\mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k-f_{k-1})|\mathscr{G}_{k-1})+\mathbb{1}_{\{S\ge n+1\}}\mathbb{1}_{\{T\ge n+1\}}(\mathbb{E}(f_{n+1}|\mathscr{G}_n)-f_n)\\
&-\mathbb{1}_{\{S\ge n+1\}}\mathbb{E}(\mathbb{1}_{\{T\ge n+2\}}(f_{n+1}-f_n)|\mathscr{G}_n)\\
=&f_{\min\{n,S,T-1\}}+\sum_{k=1}^{\min\{n,S\}}\mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k-f_{k-1})|\mathscr{G}_{k-1})=g_n.
\end{aligned}
$$

We have applied Theorem 1.2.2 on several occasions: in fourth equality, since $\{k\le S\}=\{S>k-1\}=\{S\le k-1\}^c\in\mathscr{G}_{k-1}\subseteq\mathscr{G}_k$ for each $k\in\{1,\dots,n+1\}$ and, analogously, $\{k+1\le T\}\in\mathscr{G}_k\subseteq\mathscr{G}_n$ for each $k\in\{1,\dots,n\}$, and again in fifth equality as $\{T\ge n+1\}\in\mathscr{G}_n$. Notice that the obtained inequality is also valid for $n=0$. This means that $(g_n)_{n\in\mathbb{N}_0}$ is indeed a martingale with respect to the filtration $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$. Now we can easily verify that the sequences $(b_n)_{n\in\mathbb{N}_0}$ and $(h_n)_{n\in\mathbb{N}_0}$ are martingales, as well. Indeed, directly from the Definition 1.2.3 we can verify that the sum and the difference of two martingales is also a martingale, which, along with Theorem 1.2.11 proves that $(b_n)_{n\in\mathbb{N}_0}$ is a martingale. Same conclusion follows for $(h_n)_{n\in\mathbb{N}_0}$ as $h_n=f_n-g_n-b_n$ for each $n\in\mathbb{N}_0$.

Now let us show the required estimates in stated order. We can readily check that $\|g_0\|_{\mathrm{L}^\infty(\Omega)}=\|f_0\|_{\mathrm{L}^\infty(\Omega)}<4\alpha$ and $\|g_0\|_{\mathrm{L}^1(\Omega)}=\|f_0\|_{\mathrm{L}^1(\Omega)}\le 4\|f_0\|_{\mathrm{L}^1(\Omega)}$. Notice that, by Theorem 1.2.6,

$$
\begin{aligned}
\|g_n\|_{\mathrm{L}^\infty(\Omega)}\le\|f_{\min\{n,S,T-1\}}\|_{\mathrm{L}^\infty(\Omega)}&+\left\|\sum_{k=1}^{\min\{n,S\}}\mathbb{E}(\mathbb{1}_{\{T=k\}}|f_k-f_{k-1}||\mathscr{G}_{k-1})\right\|_{\mathrm{L}^\infty(\Omega)}\\
\le\alpha+\alpha=2\alpha.
\end{aligned}
$$

The last inequality follows from the definitions of T and S, since they present the smallest number $n\in\mathbb{N}_0$ for which these inequalities are valid (given $\omega\in\Omega$). In order to estimate $\mathrm{L}^1(\Omega)$ norm of g_n, first note that

$$
\begin{aligned}
\|f_{\min\{n,S,T-1\}}\|_{\mathrm{L}^1(\Omega)}&\le\|\mathbb{1}_{\{T>n\}}f_{\min\{n,S,T-1\}}\|_{\mathrm{L}^1(\Omega)}+\|\mathbb{1}_{\{T\le n\}}f_{\min\{n,S,T-1\}}\|_{\mathrm{L}^1(\Omega)}\\
&\le\|f_{\min\{n,S,T-1\}}\|_{\mathrm{L}^1(\Omega)}+\|\mathbb{1}_{\{T\le n\}}\alpha\|_{\mathrm{L}^1(\Omega)}\le\|f_n\|_{\mathrm{L}^1(\Omega)}+\alpha\mathbb{P}(T\le n)\\
&=\|f_n\|_{\mathrm{L}^1(\Omega)}+\alpha\mathbb{P}(\max_{0\le m\le n}|f_m|>\alpha)\le 2\|f_n\|_{\mathrm{L}^1(\Omega)}.
\end{aligned}
$$

In the second row we used the definition of T and, after that, Theorem 1.2.12. In the last inequality we applied Doob's inequality, stated as Theorem 1.5.7. Also,

$$
\begin{aligned}
&\Big\|\sum_{k=1}^{\min\{n,S\}}\mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k-f_{k-1})|\mathscr{G}_{k-1})\Big\|_{\mathrm{L}^1(\Omega)} \le \int_\Omega \sum_{k=1}^{\min\{n,S\}}|\mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k-f_{k-1})|\mathscr{G}_{k-1})|d\mathbb{P}\\
&\le \int_\Omega \sum_{k=1}^{n}|\mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k-f_{k-1})|\mathscr{G}_{k-1})|d\mathbb{P} = \sum_{k=1}^{n}\|\mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k-f_{k-1})|\mathscr{G}_{k-1})\|_{\mathrm{L}^1(\Omega)}\\
&\le \sum_{k=1}^{n}\|\mathbb{1}_{\{T=k\}}(f_k-f_{k-1})\|_{\mathrm{L}^1(\Omega)} = \sum_{k=1}^{n}\|\mathbb{1}_{\{T=k\}}(f_T-f_{T-1})\|_{\mathrm{L}^1(\Omega)}\\
&= \|\mathbb{1}_{\{T\le n\}}(f_T-f_{T-1})\|_{\mathrm{L}^1(\Omega)} \le \|\mathbb{1}_{\{T\le n\}}f_T\|_{\mathrm{L}^1(\Omega)} + \|\mathbb{1}_{\{T\le n\}}f_{T-1}\|_{\mathrm{L}^1(\Omega)}\\
&\le \|f_{\min\{T,n\}}\|_{\mathrm{L}^1(\Omega)} + \alpha < 2\|f_n\|_{\mathrm{L}^1(\Omega)}.
\end{aligned}
$$

The inequality in the third row follows from Theorem 1.2.6 while the first inequality in the last row follows from Theorem 1.2.12. Notice that, comparing third and last row, we have also shown

$$
\sum_{k=1}^{n}\|\mathbb{1}_{\{T=k\}}|f_T-f_{T-1}|\|_{\mathrm{L}^1(\Omega)} \le 2\|f_n\|_{\mathrm{L}^1(\Omega)}, \tag{1.7}
$$

which we are going to need again in the rest of the proof. Overall,

$$
\begin{aligned}
\|g_n\|_{\mathrm{L}^1(\Omega)} &\le \|f_{\min\{n,S,T-1\}}\|_{\mathrm{L}^1(\Omega)} + \Big\|\sum_{k=1}^{\min\{n,S\}}\mathbb{E}(\mathbb{1}_{\{T=k\}}(f_k-f_{k-1})|\mathscr{G}_{k-1})\Big\|_{\mathrm{L}^1(\Omega)}\\
&\le 2\|f_n\|_{\mathrm{L}^1(\Omega)} + 2\|f_n\|_{\mathrm{L}^1(\Omega)} = 4\|f_n\|_{\mathrm{L}^1(\Omega)}.
\end{aligned}
$$

This completes the proof of (1.4). Notice that, for each $n\in\mathbb{N}_0$ and by the definition of b_n,

$$
\{b_n\neq 0\}\subseteq\{n\neq\min\{n,S,T\}\} = \{\min\{S,T\}<n\} = \{S<n\}\cup\{T<n\}.
$$

Let us estimate the $\mathbb{P}$-measures of these two sets. We have

$$
\begin{aligned}
\mathbb{P}(S<n) &= \mathbb{P}\Big(\sum_{k=1}^{n}\mathbb{E}(\mathbb{1}_{\{T=k\}}|f_k-f_{k-1}||\mathscr{G}_{k-1}) > \alpha\Big)\\
&\le \alpha^{-1}\Big\|\sum_{k=1}^{n}\mathbb{E}(\mathbb{1}_{\{T=k\}}|f_k-f_{k-1}||\mathscr{G}_{k-1})\Big\|_{\mathrm{L}^1(\Omega)}\\
&\le \alpha^{-1}\sum_{k=1}^{n}\|\mathbb{1}_{\{T=k\}}|f_k-f_{k-1}|\|_{\mathrm{L}^1(\Omega)} \le 2\alpha^{-1}\|f_n\|_{\mathrm{L}^1(\Omega)},
\end{aligned}
$$

by Theorem 1.2.6 and (1.7). Also,

$$
\mathbb{P}(T<n) = \mathbb{P}(\max_{0\le m\le n-1}|f_m|>\alpha) \le \alpha^{-1}\|f_{n-1}\|_{\mathrm{L}^1(\Omega)} \le \alpha^{-1}\|f_n\|_{\mathrm{L}^1(\Omega)},
$$

by Theorem 1.5.7 applied to the martingale $(\mathbb{E}(f_{n-1}|\mathscr{G}_m))_{m\in\mathbb{N}_0}$; recall that $\mathbb{E}(f_{n-1}|\mathscr{G}_m) = f_{n-1}$ for each $m \geq n-1$ and $\mathbb{E}(f_{n-1}|\mathscr{G}_m) = f_m$ otherwise. Therefore,

$$\begin{aligned}\mathbb{P}(\max_{0\leq m\leq n}|b_m|>0) &= \mathbb{P}(\cup_{m=0}^{n}\{|b_m|>0\}) \leq \mathbb{P}(\cup_{m=0}^{n}\{m\neq\min\{m,S,T\}\})\\ &\leq \mathbb{P}(n\neq\min\{n,S,T\}) \leq \mathbb{P}(\{S<m\}\cup\{T<m\})\\ &\leq \mathbb{P}(S<n)+\mathbb{P}(T<n) \leq 3\alpha^{-1}\|f_n\|_{\mathrm{L}^1(\Omega)},\end{aligned}$$

which shows (1.5). Now, for $m\in\mathbb{N}_0$ we have

$$\begin{aligned}h_{m+1}-h_m &= \mathbb{1}_{\{T=m+1<S\}}(f_T-f_{T-1}) - \mathbb{1}_{\{S\geq m+1\}}\mathbb{E}(\mathbb{1}_{\{T=m+1\}}(f_{m+1}-f_m)|\mathscr{G}_m)\\ &\leq \mathbb{1}_{\{T=m+1\}}\mathbb{1}_{\{m+1<S\}}(f_{m+1}-f_m) - \mathbb{1}_{\{S\geq m+1\}}\mathbb{E}(\mathbb{1}_{\{T=m+1\}}(f_{m+1}-f_m)|\mathscr{G}_m)\\ &= \mathbb{1}_{\{S\geq m+1\}}(\mathbb{1}_{\{T=m+1\}}(f_{m+1}-f_m) - \mathbb{E}(\mathbb{1}_{\{T=m+1\}}(f_{m+1}-f_m)|\mathscr{G}_m)).\end{aligned}$$

By this and by Theorem 1.2.6 and (1.7),

$$\begin{aligned}\sum_{m=0}^{n-1}\|h_{m+1}-h_m\|_{\mathrm{L}^1(\Omega)} &\leq \sum_{m=0}^{n-1}\|\mathbb{1}_{\{T=m+1\}}(f_{m+1}-f_m)\|_{\mathrm{L}^1(\Omega)}\\ &\quad+\sum_{m=0}^{n-1}\|\mathbb{E}(\mathbb{1}_{\{T=m+1\}}(f_{m+1}-f_m)|\mathscr{G}_m)\|_{\mathrm{L}^1(\Omega)}\\ &\leq 2\sum_{m=0}^{n-1}\|\mathbb{1}_{\{T=m+1\}}(f_{m+1}-f_m)\|_{\mathrm{L}^1(\Omega)} \leq 4\|f_n\|_{\mathrm{L}^1(\Omega)}.\end{aligned}$$

By showing (1.6) the proof of this theorem is complete, however, with the additional assumption that $|f_0|<\alpha$ $\mathbb{P}$-almost everywhere. In general case let $A := \{|f_0|<\alpha\}$. Note that $A\in\mathscr{G}_n$ for each $n\in\mathbb{N}_0$, so, by Theorem 1.2.2,

$$\mathbb{E}(f_{n+1}\mathbb{1}_{A^c}|\mathscr{G}_n) = \mathbb{1}_{A^c}\mathbb{E}(f_{n+1}|\mathscr{G}_n) = \mathbb{1}_{A^c}f_n, \tag{1.8}$$

which shows that $(f_n\mathbb{1}_{A^c})_{n\in\mathbb{N}_0}$ is also a martingale in respect to the filtration $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$. Moreover, this martingale belongs to the case for which we already obtained the statement of this theorem, so there exists its decomposition $(g_n^0)_{n\in\mathbb{N}_0}$, $(b_n^0)_{n\in\mathbb{N}_0}$ and $(h_n^0)_{n\in\mathbb{N}_0}$ with analogous properties. We claim that the required composition for the starting martingale $(f_n^0)_{n\in\mathbb{N}_0}$ is given with

$$g_n := g_n^0,\quad b_n := b_n^0 + f_n\mathbb{1}_A,\quad h_n := h_n^0,\quad n\in\mathbb{N}_0.$$

We can directly see that $(g_n)_{n\in\mathbb{N}_0}$ and $(h_n)_{n\in\mathbb{N}_0}$ are martingales with respect to $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$, but so is $(b_n)_{n\in\mathbb{N}_0}$ as sum of two martingales $(b_n^0)_{n\in\mathbb{N}_0}$ and $(f_n\mathbb{1}_A)_{n\in\mathbb{N}_0}$; the fact that the latter sequence is also a martingale follows from the analogous identities as in (1.8). Since $\|f_n\mathbb{1}_{A^c}\|_{L^1(\Omega)} \leq \|f_n\|_{L^1(\Omega)}$, we can immediately see that (1.4) and (1.6) are satisfied. As $(f_m\mathbb{1}_A)_{m\in\mathbb{N}_0}$ is a martingale, we have $\int_\Omega |f_0|\mathbb{1}_A d\mathbb{P} \leq \int_\Omega |f_n|\mathbb{1}_A d\mathbb{P}$, so

$$\begin{aligned}
&\mathbb{P}(\max_{0\leq m\leq n}|b_m|>0) \leq \mathbb{P}(\max_{0\leq m\leq n}|b_m^0|>0)+\mathbb{P}(\max_{0\leq m\leq n}|f_m|\mathbb{1}_A>0)\\
&\leq 3\alpha^{-1}\|f_n\mathbb{1}_{A^c}\|_{L^1(\Omega)}+\mathbb{P}(\mathbb{1}_A>0) = 3\alpha^{-1}\|f_n\mathbb{1}_{A^c}\|_{L^1(\Omega)}+\mathbb{P}(A)\\
&\leq 3\alpha^{-1}\int_\Omega |f_n|\mathbb{1}_{A^c}d\mathbb{P}+\alpha^{-1}\int_\Omega |f_0|\mathbb{1}_A d\mathbb{P} \leq 3\alpha^{-1}\int_\Omega |f_n|\mathbb{1}_{A^c}d\mathbb{P}+\alpha^{-1}\int_\Omega |f_n|\mathbb{1}_A d\mathbb{P}\\
&\leq 3\alpha^{-1}\int_\Omega |f_n|d\mathbb{P} = 3\alpha^{-1}\|f_n\|_{L^1(\Omega)}.
\end{aligned}$$

Therefore, (1.5) is valid and so is the statement of this theorem. ∎

The previous theorem gives a decomposition of a martingale into three martingales which intuitively present, in order, good, bad and harmless part. The first part can be considered good as it belongs to $L^p(\Omega)$ for each $p \in [1,\infty]$, which is a property we would hope for from the martingale that we started with (the fact that we cannot guarantee that is a reason to perform the decomposition). The bad part of the martingale cannot affect the starting martingale much because the measure of its support is not large. In the end, there is the third part remaining, which we cannot bound in $L^p(\Omega)$ and the support of which we cannot estimate, but still we have the last inequality guaranteeing the control of its total variation in the L^1 norm; therefore it is still harmless in a certain way.

1.3. ERGODIC THEORY

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. For $n \in \mathbb{N}_0$ and for a complex $\mathscr{F}$-measurable function f we define *a Cesàro average* A_n with respect to the iterates of a $(\mathscr{F}, \mathscr{F})$-measurable transformation $T : \Omega \to \Omega$ as

$$A_n f := \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i \tag{1.9}$$

and additionally $A_0 f \equiv 0$. In this section as well as throughout Chapter 3 we will be interested in the transformation T such that it *preserves measure* $\mathbb{P}$, meaning that T is $(\mathscr{F}, \mathscr{F})$-measurable and that for each $A \in \mathscr{F}$ we have $\mathbb{P}(T^{-1}(A)) = \mathbb{P}(A)$.

One of the biggest and most interesting results in the ergodic theory is the convergence of sequence $(A_n f)_{n \in \mathbb{N}_0}$, $\mathbb{P}$-almost surely and in $\mathrm{L}^p(\Omega)$ for $p \in [1, \infty\rangle$. On the way to showing this result, one can observe a maximal operator which, for $f \in \mathrm{L}^1(\Omega)$, is defined as

$$M_C f := \sup_{n \in \mathbb{N}} |A_n f|.$$

In Section 1.5 we are going to observe some other maximal operators and deduce their (strong or weak) L^p estimates. Here we are going to present the same result for the operator M_C, in order to give an additional illustration to the ideas that we plan to use in these thesis, such as the Marcinkiewicz's interpolation theorem and an idea from [6] known as a Calderón's transference principle.

Theorem 1.3.1. For each $p \in \langle 1, \infty]$ we have

$$\|M_C f\|_{\mathrm{L}^{1,\infty}(\Omega)} \lesssim \|f\|_{\mathrm{L}^1(\Omega)} \quad \text{and} \quad \|M_C f\|_{\mathrm{L}^p(\Omega)} \lesssim_p \|f\|_{\mathrm{L}^p(\Omega)}.$$

Proof. Trivially,

$$\|A_n f\|_{\mathrm{L}^\infty(\Omega)} \leq \frac{1}{n} \sum_{i=0}^{n-1} \|f \circ T^i\|_{\mathrm{L}^\infty(\Omega)} = \|f\|_{\mathrm{L}^\infty(\Omega)},$$

as T is $\mathbb{P}$-measure invariant, so $\|f \circ T^i\|_{\mathrm{L}^\infty(\Omega)} = \|f\|_{\mathrm{L}^\infty(\Omega)}$ for each $i \in \mathbb{N}_0$. Therefore,

$$\|M_C f\|_{\mathrm{L}^\infty(\Omega)} \leq \|f\|_{\mathrm{L}^\infty(\Omega)}.$$

If we obtain the inequality $\|M_C f\|_{\mathrm{L}^{1,\infty}(\Omega)} \lesssim \|f\|_{\mathrm{L}^1(\Omega)}$ then the claim of this theorem will follow from Theorem 1.1.6.

Let $g:\mathbb{Z}\to\mathbb{R}$ and $n\in\mathbb{N}$ be arbitrary. For $k\in\mathbb{Z}$ define discrete averages $\tilde{A}_n$ as

$$\tilde{A}_n g(k) := \frac{1}{n}\sum_{i=0}^{n-1} g(k+i).$$

Furthermore, let $h:\mathbb{R}\to\mathbb{R}$ be a function given with $h:=\sum_{k\in\mathbb{Z}} g(k)\mathbb{1}_{[k,k+1\rangle}$. Note that, if $\sum_{k\in\mathbb{Z}}|g(k)|<\infty$, by Lebesgue domination theorem,

$$\|h\|_{\mathrm{L}^1(\mathbb{R})} = \int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}|g(k)|\mathbb{1}_{[k,k+1\rangle}(x)d\lambda(x) = \sum_{k\in\mathbb{Z}}|g(k)|\int_{[k,k+1\rangle}d\lambda(x) = \|g\|_{\ell^1(\mathbb{Z})}, \tag{1.10}$$

where λ is the Lebesgue measure on $\mathbb{R}$ and $\ell^1(\mathbb{Z})$ is the Lebesgue space $\mathrm{L}^1(\mathbb{Z})$ equipped with counting measure. Similarly,

$$\frac{1}{n}\int_{[k,k+n\rangle}h(x)d\lambda(x) = \frac{1}{n}\sum_{i=0}^{n-1}g(k+i) = \tilde{A}_n g(k). \tag{1.11}$$

Let us introduce *the uncentered Hardy-Littlewood maximal operator M_{nc}* with

$$M_{\mathrm{nc}}f(x) := \sup_{\substack{y'\in\mathbb{R}, r\in\langle 0,\infty\rangle\\ x\in B(y',r)}}[|f|]_{B(y',r)} = \sup_{\substack{y'\in\mathbb{R}, r\in\langle 0,\infty\rangle\\ x\in B(y',r)}}\frac{1}{|B(y',r)|}\int_{B(y',r)}|f(y)|dy \quad \text{for } x\in\mathbb{R}^r,$$

where $B(y',r):=\{y\in\mathbb{R}:|y-y'|<r\}=\langle y'-r,y'+r\rangle$ is an open ball in $\mathbb{R}$ (given Euclidean metric) with center y' and radius r. The reader can compare this operator with the one defined in (1.17), with slight difference being the collection of sets over which the supremum is taken (or, from a different point of view, the only difference being the metric on space $\mathbb{R}$). With a similar proof as in Theorem 1.5.1 (see details in [57]) we can show that

$$\|M_{\mathrm{nc}}h\|_{\mathrm{L}^{1,\infty}(\mathbb{R})} \lesssim \alpha^{-1}\|h\|_{\mathrm{L}^1(\mathbb{R})}. \tag{1.12}$$

Additionally, note the following. Take any $\alpha\in\langle 0,\infty\rangle$ and assume that $n\in\mathbb{N}_0$ and $k\in\mathbb{Z}$ is such that $|\tilde{A}_n g(k)|>\alpha$. By (1.11) we see that $M_{\mathrm{nc}}h(x)>\alpha$ for each $x\in\langle k,k+1\rangle$ (by taking the ball $\langle k,k+n\rangle$ in the supremum that defines M_{nc}). With this, we can notice that

$$\mathrm{card}(\{k\in\mathbb{Z}:\sup_{n\in\mathbb{N}}|\tilde{A}_n g(k)|>\alpha\}) \le \lambda(\{M_{\mathrm{nc}}h>\alpha\}).$$

By (1.12) and (1.10),

$$\mathrm{card}(\{k\in\mathbb{Z}:\sup_{n\in\mathbb{N}}|\tilde{A}_n g(k)|>\alpha\}) \lesssim \alpha^{-1}\|h\|_{\mathrm{L}^1(\mathbb{R})} = \alpha^{-1}\|g\|_{\ell^1(\mathbb{Z})}. \tag{1.13}$$

Now, for $\omega \in \Omega$ and $N \in \mathbb{N}$ let us define $g_{\omega,N} : \mathbb{Z} \to \mathbb{R}$ along the trajectory of ω with

$$g_{\omega,N}(n) := f(T^n\omega)\mathbb{1}_{\{0,1,\dots,2N-1\}}(n), \quad n \in \mathbb{Z}. \tag{1.14}$$

Note that, for $n \in \{1,2,\dots,N\}$ and $k \in \{0,1,\dots,N-1\}$,

$$A_n f(T^k\omega) = \frac{1}{n}\sum_{i=0}^{n-1} f(T^{i+k}\omega) = \frac{1}{n}\sum_{i=0}^{n-1} g_{\omega,N}(k+i) = \tilde{A}_n g_{\omega,N}(k).$$

Now, by $\mathbb{P}$-invariance of T, Fubini-Tonelli theorem, (1.13) and (1.14) we conclude that

$$\begin{aligned}
\mathbb{P}(\{\max_{n\in\{1,2,\dots,N\}} |A_n f| > \alpha\}) &= \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{P}(\{\omega\in\Omega : \max_{n\in\{1,2,\dots,N\}} |A_n f(T^k\omega)| > \alpha\}) \\
&= \frac{1}{N}\int_\Omega \operatorname{card}(\{k\in\{0,1,\dots,N-1\} : \max_{n\in\{1,2,\dots,N\}} |\tilde{A}_n g_{\omega,N}(k)| > \alpha\}) d\mathbb{P}(\omega) \\
&\lesssim \frac{1}{N}\int_\Omega \alpha^{-1}\|g_{\omega,N}\|_{\ell^1(\mathbb{Z})} d\mathbb{P}(\omega) = \alpha^{-1}\cdot\frac{1}{N}\sum_{i=0}^{2N-1}\int_\Omega |f(T^n\omega)| d\mathbb{P}(\omega) = 2\alpha^{-1}\|f\|_{\mathrm{L}^1(\Omega)}.
\end{aligned}$$

With this, the statement of this theorem follows. ■

The following two corrolaries are the results on convergence of the sequence of Cesàro averages, proven by Birkhoff in [4] and by von Neumann in [43]. Interestingly, the limit turns out to be the conditional expectation $\mathbb{E}(f|\mathscr{C})$ where $\mathscr{C}$ is σ-algebra on X consisting of all T*-invariant sets*; in other words, sets $A \in \mathscr{C}$ for which $T^{-1}(A) = A$ $\mathbb{P}$-almost surely; details can be found in [19].

Corollary 1.3.2. *(Birkhoff's pointwise ergodic theorem)* For each $f \in \mathrm{L}^1(\Omega)$ the sequence $(A_n f)_{n\in\mathbb{N}_0}$ converges $\mathbb{P}$-almost surely.

Corollary 1.3.3. *(von Neumann's mean ergodic theorem)* For each $p \in [1,\infty\rangle$ and $f \in \mathrm{L}^p(\Omega)$ the sequence $(A_n f)_{n\in\mathbb{N}_0}$ converges in $\mathrm{L}^p(\Omega)$.

A similarly sounding problem is the convergence of the sequence $(B_n(f,g))_{n\in\mathbb{N}_0}$ of bilinear ergodic averages

$$B_n(f,g) := \frac{1}{N}\sum_{i=0}^{n-1}(f\circ T^n)(g\circ S^n),$$

where S and T are commuting measure-preserving transformations on $(\Omega,\mathscr{F},\mathbb{P})$ and $f,g \in \mathrm{L}^\infty(\Omega)$. This, however, is still pretty much an open question. The L^2-convergence was proven by Conze and Lesigne in [7]. Another proof of this result, in a more quantitative manner, can be found in [16]. The proof from that article reduces estimates for B_n to bounds for paraproduct-like operators similar to those defined in (3.1) and (3.2). In Chapter 3 we will be interested in L^p convergence of those two paraproducts while the convergence $\mathbb{P}$-almost surely remains open; because of the similarities with proofs from the latter mentioned article, it is reasonable to believe that these two problems have similar answers.

1.4. DYADIC SETTING

For $r \in \mathbb{N}$, we define *the set of dyadic cubes in* $\mathbb{R}^r$ as

$$\mathscr{C}_r := \left\{ \prod_{i=1}^{r} \left[2^k l_i, 2^k (l_i+1) \right\rangle : k, l_i \in \mathbb{Z}, i = 1, 2, \dots, r \right\}. \tag{1.15}$$

The elements of $\mathscr{C}_r$ will usually be denoted as $I_1 \times I_2 \times \cdots \times I_r = \prod_{i=1}^r I_i$, with $I_1, \dots, I_r \in \mathscr{C}_1$. For $I \in \mathscr{C}_1$, let

$$\mathbb{h}_I^0 := \frac{1}{|I|} \mathbb{1}_I, \;\; \mathbb{h}_I^1 := \frac{1}{|I|} \left(\mathbb{1}_{I_L} - \mathbb{1}_{I_R} \right), \tag{1.16}$$

where I_L and I_R are, in order, left and right halves of the interval I; more precisely, if $I = [a, b\rangle$ for some $a, b \in \mathbb{R}$, then $I_L := \left[a, \frac{a+b}{2}\right\rangle$ and $I_R := \left[\frac{a+b}{2}, b\right\rangle$. Function h_I^0 is simply the L^1-normalized characteristic function of I, while h_I^1 is the so-called *Haar function*, normalized in the L^1 sense, as well. We will also call these, in order, *non-cancellative* and *cancellative* Haar functions.

For each $r \in \mathbb{N}$ and $Q \in \mathscr{C}_r$ there exist exactly 2^r disjoint cubes $Q_1, \dots, Q_{2^r} \in \mathscr{C}_r$ such that $|Q_1| = \cdots = |Q_{2^r}| = 2^{-r}|Q|$ and $Q_1, \dots, Q_{2^r} \subseteq Q$. These $Q_i, i \in \{1, \dots, 2^r\}$ are called the *children* of Q, while Q is the *parent* of $Q_1, \dots, Q_{2^r}$. The family of children of a cube Q will be denoted as $\mathscr{C}(Q)$.

Definition 1.4.1. Let $r \in \mathbb{N}$. A *tree* is a family $\mathscr{T} \subseteq \mathscr{C}_r$ for which there exists $Q_{\mathscr{T}} \in \mathscr{T}$ such that $Q \subseteq Q_{\mathscr{T}}$ for every $Q \in \mathscr{T}$; such $Q_{\mathscr{T}}$ is called a *root* of the tree $\mathscr{T}$.

A tree $\mathscr{T}$ is called *convex* if for every $Q_1, Q_3 \in \mathscr{T}$ and $Q_2 \in \mathscr{C}_r$ the inclusion $Q_1 \subseteq Q_2 \subseteq Q_3$ implies $Q_2 \in \mathscr{T}$.

A *leaf* of the tree $\mathscr{T}$ is any $Q \in \mathscr{C}_r \backslash \mathscr{T}$ with the parent $Q' \in \mathscr{T}$. A family of these cubes will be marked as $\mathscr{L}(\mathscr{T})$.

If we consider families of dyadic subcubes of $[0,1\rangle$ of side lengths between 2^{-k} and 1, we get an interesting example of (forward) filtration.

Definition 1.4.2. *A dyadic filtration on* $[0,1\rangle^r$ is a (forward) filtration $(\mathscr{D}_k^r)_{i \in \mathbb{N}_0}$ given as

$$\mathscr{D}_k^r := \sigma\left(\left\{ \prod_{i=1}^{r} \left[2^{-k} l_i, 2^{-k} (l_i+1) \right\rangle : l_i \in \{0, \dots, 2^k - 1\}, i = 1, 2, \dots, r \right\} \right), \quad k \in \mathbb{N}_0.$$

For our proofs it will be enough to consider a dyadic filtration for $r = 1$ in which case we will write $\mathscr{D}_k = \mathscr{D}_k^r$ for $k \in \mathbb{N}_0$. From this definition we can see that this filtration has some practical properties that will also appear to be useful later. For example, each σ-algebra $\mathscr{D}_k^r$ is finitely generated and each of its atoms can be presented as a disjoint union of exactly 2^r atoms from $\mathscr{D}_{k+1}^r$; precisely,

$$\prod_{i=1}^{r}\left[2^{-k}l_i, 2^{-k}(l_i+1)\right\rangle = \bigcup_{(j_1,\dots,j_r)\in\{0,1\}^r}\prod_{i=1}^{r}\left[2^{-k-1}(2l_i+j_i), 2^{-k-1}(2l_i+j_i+1)\right\rangle.$$

1.5. MAXIMAL AND SQUARE FUNCTION OPERATORS

In this section we are going to define several variants of maximal function operators. As there are various results in harmonic analysis about these operators being bounded, it is a common trick to estimate a value of a certain expression with a specific maximal operator in order to obtain the required result. Another interesting thing is that the boundedness of those operators is sometimes equivalent with other interesting problems in harmonic analysis such as the convergence of certain sequence of functions; see Stein's maximal principle [55].

The most well known operator is *the Hardy-Littlewood maximal operator*, which maps a locally integrable function into a function defined by the supremum of all averages over a family of sets of our interest. We are interested in the dyadic variant of this operator, formally defined as

$$M_{\text{dyadic}}F(x) := \sup_{\substack{Q\in\mathscr{C}_r \\ x\in Q}} [|F|]_Q = \sup_{\substack{Q\in\mathscr{C}_r \\ x\in Q}} \frac{1}{|Q|}\int_Q |F(y)|dy \quad \text{for } x\in\mathbb{R}^r. \tag{1.17}$$

We can show the following.

Theorem 1.5.1. For each $p\in\langle 1,\infty]$ we have

$$\|M_{\text{dyadic}}F\|_{\mathrm{L}^p(\mathbb{R}^r)} \lesssim_p \|F\|_{\mathrm{L}^p(\mathbb{R}^r)}.$$

Proof. For each $Q\in\mathscr{C}_r$ we have $\frac{1}{|Q|}\int_Q |F(y)|dy \le \|F\|_{\mathrm{L}^\infty(\mathbb{R}^r)}$, therefore

$$\|M_{\text{dyadic}}F\|_{\mathrm{L}^\infty(\mathbb{R}^r)} \le \|F\|_{\mathrm{L}^\infty(\mathbb{R}^r)}.$$

If we show that

$$\|M_{\text{dyadic}}F\|_{\mathrm{L}^{1,\infty}(\mathbb{R}^r)} \lesssim \|F\|_{\mathrm{L}^1(\mathbb{R}^r)},$$

then we can apply Theorem 1.1.6 in order to complete the proof of the theorem. Let $\alpha\in\langle 0,\infty\rangle$, $E_\alpha := \{x\in\mathbb{R}^r : M_{\text{dyadic}}F(x) > \alpha\}$ and $x\in E_\alpha$. Then there exists a dyadic cube $x\in Q_x\in\mathscr{C}_r$ such that $[|F|]_{Q_x} > \alpha$. A collection $\{Q_x : x\in E_\alpha\}$ is contained in $\mathscr{C}_r$ which is countable, so there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in E_α such that $\{Q_x : x\in E_\alpha\} = \{Q_{x_n} :$

$n \in \mathbb{N}\}$. Moreover, we can take subsequence $(n_m)_{m\in\mathbb{N}}$ such that $\cup_{n\in\mathbb{N}} Q_{x_n} = \cup_{m\in\mathbb{N}} Q_{x_{n_m}}$ and that $Q_{x_{n_m}}, m \in \mathbb{N}$ are mutually disjoint. Indeed, we can take those points x_{n_m} so that $\{Q_{x_{n_m}} : m \in \mathbb{N}\}$ is a family of maximal cubes in $\{Q_{x_n} : n \in \mathbb{N}\}$ - those dyadic cubes that are not subset of any other cube in that collection. As we know, if two dyadic cubes are not disjoint, then one is contained in the other. This ensures that the maximal cubes are indeed disjoint and also that they contain all of the cubes from the collection. Now we can estimate

$$\begin{aligned} |E_\alpha| \le |\cup_{x\in E_\alpha} Q_x| = \left|\cup_{m\in\mathbb{N}} Q_{x_{n_m}}\right| = \sum_{m\in\mathbb{N}} \left|Q_{x_{n_m}}\right| \le \sum_{m\in\mathbb{N}} \frac{1}{\alpha} \int_{Q_{x_{n_m}}} |F(y)|dy \\ = \frac{1}{\alpha} \int_{\cup_{m\in\mathbb{N}} Q_{x_{n_m}}} |F(y)|dy \le \frac{1}{\alpha} \int_{\mathbb{R}^r} |F(y)|dy. \end{aligned}$$

Multiplying this inequality with α and taking supremum over all $\alpha \in \langle 0, \infty\rangle$ gives us the required inequality. ■

Let w be a measurable strictly positive function on $\mathbb{R}^r$. We will also consider a generalized version of Hardy-Littlewood maximal operator, known as *weighted maximal operator* and defined as

$$M_w F(x) := \sup_{\substack{Q\in\mathscr{C}_r \\ x\in Q}} \frac{[|F|w]_Q}{[w]_Q} \quad \text{for } x \in \mathbb{R}^r.$$

This operator is also bounded; the result can be found in [42].

Theorem 1.5.2. For each $p \in \langle 1, \infty]$ we have

$$\|M_w F\|_{\mathrm{L}^p(w)} \lesssim_p \|F\|_{\mathrm{L}^p(w)}.$$

It is worth noticing that this estimate is actually a Doob's martingale inequality in disguise. Indeed, if the measure ν is the measure corresponding to the weighted L^p space $\mathrm{L}^p(w)$ is probabilistic and if $(\mathscr{F}_m)_{m\in\mathbb{N}_0}$ is a dyadic filtration on the same probabilistic space, then the estimate from Theorem 1.5.2 can actually be rewritten as the one in Theorem 1.5.7.

Now we are going to mention several operators expressed as ℓ^2 quantities made of differences of various sequences. Intuitively, they sum up the jumps between neighbouring elements of the sequence.

The first operator we are going to mention is

$$f \mapsto \Big(\sum_{i=1}^{\infty} |A_{N_{i+1}}f - A_{N_i}f|^2\Big)^{1/2},$$

for an arbitrary strictly increasing sequence of positive integers $(N_i)_{i\in\mathbb{N}}$. We will state the L^p boundedness covered by two papers. The first bound was shown by Jones, Ostrovskii, and Rosenblatt [29, Theorem 2.6] for $p \in \langle 1,2]$. After that, Jones, Kaufman, Rosenblatt, and Wierdl [28, Theorem 4.6 for A_1] extended the result for the range $p \in \langle 2,\infty\rangle$.

Theorem 1.5.3. For each $p \in \langle 1,\infty\rangle$ we have

$$\sup_{\substack{N_i\in\mathbb{N} \text{ for } i\in\mathbb{N} \\ (N_i)_{i\in\mathbb{N}} \text{ is strictly increasing}}} \Big\| \Big(\sum_{i=1}^{\infty} |A_{N_{i+1}}f - A_{N_i}f|^2\Big)^{1/2} \Big\|_{\mathrm{L}^p(\Omega)} \lesssim_p \|f\|_{\mathrm{L}^p(\Omega)}.$$

Since T preserves measure $\mathbb{P}$, the statement of this theorem in case $p=2$ follows from the following proposition.

Proposition 1.5.4. Let U be a unitary operator on complex Hilbert space H. For $n\in\mathbb{N}$ we define

$$T_n v := \frac{1}{n}\sum_{k=0}^{n-1} U^k v, \quad v\in H.$$

We have

$$\Big(\sup_{\substack{M\in\mathbb{N} \\ 1\le n_0<\cdots<n_M}} \sum_{j=1}^{M} \|T_{n_j}v - T_{n_{j-1}}v\|_H^2\Big)^{\frac{1}{2}} \lesssim \|v\|_H, \quad v\in H. \tag{1.18}$$

Proof. First let us assume that $H=\mathbb{C}$, $v=1$ and $U:\mathbb{C}\to\mathbb{C}$ is given by $Uz = e^{i\theta}z$ for $\theta\in[0,2\pi\rangle$. In case of $\theta=0$ the operators U and T_n, $n\in\mathbb{N}$ are identity operators on H; therefore the left side of the inequality (1.18) equals zero, so the inequality is trivially valid. If $\theta\neq 0$, then $T_n 1 = \frac{1}{n}\sum_{k=0}^{n-1}(e^{i\theta})^k = \frac{1-e^{ni\theta}}{n(1-e^{i\theta})}$. The inequality (1.18) then turns into

$$\sup_{\substack{M,n_0,n_1,\ldots,n_M\in\mathbb{N} \\ n_0<n_1<\cdots<n_M}} \sum_{j=1}^{M} \left| \frac{1-e^{n_j i\theta}}{n_j(1-e^{i\theta})} - \frac{1-e^{n_{j-1} i\theta}}{n_{j-1}(1-e^{i\theta})} \right|^2 \lesssim 1. \tag{1.19}$$

Notice that the expression on the left side of the inequality is invariant under reflection $\theta\mapsto 2\pi-\theta$, so it is enough to observe the case $\theta\in\langle 0,\pi\rangle$. As $|1-e^{i\theta}|^2 = 4\sin^2\frac{\theta}{2} \geq 4\big(\frac{2}{\pi}\cdot\frac{\theta}{2}\big)^2 = \frac{4}{\pi^2}\theta^2$, it is enough to prove

$$\sup_{\substack{M,n_0,n_1,\ldots,n_M\in\mathbb{N} \\ n_0<n_1<\cdots<n_M}} \sum_{j=1}^{M} |F(n_j\theta) - F(n_{j-1}\theta)|^2 \lesssim 1$$

where $F(t) := \frac{1-e^{it}}{t}$. Take any $M \in \mathbb{N}$ and $n_0, \dots, n_M \in \mathbb{N}, n_0 < n_1 < \cdots < n_M$. Let

$$J_1 := \Big\{ j \in \{1, \dots, M\} : n_j - n_{j-1} < \frac{1}{\theta} \Big\} \text{ and}$$
$$J_2 := \Big\{ j \in \{1, \dots, M\} : n_j - n_{j-1} \geq \frac{1}{\theta} \Big\}.$$

As $F'(t) = \frac{-ite^{it}-1+e^{it}}{t^2}$, by Theorem 1.1.1 applied to the normalized Lebesgue measure on $\langle n_{j-1}\theta, n_j\theta \rangle$ and by definition of J_1,

$$\begin{aligned}
\sum_{j\in J_1} |F(n_j\theta) - F(n_{j-1}\theta)|^2 &= \sum_{j\in J_1} \left| \int_{n_{j-1}\theta}^{n_j\theta} F'(t)dt \right|^2 \leq \sum_{j\in J_1} (n_j - n_{j-1})\theta \int_{n_{j-1}\theta}^{n_j\theta} |F'(t)|^2 dt \\
&< \sum_{j\in J_1} \int_{n_{j-1}\theta}^{n_j\theta} |F'(t)|^2 dt \leq \int_0^\infty |F'(t)|^2 dt \\
&= \int_0^\infty \frac{(t\sin t - 1 + \cos t)^2 + (-t\cos t + \sin t)^2}{t^4} dt \\
&= \int_0^\infty \frac{t^2 + 2 - 2\cos t - 2t\sin t}{t^4} dt \lesssim \int_0^\infty \frac{\min\{t^4, t^2\}}{t^2} dt \lesssim 1.
\end{aligned}$$

Indeed, for $t \in \langle 0, 1 \rangle$ we have

$$t^2 + 2 - 2\cos t - 2t\sin t \leq t^2 + 2 - 2\Big(1 - \frac{1}{2}t^2 + \Big(\frac{1}{24} - \frac{1}{720}\Big)t^4\Big) - 2t\Big(t - \frac{1}{6}t^3\Big) = \frac{91}{360}t^4,$$

while for $t > 1$ we estimate

$$t^2 + 2 - 2\cos t - 2t\sin t \leq t^2 + 2 + 2 + 2t \leq 7t^2.$$

As for the second case, denote all elements of J_2 as $j_1, j_2, \dots, j_L$. We can see that, for each $l \in \{2, \dots, L\}$ we have $n_{j_l} \geq \frac{1}{\theta} + n_{j_l - 1} \geq \frac{1}{\theta} + n_{j_{l-1}}$ and, if we apply the same inequality for additional $l-2$ times, we get $n_{j_l} \geq \frac{l-1}{\theta} + n_{j_1} \geq \frac{l-1}{\theta} + \frac{1}{\theta} + n_{j_1 - 1} \geq \frac{l}{\theta}$. Additionally, note that $|F(t)| = \frac{2|\sin\frac{t}{2}|}{|t|} \leq \min\big\{\frac{1}{|t|}, 1\big\}$ for each $t \in \mathbb{R}\backslash\{0\}$. Now,

$$\begin{aligned}
\sum_{j\in J_2} |F(n_j\theta) - F(n_{j-1}\theta)|^2 &= \sum_{j\in J_2} (|F(n_j\theta)|^2 - 2|F(n_j\theta)||F(n_{j-1}\theta)| + |F(n_{j-1}\theta)|^2) \\
&\leq \sum_{j\in J_2} 2(|F(n_j\theta)|^2 + |F(n_{j-1}\theta)|^2) \\
&\leq 2\Big(\sum_{l=1}^{L} \Big(\frac{2}{n_{j_l}\theta}\Big)^2 + \sum_{l=2}^{L} \Big(\frac{2}{n_{j_l - 1}\theta}\Big)^2 + |F(n_0\theta)|^2 \Big) \\
&\leq 2\Big(\sum_{l=1}^{L} \frac{8}{l^2} + \sum_{l=2}^{L} \frac{8}{l^2} + 1 \Big) \lesssim 1.
\end{aligned}$$

This gives

$$\sum_{j=1}^{M}|F(n_j\theta)-F(n_{j-1}\theta)|^2=\sum_{j\in J_1}|F(n_j\theta)-F(n_{j-1}\theta)|^2+\sum_{j\in J_2}|F(n_j\theta)-F(n_{j-1}\theta)|^2\lesssim 1,$$

which is what we were required to prove.

To prove this proposition in general case, first we will mention some definitions and agreements from the spectral theory. In general, a projection-valued measure is a function P defined on a σ-algebra $\mathscr{F}$ on set H such that, for each $E\in\mathscr{F}$, $P(E)$ is a projection operator on space H and which also satisfies $P(X)=\mathbb{1}_H$ and $P(\cup_{n=1}^{\infty}E_n)=\sum_{n=1}^{\infty}P(E_n)$ for each pairwise-disjoint sets $E_n\in\mathscr{F}, n\in\mathbb{N}$. It is easy to notice that, for each $v_1,v_2\in H$ the mapping $E\mapsto\langle P(E)v_1,v_2\rangle_H$ is a complex measure. For $E\in\mathscr{F}$ we can denote

$$\int_E d\langle Pv_1,v_2\rangle_H(\alpha)=\langle P(E)v_1,v_2\rangle_H,$$

which can be generalized to

$$\int_E f(\alpha)d\langle Pv_1,v_2\rangle_H(\alpha)=\left\langle\left(\int_E f(\alpha)dP(\alpha)\right)v_1,v_2\right\rangle_H$$

where f is the $\mathscr{F}$-measurale function. Similarly, we can introduce the identification

$$\int_E f(\alpha)dP(\alpha)=f(P(E)).$$

Details can be found in [20].

Now we can apply the spectral theorem. Let $S^1:=\{\alpha\in\mathbb{C}:|\alpha|=1\}$. There exists a regular Borel projector-valued measure P on S^1 such that for any bounded Borel function f we have

$$f(U)=\int_{S^1}f(\alpha)dP(\alpha).$$

Specifically, for $f(\alpha)=\alpha^n$ and $f(\alpha)=\overline{\alpha}^n$, $n\in\mathbb{N}$, we have

$$U^n=\int_{S^1}\alpha^n dP(\alpha),(U^n)^*=\int_{S^1}\overline{\alpha^n}dP(\alpha).$$

Note that

$$\begin{aligned}C\|v\|_H^2-\|T_{n_{j-1}}v-T_{n_j}v\|_H^2&=\left\langle\left(100I-\sum_{j=1}^{M}(T_{n_{j-1}}-T_{n_j})^*(T_{n_{j-1}}-T_{n_j})\right)v,v\right\rangle_H\\&=\int_{S_1}\underbrace{\left(C-\sum_{j=1}^{M}\left|\frac{1}{n_{j-1}}\sum_{k=0}^{n_{j-1}-1}\alpha^k-\frac{1}{n_j}\sum_{k=0}^{n_j-1}\alpha^k\right|^2\right)}_{\geq 0}d\langle Pv,v\rangle_H(\alpha)\geq 0,\end{aligned}$$

where C is the constant that is implicit in (1.19) and the inequality follows from (1.19). The proof of inequality (1.18) and therefore the proof of this proposition is complete. ■

Another such operator is the martingale square function operator, defined as

$$F \mapsto \Big(\sum_{k=0}^{\infty} |\mathbb{E}(F|\mathscr{G}_{k+1}) - \mathbb{E}(F|\mathscr{G}_k)|^2 \Big)^{1/2}.$$

The following theorem is a result of Burkholder and can be found in [5].

Theorem 1.5.5. For each $q \in \langle 1, \infty \rangle$ we have

$$\Big\| \Big(\sum_{k=0}^{\infty} |\mathbb{E}(F|\mathscr{G}_{k+1}) - \mathbb{E}(F|\mathscr{G}_k)|^2 \Big)^{1/2} \Big\|_{\mathrm{L}^q(\Omega)} \lesssim_q \|F\|_{\mathrm{L}^q(\Omega)}.$$

Another useful estimate was shown by Jones, Kaufman, Rosenblatt, Wierdl in [28, Theorem C]. Here, $(\mathscr{D}_j)_{j \in \mathbb{N}_0}$ stands for the dyadic filtration on $[0,1\rangle$ from Definition 1.4.2 and $\|\cdot\|_{\mathrm{L}^p_x(\mathbb{R})}$ stands for the norm on the space $\mathrm{L}^p(\mathbb{R})$ taken over the variable x.

Theorem 1.5.6. For each $p \in \langle 1, \infty \rangle$ we have

$$\Big\| \Big(\sum_{j=0}^{\infty} \Big| 2^j \int_0^{2^{-j}} h(x+y)\,\mathrm{d}y - \mathbb{E}(h|\mathscr{D}_j)(x) \Big|^2 \Big)^{1/2} \Big\|_{\mathrm{L}^p_x(\mathbb{R})} \lesssim_p \|h\|_{\mathrm{L}^p(\mathbb{R})}.$$

The next result can be found as Theorem 6 in [56], but it is also present in any other book on basic probability theory.

Theorem 1.5.7. *(Doob's inequality)* For each $\alpha \in \langle 0, \infty \rangle$ and $p \in \langle 1, \infty]$ we have

$$\mathbb{P}\big(\sup_{m \in \mathbb{N}_0} |\mathbb{E}(F|\mathscr{F}_m)| > \alpha \big) \leq \frac{1}{\alpha} \|F\|_{\mathrm{L}^1(\Omega)},$$

$$\| \sup_{m \in \mathbb{N}_0} |\mathbb{E}(F|\mathscr{F}_m)| \|_{\mathrm{L}^p(\Omega)} \lesssim_p \|F\|_{\mathrm{L}^p(\Omega)}.$$

1.6. HYPERGRAPH SETTING

Definition 1.6.1. A *hypergraph* is an ordered pair (V,E), where V is a finite set of elements, which we call *vertices*, and E is a collection of nonempty subsets of V; the elements of E are called *edges*. Let $r \in \mathbb{N}$. A hypergraph (V,E) is called *r-partite* if there exists a partition of V into r nonempty parts $\left(V^{(i)}\right)_{1\le i\le r}$ such that one cannot find $i \in \{1,2,\dots,r\}$ and vertices $x,y \in V^{(i)}, x \neq y$ for which there would exist $e \in E$ such that $x,y \in e$. A hypergraph (V,E) is called *r-uniform* if each edge $e \in E$ has the cardinality $|e| = r$.

Notice that every edge e of an r-partite r-uniform graph (V,E) with an associated r-partition of the vertex set $V = \bigcup_{i=1}^{r} V^{(i)}$ satisfies $\operatorname{card}(e \cap V^{(i)}) = 1$ for every $i \in \{1,2,\dots, r\}$. In other words, each edge contains exactly one vertex from each of the vertex-partition parts. In this situation each edge can be identified with an element of $V^{(1)} \times V^{(2)} \times \cdots \times V^{(r)} = \prod_{i=1}^{r} V^{(i)}$.

Definition 1.6.2. A *labeled hypergraph* is any hypergraph (V,E) along with sets L_V and L_E, an injective function $l_V : V \to L_V$ and an arbitrary function $l_E : E \to L_E$. The elements of sets L_V and L_E will be called, in order, *vertex labels* and *edge labels*. Note that vertex labels are required to be different, but we allow the repetition of edge labels.

Given an r-partite hypergraph (V,E) and the corresponding partition $\left(V^{(i)}\right)_{1\le i\le r}$ of V, we will usually denote vertices as $V^{(i)} = \left\{v_1^{(i)}, v_2^{(i)}, \dots, v_{n_i}^{(i)}\right\}$ for each $i \in \{1,2,\dots,r\}$. Similarly, we will write $L_V := \cup_{i=1}^{r} L_V^{(i)}$ and $L_V^{(i)} := \{x_j^{(i)} : j \in \mathbb{N}\}$ for each $i \in \{1,\dots,r\}$; with this notation, we will assume that $l_V(V^{(i)}) \subseteq L_V^{(i)}$ for each $i \in \{1,\dots,r\}$.

With this, we will often denote $n_i := \operatorname{card}(V^{(i)})$ and $n := \sum_{i=1}^{r} n_i$. For shorter notation we may write $x_v := l_V(v)$ for each $v \in V$ and also

$$\mathbb{x} = \left(x_{v_1^{(1)}}, \dots, x_{v_{n_1}^{(1)}}, \dots, x_{v_1^{(r)}}, \dots, x_{v_{nr}^{(r)}}\right) \quad \text{and} \quad \mathbb{x}_e := \left(x_{v^{(1)}}, \dots, x_{v^{(r)}}\right),$$

for $e = (v^{(1)}, \dots, v^{(r)}) \in E$. The elements of sets L_V and L_E will be substituted with real variables and real-valued functions.

Let $H = (V,E)$ be a labeled hypergraph with the label functions l_V and l_E and the set of vertex labels marked as $L_V = \cup_{i=1}^{r} L_V^{(i)}$ such that, for each $i \in \{1,\dots,r\}$, $l_V\left(V^{(i)}\right) \subseteq L_V^{(i)}$. With a slight deviation from the previous notation, this time we will write $L_V^{(i)} = \{x_{j,k}^{(i)} :$

$j, k \in \mathbb{N}\}$. In some proofs we will square certain parts of paraproduct-type terms, making certain variables appear more than once. To keep the practical notation of the evaluation of the expression at certain graph, we are expanding the vertex label sets with "copies", i.e. as certain variable $x_j^{(i)}$ can appear more than once (but at most $n_i = \mathrm{card}(V^{(i)})$ times), so we will mark its copies with $x_{j,k}^{(i)}, k \in \mathbb{N}$. It will also be practical to denote $L_j^{(i)} := \{x_{j,k}^{(i)} : k \in \mathbb{N}\}$.

Let us introduce a requirement on l_V and l_E so that they produce "properly" labelled hypergraphs, i.e., those that will appear later. Take any label variable $x_{i_k,j_k}^{(k)}$ for $k \in \{1,\dots,r\}, i_k, j_k \in \mathbb{N}$. As l_V is an injective function, whenever $x_{i_k,j_k}^{(k)} \in \mathrm{Im}(l_V)$ we can define $v_{i_k,j_k}^{(k)} := l_V^{-1}(x_{i_k,j_k}^{(k)})$. For the set of edge labels we choose $L_E := \{F_{i_1,\dots,i_r} : i_1,\dots,i_r \in \mathbb{N}\}$. With this notation, we will require the following condition to be satisfied:

$$l_E((v_{i_1,j_1}^{(1)},\dots,v_{i_r,j_r}^{(r)})) = F_{i_1,\dots,i_r} \tag{1.20}$$

for each choice of indices i_k and j_k, $k = 1,\dots,r$. This means that any two edges with the same first lower indices of their vertices receive the same label from the set L_E. Otherwise, the two edges receive different labels. To make this agreement clearer and easier to understand, an example of the 3-partite 3-uniform labeled hypergraph is given as a Figure 1.1.

Additionally, we will restrict our attention to hypergraphs that are "proper" in the sense that we are about to define. When two variables x_{v_1} and x_{v_2} have the same first lower indices (better said, when they appear to be the copies of the same variable), we will require that their vertices v_1 and v_2 play identical roles in the hypergraph, i.e. interchanging them leads to the hypergraph isomorphic to the original one. To be precise, for every $v_1, v_2 \in V$ we require

$$\begin{aligned} &l_V(v_1), l_V(v_2) \in L_j^{(i)} \text{ for some } i \in \{1,\dots,r\}, j \in \mathbb{N} \\ &\qquad \Longrightarrow \text{ for each } e \in E \text{ we have } (v_1 \in e \implies (e \backslash \{v_1\}) \cup \{v_2\} \in E). \end{aligned} \tag{1.21}$$

Notice that this property is trivially satisfied in case of the complete hypergraph.

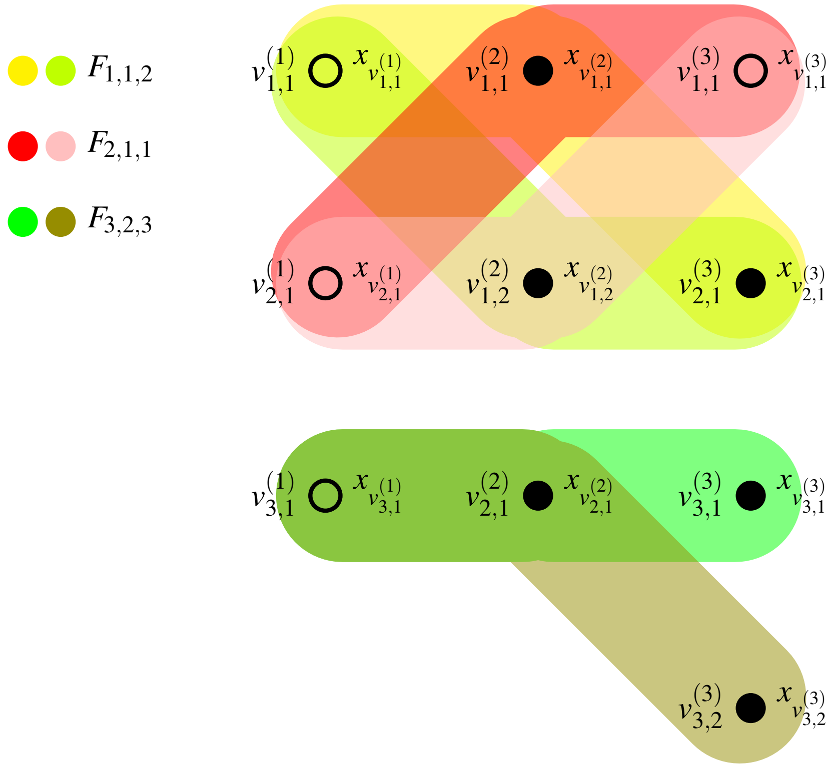

Figure 1.1: An example of the 3-partite 3-uniform labeled hypergraph. Empty circles present selected vertices.

1.7. REMOVAL LEMMAS

Hypergraph removal lemmas are general results of the following type: if a (hyper)graph contains a small portion of sub(hyper)graphs isomorphic to a given pattern, then it is possible to remove a small portion of its edges in order to make this pattern disappear completely. The most general hypergraph removal result was shown by Gowers [22] and many alternative proofs followed, for instance the one by Tao [59]. We will give one possible formulation a bit later, as Lemma 1.7.8. An expository paper by Tao [60] shows that, in fact, graph/hypergraph removal results belong to the realm of mathematical analysis (rather than combinatorics) as both their formulations and their proofs carry over to $[0,1]$ with the Lebesgue measure. Obtaining "reasonable" bounds in graph/hypergraph removal results are some of the greatest and the most important problems in combinatorics and graph theory today, with applications to theoretical computer science; see the paper by Alon [2]. Many particular cases have been studied in the recent literature, but almost all of them were concerned with graphs and not with more general hypergraphs.

In this work we do not prove any new removal results. However, their formulations and the ideas around their proofs will be quite relevant in the next chapter, so we spend a few words on them here. Although not obvious at first, techniques used in Chapter 2 were inspired by Tao's proof of removal lemma. They turned out to be a direct inspiration for Kovač in his work [34], while the proofs in Chapter 2 generalize concepts from that article.

Many elements of proof of the hypergraph removal lemma are actually a single-scale variants of arguments from Chapter 2. The best example is so-called *box-Gowers-Cauchy-Schwarz* inequality. We are going to observe techniques from [60] applied to the space $([0,1],\mathscr{B}([0,1]),\lambda)$, where λ is the Lebesgue measure on $[0,1]$; additionally, $\lambda^{(2)}$ is the Lebesgue measure on $[0,1]^2$.

Definition 1.7.1. *The box-Gowers norm* of a real-valued function $f \in \mathrm{L}^\infty([0,1]^2, \mathscr{B}([0,1]^2),\lambda^{(2)})$ is defined as

$$\|f\|_{\square^2} := \Big(\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}\int_{[0,1]} f(x,y)f(x,y')f(x',y)f(x',y')dydy'dxdx'\Big)^{\frac{1}{4}}.$$

The box-Gowers inner product of real-valued functions $f_{00}, f_{01}, f_{10}, f_{11} \in \mathrm{L}^\infty([0,1]^2,$

$\mathscr{B}([0,1]^2), \lambda^{(2)})$ is given by

$$\begin{aligned}
&[f_{00}, f_{01}, f_{10}, f_{11}]_{\square^2} \\
&\qquad := \int_{[0,1]} \int_{[0,1]} \int_{[0,1]} \int_{[0,1]} f_{00}(x,y) f_{01}(x,y') f_{10}(x',y) f_{11}(x',y') dy dy' dx dx'.
\end{aligned}$$

Theorem 1.7.2. *(The box-Gowers-Cauchy-Schwarz inequality)* We have

$$\left|[f_{00}, f_{01}, f_{10}, f_{11}]_{\square^2}\right| \leq \|f_{00}\|_{\square^2} \|f_{01}\|_{\square^2} \|f_{10}\|_{\square^2} \|f_{11}\|_{\square^2}.$$

Proof. By Theorem 1.1.3 and by Fubini-Tonelli theorem we have

$$\begin{aligned}
&\left| \int_{[0,1]} \int_{[0,1]} \int_{[0,1]} \int_{[0,1]} f_{00}(x,y) f_{01}(x,y') f_{10}(x',y) f_{11}(x',y') dy dy' dx dx' \right| \\
&\quad \leq \int_{[0,1]} \int_{[0,1]} \left| \int_{[0,1]} f_{00}(x,y) f_{01}(x,y') dx \right| \cdot \left| \int_{[0,1]} f_{10}(x',y) f_{11}(x',y') dx' \right| dy dy' \\
&\quad \leq \left(\int_{[0,1]} \int_{[0,1]} \left(\int_{[0,1]} f_{00}(x,y) f_{01}(x,y') dx \right)^2 dy dy' \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{[0,1]} \int_{[0,1]} \left(\int_{[0,1]} f_{10}(x',y) f_{11}(x',y') dx' \right)^2 dy dy' \right)^{\frac{1}{2}} \\
&\quad = \left(\int_{[0,1]} \int_{[0,1]} \left(\int_{[0,1]} f_{00}(x,y) f_{01}(x,y') dx \right) \left(\int_{[0,1]} f_{00}(x',y) f_{01}(x',y') dx' \right) dy dy' \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{[0,1]} \int_{[0,1]} \left(\int_{[0,1]} f_{10}(x',y) f_{11}(x',y') dx' \right) \left(\int_{[0,1]} f_{10}(x,y) f_{11}(x,y') dx \right) dy dy' \right)^{\frac{1}{2}} \\
&\quad = \left(\int_{[0,1]} \int_{[0,1]} \left(\int_{[0,1]} f_{00}(x,y) f_{00}(x',y) dy \right) \right. \\
&\qquad\qquad\qquad \left. \cdot \left(\int_{[0,1]} f_{01}(x,y') f_{01}(x',y') dy' \right) dx dx' \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{[0,1]} \int_{[0,1]} \left(\int_{[0,1]} f_{10}(x',y) f_{10}(x,y) dy \right) \right. \\
&\qquad\qquad\qquad \left. \cdot \left(\int_{[0,1]} f_{11}(x',y') f_{11}(x,y') dy' \right) dx dx' \right)^{\frac{1}{2}} \\
&\quad \leq \left(\int_{[0,1]} \int_{[0,1]} \left(\int_{[0,1]} f_{00}(x,y) f_{00}(x',y) dy \right)^2 dx dx' \right)^{\frac{1}{4}} \\
&\qquad \cdot \left(\int_{[0,1]} \int_{[0,1]} \left(\int_{[0,1]} f_{01}(x,y') f_{01}(x',y') dy' \right)^2 dx dx' \right)^{\frac{1}{4}} \\
&\qquad \cdot \left(\int_{[0,1]} \int_{[0,1]} \left(\int_{[0,1]} f_{10}(x',y) f_{10}(x,y) dy \right)^2 dx dx' \right)^{\frac{1}{4}}
\end{aligned}$$

$$\cdot\left(\int_{[0,1]}\int_{[0,1]}\left(\int_{[0,1]}f_{11}(x',y')f_{11}(x,y')dy'\right)^2 dxdx'\right)^{\frac{1}{4}}$$
$$=\left(\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}f_{00}(x,y)f_{00}(x',y)f_{00}(x,y')f_{00}(x',y')dydy'dxdx'\right)^{\frac{1}{4}}$$
$$\cdot\left(\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}f_{01}(x,y)f_{01}(x',y)f_{01}(x,y')f_{01}(x',y')dydy'dxdx'\right)^{\frac{1}{4}}$$
$$\cdot\left(\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}f_{10}(x,y)f_{10}(x',y)f_{10}(x,y')f_{10}(x',y')dydy'dxdx'\right)^{\frac{1}{4}}$$
$$\cdot\left(\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}f_{11}(x,y)f_{11}(x',y)f_{11}(x,y')f_{11}(x',y')dydy'dxdx'\right)^{\frac{1}{4}}$$
$$=\|f_{00}\|_{\square^2}\|f_{01}\|_{\square^2}\|f_{10}\|_{\square^2}\|f_{11}\|_{\square^2}.$$

This gives us the desired inequality. ■

To notice the connection between this inequality and the one from Lemma 2.2.4 it is good to observe that, by Theorem 1.1.3,

$$\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}f(x,y)f(x,y')f(x',y)f(x',y')dydy'dxdx'$$
$$\leq\left(\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}f(x,y)^2f(x',y')^2dydy'dxdx'\right)^{\frac{1}{2}}$$
$$\cdot\left(\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}f(x,y')^2f(x',y)^2dydy'dxdx'\right)^{\frac{1}{2}}$$
$$=\left(\int_{[0,1]}\int_{[0,1]}f(x,y)^2dydx\right)^2,$$

or, written in a slightly different way,

$$\|f\|_{\square^2}\leq\|f\|_{\mathrm{L}^2([0,1]^2)}. \tag{1.22}$$

Now, for any functions $f_{00}, f_{01}, f_{10}, f_{11}$ as before by this and Theorem 1.7.2 we have

$$\left|[f_{00},f_{01},f_{10},f_{11}]_{\square^2}\right|\leq\|f_{00}\|_{\mathrm{L}^2([0,1]^2)}\|f_{01}\|_{\mathrm{L}^2([0,1]^2)}\|f_{10}\|_{\mathrm{L}^2([0,1]^2)}\|f_{11}\|_{\mathrm{L}^2([0,1]^2)},$$

which is just a special case $r=2, n_1=n_2=2$ of Lemma 2.2.4 from the following chapter. Further particular cases of Lemma 2.2.4 were proved and used in the papers [23] and [53].

Theorem 1.7.2 also helps us in verifying the rightfulness of calling $\|\cdot\|_{\square^2}$ a norm.

Indeed, we can see that, for each real-valued measurable function f,

$$\int_{[0,1]}\int_{[0,1]}\int_{[0,1]}\int_{[0,1]} f(x,y)f(x,y')f(x',y)f(x',y')dydy'dxdx'$$
$$= \int_{[0,1]}\int_{[0,1]}\left(\int_{[0,1]} f(x,y)f(x',y)dy\right)^2 dxdx' \geq 0,$$

therefore $\|f\|_{\square^2}$ is nonnegative (more importantly, the above integral expression raised to the power of $\frac{1}{4}$ is well defined). It is easy to notice that $\|cf\|_{\square^2} = |c|\|f\|_{\square^2}$ for every $c \in \mathbb{R}$. The triangle inequality follows from Theorem 1.7.2; indeed,

$$\|f+g\|_{\square^2}^4 \leq \|f\|_{\square^2}^4 + 4\|f\|_{\square^2}^3\|g\|_{\square^2} + 6\|f\|_{\square^2}^2\|g\|_{\square^2}^2 + 4\|f\|_{\square^2}\|g\|_{\square^2}^3 + \|g\|_{\square^2}^4$$
$$= \left(\|f\|_{\square^2} + \|g\|_{\square^2}\right)^4.$$

Trivially, for $f \equiv 0$ $\lambda^{(2)}$-almost everywhere we have $\|f\|_{\square^2} = 0$. Now, fix $g \in \mathrm{L}^\infty([0,1])$ and $h \in \mathrm{L}^\infty([0,1])$. By the box-Gowers-Cauchy-Schwarz inequality for $f_{00} = f$, $f_{01}(x,y')$ $= g(x)$, $f_{10}(x',y) = h(y)$ and $f_{11} \equiv 1$, $x,y \in [0,1]$,

$$\left|\int_{[0,1]}\int_{[0,1]} f(x,y)g(x)h(y)dxdy\right| \leq \|f\|_{\square^2}\|g\|_{\mathrm{L}^\infty([0,1])}\|h\|_{\mathrm{L}^\infty([0,1])}.$$

Here we also used the inequality $\|f\|_{\square^2} \leq \|f\|_{\mathrm{L}^\infty([0,1]^2)}$ that follows from (1.22). If we assume that $\|f\|_{\square^2} = 0$, then we have that the bilinear form

$$(g,h) \mapsto \int_{[0,1]}\int_{[0,1]} f(x,y)g(x)h(y)dxdy$$

gives value zero idenitically over all g and h. Since $h \in \mathrm{L}^\infty([0,1])$ was arbitrarily chosen, we conclude that, for each choice of function g, the function

$$y \mapsto \int f(x,y)g(x)dx$$

is zero λ-almost everywhere. Moreover, we can conclude that it equals zero on the set $E \in \mathscr{B}([0,1])$ such that $\lambda(E^c) = 0$, where E does not depend on the choice of g, if taken from the collection

$$\left\{\sum_{i=1}^n \alpha_i \mathbb{1}_{\langle a_i,b_i\rangle} : n \in \mathbb{N}, \alpha_i \in \mathbb{Q}, a_1,\ldots,a_n,b_1,\ldots,b_n \in [0,1]\cap\mathbb{Q}\right\}$$

because it is countable by definition. As it is also dense in $\mathrm{L}^2([0,1])$, we conclude that the above function is zero λ-almost everywhere for any choice of $g \in \mathrm{L}^\infty([0,1])$, which gives

$f(x,y) = 0$ for almost every choice of $x \in [0,1]$ and for almost every choice of $y \in [0,1]$. Overall, $f \equiv 0$ $\lambda^{(2)}$-almost everywhere. We conclude that $\|\cdot\|_{\square^2}$ indeed satisfies the properties required from a norm.

Now we are going to mention two interesting classical results in combinatorics and show their connection with integral forms similar to those that we observe. *An arithmetic progression* is any k-tuple of integers

$$(a, a+k, \ldots, a+(k-1)d),$$

where $a, d \in \mathbb{Z}$ and $k \in \mathbb{N}$. The number k is also called *length* of the arithmetic progression. If $d = 0$, then the arithmetic progression is *trivial*.

One of the most fundamental results of combinatorial number theory was proven by Roth in [49].

Theorem 1.7.3. *(Roth's theorem)* For each $\delta \in \langle 0, \infty\rangle$ there exists $n_0 \in \mathbb{N}$ depending only on δ with the following property: if $n \in \mathbb{N}, n \geq n_0$ and if $A \subseteq \{1,2,\ldots,n\}$ satisfies $\operatorname{card}(A) \geq \delta n$, then A contains a nontrivial arithmetic progression of length 3.

Another interesting pattern to observe is *a corner*, defined as an ordered triple

$$((a,b),(a+d,b),(a,b+d))$$

where $a, b, d \in \mathbb{Z}$. Again, if $d = 0$, then the corner is considered to be *trivial*. Ajtai and Szemerédi [1] showed the following.

Theorem 1.7.4. *(A corners theorem)* For each $\delta \in \langle 0, \infty\rangle$ there exists $n_0 \in \mathbb{N}$ depending only on δ with the following property: if $n \in \mathbb{N}, n \geq n_0$ and if $A \subseteq \{1,2,\ldots,n\}^2$ satisfies $\operatorname{card}(A) \geq \delta n^2$, then A contains a nontrivial corner.

Interestingly, one can show that Theorem 1.7.3 follows from 1.7.4. Indeed, pick $\delta \in \langle 0, \infty\rangle$ and choose n_0 from Theorem 1.7.4 applied for the parameter $\frac{\delta}{4}$ in the place of δ. Choose $n \in \mathbb{N}$, $n \geq n_0$ and $A \subseteq \{1,2,\ldots,n\}$ such that $\operatorname{card}(A) \geq \delta n$. Define

$$\tilde{A} := \{(a,b) \in \{1,2,\ldots,2n\}^2 : b - a \in A\}.$$

For each $k \in A$ we have $(1,k+1),(2,k+2),\ldots,(2n-k,2n) \in \tilde{A}$; there are $2n-k \geq n$ such ordered pairs. Since $\operatorname{card}(A) \geq \delta n$, we conclude $\operatorname{card}(\tilde{A}) \geq \delta n^2 = \frac{\delta}{4} \cdot (2n)^2$. By

Theorem 1.7.4, there exist $a,b,d \in \{1,2,\dots,2n\}$ such that $(a,b),(a+d,b),(a,b+d) \in \tilde{A}$. By definition of $\tilde{A}$ we have $b-a-d, b-a, b-a+d \in A$ which is exactly a nontrivial arithmetic progression of length 3.

Let us for a moment observe a standard (simple undirected) graph G, which means that its edges contain exactly two (different) vertices.

Lemma 1.7.5. *(A triangle removal lemma [50])* For each $\varepsilon \in \langle 0,\infty\rangle$ there exists $\delta \in \langle 0,\infty\rangle$ depending only on ε with the following property: if G is a graph with n vertices that contains fewer than δn^3 triangles, then it is possible to delete less than εn^2 edges from G to create a triangle-free graph.

Intuitively, if a graph has relatively small number of triangles, then we can eliminate all of its triangles by removing a relatively small number of edges. This lemma implies Theorem 1.7.3 as well as Theorem 1.7.4.; see [50] and [54]. We will also elaborate the former implication below.

A triangle removal lemma can be stated in the following, equivalent way.

Corollary 1.7.6. For each $\varepsilon \in \langle 0,\infty\rangle$ there exists $\delta \in \langle 0,\infty\rangle$ depending only on ε with the following property: if G is a graph with n vertices that contains at least εn^2 edge-disjoint triangles, then G must contain at least δn^3 vertices.

From this variant of the triangle removal lemma we can deduce the corners theorem as well as Roth's theorem. The proof of Corrolary 1.7.6 implying Theorem 1.7.3 is also interesting as it shows the connection of two observed objects from the graph theory and number theory, respectively. Assume the contrary of Theorem 1.7.3 and let $\delta \in \langle 0,\infty\rangle$ be the parameter from the negation of the statement. For arbitrary $n_0 \in \mathbb{N}$ let $n \in \mathbb{N}$, $n \geq n_0$ and let $A \subseteq \{1,2,\dots,n\}$ be such that $\operatorname{card}(A) \geq \delta n$ and that it does not contain a nontrivial arithmetic progression of length 3. Let us construct a graph $G=(V,E)$ with $V=\{1,2,\dots,3n\}\times\{1,2,3\}$ and all the edges can take only one of the following forms:

- $\{(i,1),(j,2)\} \in E$ if and only if $j-i \in A$,
- $\{(j,2),(k,3)\} \in E$ if and only if $k-j \in A$,
- $\{(i,1),(k,3)\} \in E$ if and only if $\frac{k-i}{2} \in A$.

This way we constructed a 3-partite graph in which vertices of each triangle take form $(i,1)$, $(j,2)$ and $(k,3)$; in other words, we take exactly one vertex from each of partition sets. Trivially, we have $\frac{k-i}{2} = j-i+\left(\frac{k+i}{2}-j\right)$ and $k-j = j-i+2\cdot\left(\frac{k+i}{2}-j\right)$, which means that, if $(i,1),(j,2),(k,3)$ forms a triangle in G, then A contains an arithmetic progression of length 3. By assumption, this progression has to be trivial, meaning $\frac{k+i}{2}-j=0$. Now we can notice that G can contain at least δn^2 triangles. Indeed, for any choice of $i\in\{1,2,\ldots,n\}$ and any of at least δn choices of $a\in A$ we can then take $j=i+a$ and k that is uniquely determined by $\frac{k+i}{2}-j=0$. Also, all of these triangles are edge-disjoint because any edge in G, either $\{(i,1),(j,2)\}$, $\{(j,2),(k,3)\}$ or $\{(i,1),(k,3)\}$, determines unique parameters $i\in\{1,2,\ldots,n\}$ and $a\in A$, so no two triangles can have a mutual edge.

By Corollary 1.7.6 applied for $\varepsilon=\frac{\delta}{81}$ the graph G contains at least $729\delta' n^3$ triangles for certain $\delta'\in\langle 0,\infty\rangle$ that does not depend on n. On the other hand, as A contains at most n arithmetic progressions of length 3, we can have at most $3n^2$ triangles, as we have $3n$ choices for vertex $(i,1)$ after which j is determined with at most n choices and then, again, k is uniquely determined. This means that we must have $729\delta' n^3\le 3n^2$ or, equivalently, $n\le\frac{1}{243\delta'}$. However, as the choice of n_0 was arbitrary, n can be arbitrarily large, so the uniform upper bound for n leads us to contradiction. Theorem 1.7.3 follows.

Now we will give an analytic formulation of Lemma 1.7.5, which was stated as Lemma 6.5 in [60].

Lemma 1.7.7. For each $\varepsilon\in\langle 0,\infty\rangle$ there exists $\delta\in\langle 0,\infty\rangle$ depending only on ε such that the following is satisfied. Let $f,g,h:[0,1]^2\to[0,1]$ be measurable functions such that

$$\int_{[0,1]}\int_{[0,1]}\int_{[0,1]} f(x_1,x_2)g(x_2,x_3)h(x_3,x_1)\,dx_3dx_2dx_1\le\delta. \tag{1.23}$$

Then there exist $E_{1,2},E_{2,3},E_{3,1}\in\mathscr{B}([0,1]^2)$ such that

$$\mathbb{1}_{E_{1,2}}(x_1,x_2)\mathbb{1}_{E_{2,3}}(x_2,x_3)\mathbb{1}_{E_{3,1}}(x_3,x_1)=0 \quad \text{for each } x_1,x_2,x_3\in[0,1] \tag{1.24}$$

and

$$\int_{[0,1]}\int_{[0,1]} f(x_1,x_2)\mathbb{1}_{E_{1,2}^c}(x_1,x_2)\,dx_1dx_2+\int_{[0,1]}\int_{[0,1]} g(x_2,x_3)\mathbb{1}_{E_{2,3}^c}(x_2,x_3)\,dx_2dx_3$$
$$+\int_{[0,1]}\int_{[0,1]} h(x_3,x_1)\mathbb{1}_{E_{3,1}^c}(x_3,x_1)\,dx_3dx_1\le\varepsilon. \tag{1.25}$$

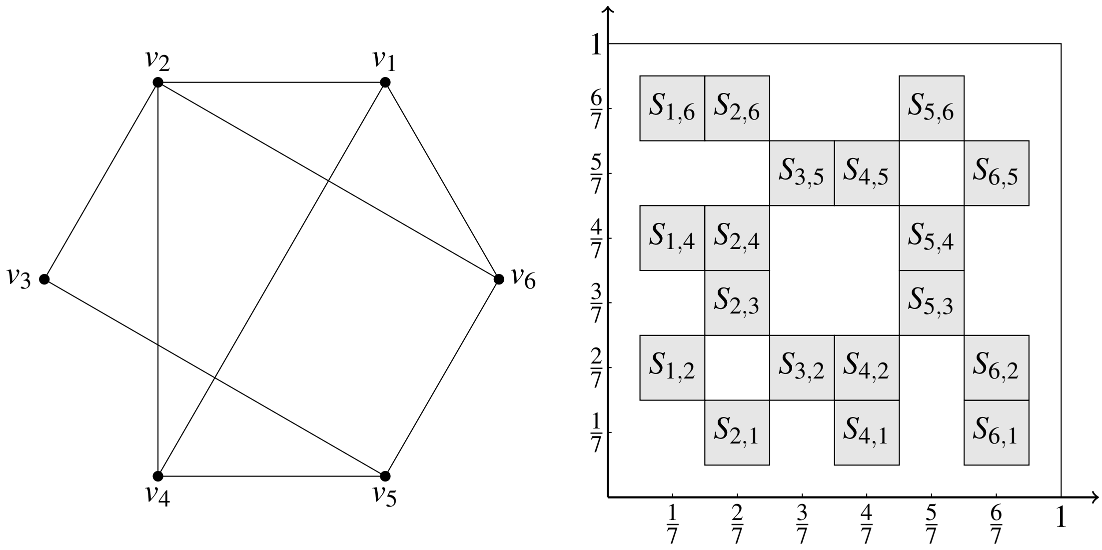

Figure 1.2: An example of a graph with 6 vertices and the corresponding plot of squares $S_{i,j}$.

Moreover, if there exist σ-algebras $\mathscr{U},\mathscr{V},\mathscr{W} \subseteq \mathscr{B}([0,1])$ such that f is $(\mathscr{U}\times\mathscr{V})$-measurable, g is $(\mathscr{V}\times\mathscr{W})$-measurable, and h is $(\mathscr{W}\times\mathscr{U})$-measurable, then one can achieve $E_{1,2}\in\mathscr{U}\times\mathscr{V}$, $E_{2,3}\in\mathscr{V}\times\mathscr{W}$, $E_{3,1}\in\mathscr{W}\times\mathscr{U}$.

Interestingly, this lemma implies Lemma 1.7.5. Suppose that $\varepsilon > 0$ is given. Let $G=(V,E)$ be a graph with $\operatorname{card}(V)=n$ vertices and with fewer than δn^3 triangles, where $\delta>0$ depends on ε and it will be chosen a bit later. Let us denote $V=\{v_1,v_2,\ldots,v_n\}$ and let us define

$$f=g=h:=\sum_{\substack{(i,j)\in\{1,2,\ldots,n\}\\ \{v_i,v_j\}\in E}} \mathbb{1}_{S_{i,j}}.$$

where $S_{i,j}:=\left[\frac{2i-1}{2(n+1)},\frac{2i+1}{2(n+1)}\right]\times\left[\frac{2j-1}{2(n+1)},\frac{2j+1}{2(n+1)}\right]$. Visually, these functions take value 1 on $\operatorname{card}(E)$ squares $S_{i,j}$ with area $|S_{i,j}|=\frac{1}{(n+1)^2}$ and with center $\left(\frac{i}{n+1},\frac{j}{n+1}\right)$, where $(i,j)\in\{1,2,\ldots,n\}^2$ is any pair such that $\{v_i,v_j\}\in E$; otherwise, f,g and h equal 0. It is easy to check that these squares are mutually disjoint up to its borders and that they are subsets of the domain square $[0,1]^2$ when $n\geq 3$, which does not restrict us since we consider the existence of triangles in the graph G. To understand this construction easily, see Figure 1.2. Also note that the last condition of Lemma 1.7.7 is satisfied with $\mathscr{U}=\mathscr{V}=\mathscr{W}=\sigma\left(\left\{\left[\frac{2i-1}{2(n+1)},\frac{2i+1}{2(n+1)}\right] : i\in\{1,2,\ldots,n\}\right\}\right)$.

Let $\delta'\in\langle 0,\infty\rangle$ be from Lemma 1.7.7 applied for $\frac{\varepsilon}{2}$ in the role of parameter ε and set $\delta := \delta'/6$. Notice that the function $(x_1,x_2,x_3) \mapsto f(x_1,x_2)g(x_2,x_3)h(x_3,x_1)$ equals 1 exactly when there exist squares $S_{i,j}$,$S_{j,k}$ and $S_{k,i}$ such that $(x_1,x_2) \in S_{i,j}$, $(x_2,x_3) \in S_{j,k}$, $(x_3,x_1) \in S_{k,i}$ and $(v_i,v_j),(v_j,v_k),(v_k,v_i) \in E$, therefore detecting all the triangles in G, each of them precisely 6 times. The integration of this function gives

$$\begin{aligned}
&\int_{[0,1]}\int_{[0,1]}\int_{[0,1]} f(x_1,x_2)g(x_2,x_3)h(x_3,x_1)dx_3dx_2dx_1 \\
&= \sum_{\substack{i,j,k\in\{1,2,\dots,n\} \\ \{v_i,v_j\},\{v_j,v_k\},\{v_k,v_i\}\in E}} \Big(\int_{[(2i-1)/2(n+1),(2i+1)/2(n+1)]} dx_1\Big) \\
&\quad \Big(\int_{[(2j-1)/2(n+1),(2j+1)/2(n+1)]} dx_2\Big)\Big(\int_{[(2k-1)/2(n+1),(2k+1)/2(n+1)]} dx_3\Big) \\
&\le 6\delta n^3 \frac{1}{(n+1)^3} < 6\delta = \delta',
\end{aligned}$$

so (1.23) is satisfied with δ' in the place of δ. Take $E_{1,2},E_{2,3},E_{3,1} \in \mathscr{U}\times\mathscr{U} \subseteq \mathscr{B}([0,1]^2)$ from Lemma 1.7.7 and define E' as the subset of E which satisfies the following condition:

$$\{v_i,v_j\} \in E' \quad \text{if and only if} \quad S_{i,j}\cap E_{1,2}\cap E_{2,3}\cap E_{3,1} = \emptyset.$$

Note that, by the $(\mathscr{U}\times\mathscr{U})$-measurability condition, for each set $E_{k,l}$ and each square $S_{i,j}$ we have either $S_{i,j} \subseteq E_{k,l}$ or $S_{i,j}\cap E_{k,l} = \emptyset$. We claim that Lemma 1.7.5 is satisfied if we delete edges from the set E'. First, notice that

$$\begin{aligned}
\operatorname{card}(E') &= \frac{(n+1)^2}{2} \sum_{\substack{i,j\in\{1,2,\dots,n\} \\ \{v_i,v_j\}\in E'}} \iint_{S_{i,j}} dxdy \\
&\le \frac{(n+1)^2}{2}\Big(\int_{[0,1]}\int_{[0,1]} f(x_1,x_2)\mathbb{1}_{E_{1,2}^c}(x_1,x_2)dx_1dx_2 \\
&\quad + \int_{[0,1]}\int_{[0,1]} g(x_2,x_3)\mathbb{1}_{E_{2,3}^c}(x_2,x_3)dx_2dx_3 + \int_{[0,1]}\int_{[0,1]} h(x_3,x_1)\mathbb{1}_{E_{3,1}^c}(x_3,x_1)dx_3dx_1\Big) \\
&\le \frac{(n+1)^2}{2}\cdot\frac{\varepsilon}{2} \le \varepsilon n^2,
\end{aligned}$$

by (1.25) and by the choice of the parameter. This means that, by the suggested procedure, we would indeed delete less than εn^2 edges. To check that the newly constructed graph would be triangle-free, note that, if there existed a triangle $\{v_i,v_j\},\{v_j,v_k\},\{v_k,v_i\} \in E\setminus E'$, $i<j<k$, then we would have $S_{i,j} \subseteq E_{1,2}$, $S_{j,k} \subseteq E_{2,3}$, $S_{k,i} \subseteq E_{3,1}$, so that $\big(\frac{i}{n+1},\frac{j}{n+1}\big) \in E_{1,2}$, $\big(\frac{j}{n+1},\frac{k}{n+1}\big) \in E_{2,3}$ and $\big(\frac{i}{n+1},\frac{k}{n+1}\big) \in E_{3,1}$, which would contradict (1.24).

Observations in this section offer another motivation to analyse integral forms associated with graphs and actually motivate one to generalize the concept to the hypergraph setting, in an attempt to offer a removal lemma for this more general concept. Recall the notation introduced in Section 1.6.

Lemma 1.7.8. Suppose that k, r are positive integers such that $k \leq r$. For each $\varepsilon \in \langle 0, \infty \rangle$ there exists $\delta \in \langle 0, \infty \rangle$ depending only on ε such that the following holds. Let $H = (V, E)$ be an r-partite k-uniform hypergraph and let $\mathbf{F} = (F_e)_{e \in E}$ be a tuple of measurable functions from $[0,1]^k$ to $[0,1]$ such that

$$\int_{[0,1]^n} \prod_{e \in E} F_e(\mathbb{x}_e) d\mathbb{x} \leq \delta$$

where $n = \mathrm{card}(V)$. Then there exist a tuple $(E_e)_{e \in E}$ of elements in $\mathscr{B}([0,1]^k)$ such that

$$\prod_{e \in E} \mathbb{1}_{E_e}(\mathbb{x}_e) = 0 \quad \text{for each } \mathbb{x} \in [0,1]^n$$

and

$$\sum_{e \in E} \int_{[0,1]^r} F_e(\mathbb{x}_e) \mathbb{1}_{E_e^c}(\mathbb{x}_e) d\mathbb{x}_e \leq \varepsilon.$$

This is just a slight reformulation of Tao's hypergraph removal lemma from [59]. Its proof is quite involved and the reader can find it in [59]. We will not need it later in the text.

2. T(1) THEOREM FOR DYADIC SINGULAR INTEGRAL FORMS

In this chapter we will study dyadic singular integral forms associated with r-partite r-uniform hypergraphs. We will prove L^p boundedness of singular integral forms with Haar functions in integral expressions and estimates for local entagled dyadic paraproducts. Then we will prove characterizations of L^p boundedness of singular integral forms with general dyadic Calderón-Zygmund kernels.

2.1. CHARACTERIZATIONS OF L^p BOUNDEDNESS

Starting with an r-partite r-regular hypergraph $H=(V,E)$, let $(V^{(i)})_{i=1}^{r}$ be an r-partition of the set of vertices V. We assume that there are no isolated vertices, i.e., each vertex from V belongs to some edge from E. Moreover, we also assume that each partition set $V^{(i)}$ contains at least two vertices. Also, let $(H_l)_{l=1}^{k}$, given with $H_l=(V_l,E_l)$ for each $l\in\{1,\dots,k\}$, be connected components of H, meaning that there exist partitions $\big(V_j\big)_{1\le j\le m}$ of V and $\big(E_j\big)_{1\le j\le m}$ of E such that each subhypergraph $\big(V_j,E_j\big)$, $j\in\{1,2,\dots,m\}$ is connected (i.e. for each $x,y\in V_j$ there exist $n\in\mathbb{N}$, $v_1,\dots,v_{n-1}\in V_j$ and $e_1,\dots,e_n\in E_j$ such that $x,v_1\in e_1,v_1,v_2\in e_2,\dots,v_{n-1},y\in e_n$) and maximal (i.e. it is not contained in any other connected subhypergraph of (V,E)). Throughout this chapter we are going to impose the assumption that each of these connected components constitutes a complete hypergraph. This is a serious restriction when compared with the two-dimensional case from [37], but it keeps us far from unresolved issues related to the so-called triangular

Hilbert transforms; see the comments in [34]. For each such l and for each $i \in \{1,\ldots,r\}$ we define $V_l^{(i)} := V_l \cap V^{(i)}$. This makes $(V_l^{(i)})_{i=1}^r$ an r-partition of the set V_l, which goes along with the hypergraph H_l being r-partite as well. For each $e \in E$, taking the unique $l \in \{1,\ldots,k\}$ such that $e \in E_l$, we define

$$d_e := \max_{1\leq i\leq r} \prod_{\substack{1\leq j\leq r\\ j\neq i}} \operatorname{card}(V_l^{(j)}) = \max_{1\leq i\leq r} \frac{\prod_{1\leq j\leq r} \operatorname{card}(V_l^{(j)})}{\operatorname{card}(V_l^{(i)})}. \tag{2.1}$$

In words, d_e is the product of cardinalities of the $r-1$ largest vertex-partition parts of the connected component containing e. These quantities will turn out to be important in determining the ranges of exponents of the estimates to follow.

The following setting is a higher-dimensional multilinear generalization of the dyadic setup from the paper [3] by Auscher, Hofmann, Muscalu, Tao, and Thiele. Let us denote $n := \operatorname{card}(V)$, $n_i := \operatorname{card}(V^{(i)})$ and let $K : \mathbb{R}^n \to \mathbb{C}$ be *a perfect dyadic Calderón-Zygmund kernel*, i.e. a locally integrable, bounded and compactly supported function that is constant on each n-dimensional dyadic cube not intersecting the diagonal

$$D = \big\{(\underbrace{x^{(1)},\ldots,x^{(1)}}_{n_1 \text{ times}},\ldots,\underbrace{x^{(r)},\ldots,x^{(r)}}_{n_r \text{ times}}) \in \mathbb{R}^n\big\}$$

and that, for each $\mathbb{x} = (x_1^{(1)},\ldots,x_{n_1}^{(1)},\ldots,x_1^{(r)},\ldots,x_{n_r}^{(r)}) \in \mathbb{R}^n \backslash D$, satisfies

$$|K(\mathbb{x})| \lesssim \Big(\sum_{i=1}^{r} \sum_{1\leq j_1<j_2\leq n_i} |x_{j_1}^{(i)} - x_{j_2}^{(i)}|\Big)^{r-n}. \tag{2.2}$$

For a tuple $\mathbf{F} = (F_e)_{e\in E}$ of measurable bounded functions we define

$$\Lambda_E(\mathbf{F}) := \int_{\mathbb{R}^n} \Big(\prod_{e\in E} F_e(\mathbb{x}_e)\Big) K(\mathbb{x}) d\mathbb{x}, \tag{2.3}$$

and, for fixed $e_0 \in E$ and for each $\mathbb{x}_{e_0} \in \mathbb{R}^r$,

$$T_{e_0}\big(\mathbf{F}_{E\backslash\{e_0\}}\big)(\mathbb{x}_{e_0}) := \int_{\mathbb{R}^{n-r}} \Big(\prod_{e\in E\backslash\{e_0\}} F_e(\mathbb{x}_e)\Big) K(\mathbb{x}) \prod_{v\in V\backslash e_0} dx_v, \tag{2.4}$$

where we have denoted the tuple $(F_e)_{e\in E\backslash\{e_0\}}$ simply by $\mathbf{F}_{E\backslash\{e_0\}}$. We can notice that we have

$$\Lambda_E(\mathbf{F}) = \int_{\mathbb{R}^r} T_{e_0}\big(\mathbf{F}_{E\backslash\{e_0\}}\big)(\mathbb{x}_{e_0}) F_{e_0}(\mathbb{x}_{e_0}) d\mathbb{x}_{e_0}$$

for each $e_0 \in E$. One could say that the operators T_{e_0} for different $e_0 \in E$ are multilinear adjoints of each other. We also say that Λ_E is the *entangled dyadic form* associated with H and K.

Let us introduce the notation of the *elementary tensor product*, which, for two functions $f, g : \mathbb{R} \to \mathbb{C}$, is denoted and defined as

$$(f \otimes g)(x,y) := f(x)g(y) \text{ for each } x, y \in \mathbb{R}.$$

By the associativity of the operation $\otimes$, we will assume the notation $f_1 \otimes f_2 \otimes \cdots \otimes f_m$ as an elementary tensor product of more than two functions and also write $\otimes_{i=1}^m f_i$. For what follows, we consider all functions of the form

$$\mathbb{h}_Q^S := |Q|^{\frac{1}{2}} \bigotimes_{k=1}^{r} \bigotimes_{i=1}^{n_k} \Big(\bigotimes_{v_i^{(k)} \in S^{(i)}} \mathbb{h}^1_{I_i^{(k)}} \Big) \Big(\bigotimes_{v_i^{(k)} \in (S^{(k)})^c} \mathbb{h}^0_{I_i^{(k)}} \Big),$$

where $Q = \prod_{k=1}^r \prod_{i=1}^{n_k} I_i^{(k)} \in \mathscr{C}_n$ is arbitrary and $S = ((S^{(k)})_{k=1}^r) \neq (\emptyset)$ is an r-tuple of selected vertices from the r-partitioned set of vertices, i.e. $S^{(k)} \subseteq V^{(k)}$ for each $k \in \{1, \ldots, r\}$. Notice that these are the tensor products of L^2-normalized Haar functions with at least one of them being cancellative. This means that for a perfect dyadic Calderón-Zygmund kernel K, being a square-integrable function over $\mathbb{R}^n$, we have

$$K = \sum_{\substack{S=(S^{(i)})_{i=1}^r \\ (\forall i \in \{1,\ldots,r\}) S^{(i)} \subseteq V^{(i)} \\ (\exists i_0 \in \{1,\ldots,r\}) S^{(i_0)} \neq \emptyset}} \sum_{Q=\prod_{i=1}^r \prod_{j=1}^{n_i} I_{j_i}^{(i)} \in \mathscr{C}_n} \left\langle K, \mathbb{h}_Q^S \right\rangle_{\mathrm{L}^2(\mathbb{R}^n)} \mathbb{h}_Q^S. \tag{2.5}$$

Notice that, as K is constant on dyadic cubes not intersecting the diagonal and each of these tensor products has a cancellation in at least one of the variables, the corresponding scalar products equal zero, so we can actually consider this sum only over dyadic cubes $Q = \prod_{i=1}^r \prod_{j=1}^{n_i} I_{j_i}^{(i)}$ for which $I_{j_1}^{(i)} = I_{j_2}^{(i)}$ for each $j_1, j_2 \in \{1, \ldots, n_i\}$ and $i \in \{1, \ldots, r\}$. Using this and by assuming that functions $F_e, e \in E$ and K are bounded and compactly supported, we can present the form Λ_E as

$$\Lambda_E(\mathbf{F}) = \sum_{\substack{S=(S^{(i)})_{i=1}^r \\ (\forall i \in \{1,\ldots,r\}) S^{(i)} \subseteq V^{(i)} \\ (\exists i_0 \in \{1,\ldots,r\}) S^{(i_0)} \neq \emptyset}} \sum_{Q=\prod_{i=1}^r (I^{(i)})^{n_i} \in \mathscr{C}_n} \left\langle K, \mathbb{h}_Q^S \right\rangle_{\mathrm{L}^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Big(\prod_{e \in E} F_e(\mathbb{x}_e) \Big) \mathbb{h}_Q^S(\mathbb{x}) d\mathbb{x}, \tag{2.6}$$

as we can use the Lebesgue dominated convergence theorem.

Along with the form Λ_E for each tuple S we will define an *entangled dyadic paraproduct* as

$$\Lambda_E^S(\mathbf{F}) := \sum_{Q=\prod_{i=1}^r \left(I^{(i)}\right)^{n_i} \in \mathscr{C}_n} \left\langle K, \mathbb{h}_Q^S \right\rangle_{\mathrm{L}^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left(\prod_{e\in E} F_e(\mathbb{x}_e) \right) \mathbb{h}_Q^S(\mathbb{x}) d\mathbb{x} \tag{2.7}$$

Another useful way of writing this form will be

$$\Lambda_E^S(\mathbf{F}) = \sum_{Q=\prod_{i=1}^r \left(I^{(i)}\right)^{n_i} \in \mathscr{C}_n} |Q| \lambda_Q \left[\mathbf{F}\right]_{H,S,Q},$$

with λ_Q defined as $\lambda_Q := |Q|^{-\frac{1}{2}} \left\langle K, \mathbb{h}_Q^S \right\rangle_{\mathrm{L}^2(\mathbb{R}^n)}$ and $[\mathbf{F}]_{H,S,Q}$ is defined as in Section 2.2.

The following theorem is the main result of this work and it characterizes various bounds for Λ_E.

Theorem 2.1.1. Let (V,E) be an r-partite r-uniform hypergraph such that all its connected components are complete, with an r-partition $\left(V^{(i)}\right)_{1\le i\le r}$ of the set of vertices V such that $\min_{1\le i\le r} \operatorname{card}(V^{(i)}) \ge 2$. Furthermore, let $\left(S^{(i)}\right)_{1\le i\le r}$ be a family of selected vertices and let $(d_e)_{e\in E}$ be numbers given with (2.1). The following statements are equivalent.

(a) The weak boundedness property

$$|\Lambda_E(\mathbb{1}_Q, \ldots, \mathbb{1}_Q)| \lesssim |Q| \text{ for each } Q \in \mathscr{C}_r \tag{2.8}$$

and the T(1)-type conditions

$$\|T_e(\mathbb{1}_{\mathbb{R}^r}, \ldots, \mathbb{1}_{\mathbb{R}^r})\|_{\mathrm{BMO}(\mathbb{R}^r)} \lesssim 1 \text{ for each } e \in E \tag{2.9}$$

are valid.

(b) We have

$$\|T_{e_0}(\mathbb{1}_Q)_{e\in E\setminus\{e_0\}}\|_{\mathrm{L}^1(Q)} \lesssim |Q| \text{ for each } e_0 \in E \text{ and } Q \in \mathscr{C}_r. \tag{2.10}$$

(c) The form Λ_E satisfies the estimate

$$|\Lambda_E(\mathbf{F})| \lesssim \prod_{e\in E} \|F_e\|_{\mathrm{L}^{p_e}(\mathbb{R}^r)} \tag{2.11}$$

for all choices of exponents $d_e < p_e \le \infty, e \in E$, such that $\sum_{e\in E} \frac{1}{p_e} = 1$.

(d) The form Λ_E satisfies the estimate (2.11) for some choice of exponents $d_e < p_e \leq \infty, e \in E$, such that $\sum_{e\in E} \frac{1}{p_e} = 1$.

(e) For any measurable, bounded, and and compactly supported tuple of functions $\mathbf{F}$ there exists a sparse form $\Theta_{\mathscr{S}}$ for which we have $|\Lambda_E(\mathbf{F})| \lesssim \Theta_{\mathscr{S}}(\mathbf{F})$.

(f) Let $\mathbf{p} = (p_e)_{e\in E}$ be an arbitrary tuple of exponents from $[1,\infty]$ such that $p_e > d_e$ for each $e \in E$ and $\sum_{e\in E} \frac{1}{p_e} = 1$ and $\mathbf{w} = (w_e)_{e\in E}$ a tuple of strictly positive functions satisfying (1.3). For each tuple $\mathbf{F} = (F_e)_{e\in E}$ we have

$$|\Lambda_E(\mathbf{F})| \lesssim [\mathbf{w}]_{\mathbf{p},\mathbf{d}}^{\max_{e\in E} \frac{p_e}{p_e - d_e}} \prod_{e\in E} \|F_e\|_{\mathrm{L}^{p_e}(w_e)}.$$

The implicit constants in all of the above estimates depend on the hypergraph H, kernel K, the exponents in question, and they also mutually depend on each other.

Throughout the whole chapter we are going to work with the functions $F_e, e \in E$ that are nonnegative, as the general result will follow by representing each of these functions as a difference of its positive and negative parts.

Because of the assumption $\min_{1\leq i\leq r} \mathrm{card}(V^{(i)}) \geq 2$ we easily deduce that

$$\sum_{e\in E} \frac{1}{d_e} > 1 \tag{2.12}$$

is valid. Indeed, if there exists a connected component $H_l = (V_l, E_l)$ such that $\mathrm{card}(V_l^{(i)}) \geq 2$ for each $i \in \{1,\dots,r\}$, then by denoting $i_0 := \min_{1\leq i\leq r} \mathrm{card}(V_l^{(i)})$, since $\mathrm{card}(E_l) = \prod_{i=1}^r \mathrm{card}(V_l^{(i)})$, we have

$$\sum_{e\in E} \frac{1}{d_e} \geq \sum_{e\in E_l} \frac{1}{d_e} = \frac{\mathrm{card}(E_l)\mathrm{card}(V_l^{(i_0)})}{\prod_{i=1}^r \mathrm{card}(V_l^{(i)})} \geq 2 > 1.$$

If such component does not exist, then there exist at least two different components $H_{l_1} = (V_{l_1}, E_{l_1})$ and $H_{l_2} = (V_{l_2}, E_{l_2})$, $l_1 \neq l_2$, in which case, with $i_1 := \min_{1\leq i\leq r} \mathrm{card}(V_{l_1}^{(i)})$ and $i_2 := \min_{1\leq i\leq r} \mathrm{card}(V_{l_2}^{(i)})$, we can conclude

$$\sum_{e\in E} \frac{1}{d_e} \geq \sum_{e\in E_{l_1}} \frac{1}{d_e} + \sum_{e\in E_{l_2}} \frac{1}{d_e} = \frac{\mathrm{card}(E_{l_1})\mathrm{card}(V_l^{(i_1)})}{\prod_{i=1}^r \mathrm{card}(V_l^{(i)})} + \frac{\mathrm{card}(E_{l_2})\mathrm{card}(V_l^{(i_2)})}{\prod_{i=1}^r \mathrm{card}(V_l^{(i)})} \geq 1+1 > 1.$$

Theorem 2.1.1 would not give any estimates for forms associated with hypergraphs if (2.12) failed. In the particular case dealing with bipartite graphs (without the completeness assumption), i.e. when $r = 2$, the paper [37] proceeds by studying exceptional cases, so that all nondegenerate bipartite graphs are covered with some nonempty range of exponents. We are not able to do the same here, since higher dimensions bring an additional structural complexity, and this is another reason why we find convenient to assume that each hypergraph component is complete. Indeed, the reader can see the recent paper by Durcik and Roos [17] for an example of an open problem in dimensions $r \geq 4$, which would be resolved if we could apply our main result to the hypergraph in question.

Example 2.1.2. Let us illustrate how the twisted paraproduct form from the paper [33] can be represented as an entangled form associated to a hypergraph.

Suppose that each of the partition classes $V^{(i)}$ has precisely two vertices and suppose that the hypergraph is complete, so that indeed $E = \prod_{i=1}^{r} V^{(i)}$, $\operatorname{card}(E) = 2^r$. Thus, the set of edges is in a bijective correspondence with $\{0,1\}^r$ and we are working with a tuple of functions $\mathbf{F} = (F_{j_1,j_2,\dots,j_r})_{j_1,j_2,\dots,j_r \in \{0,1\}}$. For the kernel we take

$$K(\mathbb{x}) := \sum_{Q=\prod_{i=1}^{r} I^{(i)}} |I^{(1)}|^r \mathbb{h}^1_{I^{(1)}}(x^{(1)}_1)\mathbb{h}^1_{I^{(1)}}(x^{(1)}_2) \prod_{k=2}^{r} \mathbb{h}^0_{I^{(k)}}(x^{(k)}_1)\mathbb{h}^0_{I^{(k)}}(x^{(k)}_2),$$

where the summation is performed over all dyadic cubes contained in $[0,2^N\rangle^r$ with edge-length at least 2^{-N}, for some positive integer N. Since $\int \mathbb{h}^1_I(x)dx = 0$ for every dyadic interval I, it is easy to verify that K and the associated form Λ_E satisfy conditions from part (a) of Theorem 2.1.1 with

$$T_e(\mathbb{1}_{\mathbb{R}^r}, \dots, \mathbb{1}_{\mathbb{R}^r}) = 0$$

for each $e \in E$. Consequently, we obtain L^p estimates for Λ_E in the range $2^{r-1} < p_e \leq \infty$ for each $e \in E$, $\sum_{e\in E} \frac{1}{p_e} = 1$.

The most interesting case in [33] is obtained by taking $F_{j_1,j_2,\dots,j_r} = \mathbb{1}_{\mathbb{R}^r}$ whenever $j_1 + j_2 + \cdots + j_r \geq 2$, which leaves us with only $r+1$ nontrivial functions. For the remaining functions we need to take $p_e = \infty$, which makes the range of exponents empty unless $(r+1)\frac{1}{2^{r-1}} > 1$, i.e. unless $r \leq 2$. The case $r = 3$ (and without the requirement $d_e < p_e$ for each $e \in E$) was handled in [17], while the cases $r \geq 4$ are still open at the time of writing.

2.2. L^p BOUNDEDNESS FOR A SPECIFIC KERNEL

First we are going to prove the L^p-estimate for a dyadic singular integral form where, instead of a perfect dyadic Calderón-Zygmund kernel K, we have Haar functions defined in 1.16. The form will be defined as follows.

Let (V,E) be an r-partite r-uniform hypergraph with a fixed r-partition; denote $V^{(i)} = \{v_1^{(i)},\dots,v_{n_i}^{(i)}\}$ for each $i \in \{1,\dots,r\}$. Let $(F_e)_{e\in E}$ be a tuple of measurable, bounded and compactly supported functions from $\mathbb{R}^r$ to $\mathbb{R}$ and take $S = (S^{(i)})_{1\le i\le r}$ with $S^{(i)} \subseteq V^{(i)}$ for each $i \in \{1,\dots,r\}$ and such that there exists $i_0 \in \{1,\dots,r\}$ such that $\mathrm{card}(S^{i_0}) \geq 2$. We define

$$\Lambda_{E,S}\left((F_e)_{e\in E}\right)$$
$$:= \sum_{Q=I_1\times\cdots\times I_r\in\mathscr{C}_r} |Q| \int_{\mathbb{R}^n} \Big(\prod_{e\in E} F_e(\mathbb{x}_e)\Big) \prod_{i=1}^{r} \Big(\prod_{v^{(i)}\in S^{(i)}} \mathbb{h}_{I_i}^1(x_{v^{(i)}}) \prod_{v^{(i)}\in (S^{(i)})^c} \mathbb{h}_{I_i}^0(x_{v^{(i)}})\Big) d\mathbb{x},$$

where $n = \mathrm{card}(V)$ and $\mathbb{x}$ and $\mathbb{x}_e, e \in E$ are vectors defined in Section 1.6. For the definition of the form Λ and for the statement of the main problem we intentionally labeled functions F_e with the set of edges E and variables $x_{v^{(i)}}$ with the set of vertices V and its r-partition $(V^{(i)})_{1\le i\le r}$. As we will see later, we will introduce a short, compact notation which encodes all important information by just defining a certain labeled (r-partite and r-uniform) hypergraph, therefore making proofs easier to write and more practical to visualise. With this, a tuple S of vertices will be considered as a tuple of *selected vertices*.

Take a tuple $\mathbf{F} = (F_l)_{l\in L_E}$ of nonnegative measurable compactly supported functions. More precisely, this is a collection of functions indexed by the set L_E and these functions will be substituted in the places of edge labels in all of the following analytical expressions. For $Q = \prod_{k=1}^r I_k \in \mathscr{C}_r$ an *evaluation of a tuple* $\mathbf{F}$ *on the hypergraph* H, given S and Q is defined as the number given by

$$[\mathbf{F}]_{H,S,Q} := \int_{\mathbb{R}^n} \prod_{e\in E} F_{l_E(e)}(\mathbb{x}_e) \prod_{k=1}^{r} \Big(\prod_{v_i^{(k)}\in S^{(i)}} \mathbb{h}_{I_i}^1(x_{v_i^{(k)}}) \prod_{v_i^{(k)}\in (S^{(i)})^c} \mathbb{h}_{I_i}^0(x_{v_i^{(k)}})\Big) d\mathbb{x},$$

where $S = (S^{(i)})_{1\le i\le r}$ and $S^{(i)} \subseteq V^{(i)}$ for each $i \in \{1,2,\dots,r\}$. This expression will also be called *paraproduct-type term*. In particular, if each $S^{(i)} = \emptyset$, then the mapping $\mathscr{A} : \mathbf{F} \mapsto [\mathbf{F}]_{H,S,Q}$ will be called an *averaging paraproduct-type term*. Also, any linear combination of paraproduct-type terms will be called a *paraproduct-type expression*. Note that the form $\Lambda_{E,S}$ that we are trying to bound has much more compact notation now:

$$\Lambda_{E,S}(\mathbf{F}) = \sum_{Q\in\mathscr{C}_r} |Q|\,[\mathbf{F}]_{H,S,Q}\,,$$

where the initial labelling of edges is given by $l_E(e) := F_e$.

Theorem 2.2.1. Let (V,E) be an r-partite r-uniform hypergraph with an r-partition $\left(V^{(i)}\right)_{1\le i\le r}$ of the set of vertices V, where $V^{(i)} = \{v_1^{(i)},\dots,v_{n_i}^{(i)}\}$ for each $i\in\{1,\dots,r\}$, a family of selected vertices $\left(S^{(i)}\right)_{1\le i\le r}$ and numbers $(d_e)_{e\in E}$ given with (2.1). Then, for each tuple $(p_e)_{e\in E}$ of positive real numbers that satisfy $d_e < p_e \le \infty$ for each $e\in E$ and $\sum_{e\in E}\frac{1}{p_e} = 1$ we have

$$\left|\Lambda_{E,S}\left((F_e)_{e\in E}\right)\right| \lesssim_{(n_i),r,(p_e)} \prod_{e\in E} \|F_e\|_{\mathrm{L}^{p_e}(\mathbb{R}^r)}\,.$$

Before obtaining this result, we are going to define additional forms which are going to appear in the proof of this result, and we will show its specific estimates that will turn out to be helpful for the later proof.

Given $Q\in\mathscr{C}_r$, for an expression $\mathscr{B} = \mathscr{B}_Q(\mathbf{F})$ we define its *first-order difference* as

$$\Box\mathscr{B}_Q(\mathbf{F}) := \sum_{Q'\in\mathscr{C}(Q)} \frac{1}{2^r}\mathscr{B}_{Q'}(\mathbf{F}) - \mathscr{B}_Q(\mathbf{F})\,.$$

The operator $\Box$ can be thought of as a certain discrete version of the Laplace operator.

Theorem 2.2.2. For any cube Q, the first-order difference of the averaging paraproduct-type term $\mathscr{B}_{H,(\emptyset),Q} = [\mathbf{F}]_{H,(\emptyset),Q}$ is the paraproduct-type expression

$$\Box\mathscr{B}_{H,(\emptyset),Q} = \sum_{\substack{(\forall i\in\{1,\dots,r\})S^{(i)}\subseteq V^{(i)},\mathrm{card}(S^{(i)})\,\mathrm{even}\\ (\exists i_0\in\{1,\dots,r\})\mathrm{card}(S^{(i_0)})\neq\emptyset}} [\mathbf{F}]_{H,S,Q}\,.$$

Proof. Let $Q := \prod_{i=1}^r I_i$ and, for each dyadic interval $I\in\mathscr{C}_1$, let $\nu_I := \mathbb{1}_{I_L} - \mathbb{1}_{I_R}$. Notice that, for any $\alpha_1,\dots,\alpha_r\in\mathbb{R}$,

$$\prod_{i=1}^r\Big(\prod_{j=1}^{n_i}\big(1+\alpha_i\nu_{I_i}\big(x_{v_j^{(i)}}\big)\big)\Big) = \sum_{\substack{S^{(1)}\subseteq V^{(1)}\\ \dots\\ S^{(r)}\subseteq V^{(r)}}} \prod_{i=1}^r \prod_{v_j^{(i)}\in S^{(i)}} \alpha_i^{|S_i|}\nu_{I_i}\big(x_{v_j^{(i)}}\big).$$

Indeed, the product on the left-hand side can be multiplied out. The left-hand side can be rewritten by multiplying each term from each of the brackets, so we get each summand given on the right-hand side of the equation, depending on which of the terms $\alpha_i \nu_{I_i}(x_{v_j^{(i)}})$ we selected while multiplying. Adding up these equations for every choice of $\alpha_i = \pm 1, 1 \leq i \leq r$, multiplying the resulting equation with $\frac{\prod_{e\in E} F_{l_E(e)}(\mathbb{x}_e)}{2^r |I_1|^{n_1} \dots |I_r|^{n_r}}$ and then integrating in each variable $x_{v_j^{(i)}}$ over I_i, we get

$$\int_Q \frac{\prod_{e\in E} F_{l_E(e)}(\mathbb{x}_e)}{2^r |I_1|^{n_1} \dots |I_r|^{n_r}} \sum_{\substack{\alpha_1=\pm 1\\ \dots \\ \alpha_r=\pm 1}} \prod_{i=1}^{r} \Big(\prod_{j=1}^{n_i} (1+\alpha_i \nu_{I_i}(x_{v_j^{(i)}})) \Big) d\mathbb{x}$$
$$= \int_Q \frac{\prod_{e\in E} F_{l_E(e)}(\mathbb{x}_e)}{|I_1|^{n_1} \dots |I_r|^{n_r}} \sum_{\substack{S^{(1)}\subseteq V^{(1)}, \mathrm{card}(S^{(1)})\,\mathrm{even}\\ \dots \\ S^{(r)}\subseteq V^{(r)}, \mathrm{card}(S^{(r)})\,\mathrm{even}}} \prod_{i=1}^{r} \prod_{v_j^{(i)}\in S^{(i)}} \nu_{I_i}(x_{v_j^{(i)}}) d\mathbb{x}.$$

The right-hand side is the result of fixing the value of each but one α_i and then adding terms depending on whether the remaining α_i equals 1 or -1; for subsets S_i of odd cardinality the terms cancel out. This procedure is repeated by fixing each of the remaining scalars α_i. Note that

$$\mathbb{h}_I^1 = \nu_I \mathbb{h}_I^0, \quad \mathbb{h}_{I_L}^0 = (1+\nu_I)\mathbb{h}_I^0, \quad \mathbb{h}_{I_R}^0 = (1-\nu_I)\mathbb{h}_I^0.$$

If we rewrite the above equality as

$$\frac{1}{2^r} \sum_{\substack{\alpha_1=\pm 1\\ \dots \\ \alpha_r=\pm 1}} \int_{\mathbb{R}^n} \prod_{e\in E} F_{l_E(e)}(\mathbb{x}_e) \prod_{i=1}^{r} \prod_{j=1}^{n_i} (1+\alpha_i \nu_{I_i}(x_{v_j^{(i)}})) \mathbb{h}_{I_i}^0(x_{v_j^{(i)}}) dx_{v_j^{(i)}} d\mathbb{x}$$
$$= \sum_{\substack{S^{(1)}\subseteq V^{(1)}, \mathrm{card}(S^{(1)})\,\mathrm{even}\\ \dots \\ S^{(r)}\subseteq V^{(r)}, \mathrm{card}(S^{(r)})\,\mathrm{even}}} \int_{\mathbb{R}^n} \prod_{e\in E} F_{l_E(e)}(\mathbb{x}_e) \prod_{i=1}^{r} \prod_{v_j^{(i)}\in S^{(i)}} \nu_{I_i}(x_{v_j^{(i)}}) \mathbb{h}_{I_i}^0(x_{v_j^{(i)}}) \prod_{v_j^{(i)}\in (S^{(i)})^c} \mathbb{h}_{I_i}^0(x_{v_j^{(i)}}) dx_{v_j^{(i)}} d\mathbb{x},$$

then, using the identities above,

$$\frac{1}{2^r} \int_{\mathbb{R}^n} \prod_{e\in E} F_{l_E(e)}(\mathbb{x}_e) \prod_{i=1}^{r} \prod_{j=1}^{n_i} (\mathbb{h}_{(I_i)_L}^0(x_{v_j^{(i)}}) + \mathbb{h}_{(I_i)_R}^0(x_{v_j^{(i)}})) dx_{v_j^{(i)}} d\mathbb{x}$$
$$= \sum_{\substack{S^{(1)}\subseteq V^{(1)}, \mathrm{card}(S^{(1)})\,\mathrm{even}\\ \dots \\ S^{(r)}\subseteq V^{(r)}, \mathrm{card}(S^{(r)})\,\mathrm{even}}} \int_{\mathbb{R}^n} \prod_{e\in E} F_{l_E(e)}(\mathbb{x}_e) \prod_{i=1}^{r} \Big(\prod_{v_j^{(i)}\in S^{(i)}} \mathbb{h}_{I_i}^1(x_{v_j^{(i)}}) \prod_{v_j^{(i)}\in (S^{(i)})^c} \mathbb{h}_{I_i}^0(x_{v_j^{(i)}}) \Big) dx_{v_j^{(i)}} d\mathbb{x},$$

which can also be written as

$$\frac{1}{2^r}\sum_{Q'\in\mathscr{C}(Q)}[\mathbf{F}]_{H,(\emptyset),Q'}=\sum_{\substack{S^{(1)}\subseteq V^{(1)},\mathrm{card}(S^{(1)})\,\mathrm{even}\\ \cdots\\ S^{(r)}\subseteq V^{(r)},\mathrm{card}(S^{(r)})\,\mathrm{even}}}[\mathbf{F}]_{H,S,Q}\,.$$

Substracting the expression $[\mathbf{F}]_{H,(\emptyset),Q}$ from both sides of the equality gives the desired result. ■

Another useful result is the following lemma from [34], stated in a general notation instead of a notation using hypergraphs and selected vertices.

Lemma 2.2.3. For any $m\in\mathbb{N}$, any $I_1,\ldots,I_m\in\mathscr{C}_1$ and any nonnegative function $f:\mathbb{R}^m\to\mathbb{R}$, the expression

$$\sum_{S\subseteq\{1,\ldots,m\},\mathrm{card}(S)\,\mathrm{even}}\int_{\mathbb{R}^n}f(x_1,\ldots,x_m)\Big(\prod_{i\in S}\mathrm{h}^1_{I_i}(x_i)\prod_{i\in S^c}\mathrm{h}^0_{I_i}(x_i)\Big)d(x_1,\ldots,x_n)$$

is also nonnegative.

The next lemma follows from repeated application from Hölder's inequality throughout which we will be able to read the exact form of thresholds defined in 2.1. This is also the part where we will require for the connected subhypergraphs to be complete; more precisely, we will state the estimate for complete hypergraphs. The reader can compare it with the particular case $r=2$ appearing in [34].

Lemma 2.2.4. Let $H=(V,E)$ be a complete r-partite r-uniform labeled hypergraph. If $N:=\prod_{i=1}^r n_i$ and $M:=\max\left\{\frac{N}{n_1},\ldots,\frac{N}{n_r}\right\}=\max_{1\le i\le r}\prod_{\substack{1\le j\le r\\ j\neq i}}n_j$, then for any tuple $\mathbf{F}=(F_l)_{l\in L_E}$ of nonnegative measurable functions we have

$$\big[(F_{l_E(e)})_{e\in E}\big]_{H,(\emptyset),Q}\le\prod_{e\in E}\big[F^M_{l_E(e)}\big]_Q^{\frac{1}{M}}.$$

Proof. We can clearly assume that the edge labels are assigned via $l_E(e)=F_e$. Different labelling would simply account for repetition of the functions, while this is the most general case. We will prove the claim by the mathematical induction on $r\in\mathbb{N}$. The inequality for $r=1$ is trivial. Assume that, for some $r\ge 1$, we have the inequality from the statement of the lemma for each choice of a labeled r-partite r-uniform complete hypergraph $H_r=(V_r,E_r,l_{V_r})$ and for each choice of $(F_e)_{e\in E_r}$ and $Q_r:=\prod_{k=1}^r I_k$.

Let $H_{r+1} = (V_{r+1}, E_{r+1}, l_{V_{r+1}})$ be an arbitrary $(r+1)$-partite $(r+1)$-uniform complete hypergraph, let $(F_e)_{e\in E_{r+1}}$ be arbitrary and let $Q_{r+1} = \prod_{k=1}^{r+1} I_k \in \mathscr{C}_{r+1}$. Taken the $(r+1)$-partition $(V^{(i)})_{1\le i\le r+1}$ of the set of vertices V_{r+1} from the definition of the r-partite hypergraph, define $V_r := \cup_{i=1}^r V^{(i)}$, $E_r := \prod_{i=1}^r V^{(i)}$ and $l_{V_r} := l_{V_{r+1}}\big|_{V_r}$. Note that $H_r := (V_r, E_r, l_{V_r})$ is an r-partite r-uniform complete hypergraph, constructed from H_{r+1} by removing the last part of the partition of vertices in H_r, by shortening its edges and by removing duplicate edges. Also, let $N_r := \prod_{i=1}^r n_i, N_{r+1} := \prod_{i=1}^{r+1} n_i = N_r n_{r+1}, M_r := \max\left\{\frac{N_r}{n_1},\dots,\frac{N_r}{n_r}\right\}$ and $M_{r+1} := \max\left\{\frac{N_{r+1}}{n_1},\dots,\frac{N_{r+1}}{n_{r+1}}\right\} = \max\{M_r n_{r+1}, N_r\}$. Also, for $e = (v_{i_1}^{(1)},\dots,v_{i_{r+1}}^{(r+1)}) \in E$, let $e' := (v_{i_1}^{(1)},\dots,v_{i_r}^{(r)})$. To additionally simplify and make the expressions as clearer as possible, we will use a different notation for the Haar function, writing it as $\mathbb{h}^0_{x_{v_{i_{r+1}}^{(r+1)}}\in I_{r+1}}$, to emphasize in which variable we evaluate the function. With that, let $\mathbb{h}^0_{V_r} := \prod_{i=1}^r \prod_{v^{(i)}\in V^{(i)}} \mathbb{h}^0_{x_{v^{(i)}}\in I_i}$. We have

$$
\begin{aligned}
&\left[(F_e)_{e\in E_{r+1}}\right]_{H_{r+1},(\emptyset),Q_{r+1}} \\
&= \Big\| \big\| \prod_{e'\in E_r} \prod_{v_{i_{r+1}}^{(r+1)}\in V^{(r+1)}} F_{e'\cup\{v_{i_{r+1}}^{(r+1)}\}} \big(\mathbb{h}^0_{x_{v_{i_{r+1}}^{(r+1)}}\in I_{r+1}}\big)^{\frac{1}{|E_r|}} \big\|_{\mathrm{L}^1(\mathbb{R}^{n_{r+1}})} \mathbb{h}^0_{V_r} \Big\|_{\mathrm{L}^1(\mathbb{R}^{n-n_{r+1}})} \\
&\le \Big\| \prod_{e'\in E_r} \big\| \prod_{v_{i_{r+1}}^{(r+1)}\in V^{(r+1)}} F_{e'\cup\{v_{i_{r+1}}^{(r+1)}\}} \big(\mathbb{h}^0_{x_{v_{i_{r+1}}^{(r+1)}}\in I_{r+1}}\big)^{\frac{1}{N_r}} \big\|_{\mathrm{L}^{N_r}(\mathbb{R}^{n_{r+1}})} \mathbb{h}^0_{V_r} \Big\|_{\mathrm{L}^1(\mathbb{R}^{n-n_{r+1}})} \qquad (2.13) \\
&= \Big\| \prod_{e'\in E_r} \prod_{v_{i_{r+1}}^{(r+1)}\in V^{(r+1)}} \big\| F_{e'\cup\{v_{i_{r+1}}^{(r+1)}\}} \big(\mathbb{h}^0_{x_{v_{i_{r+1}}^{(r+1)}}\in I_{r+1}}\big)^{\frac{1}{N_r}} \big\|_{\mathrm{L}^{N_r}(\mathbb{R})} \mathbb{h}^0_{V_r} \Big\|_{\mathrm{L}^1(\mathbb{R}^{n-n_{r+1}})} \\
&= \Big[\Big(\prod_{v_{i_{r+1}}^{(r+1)}\in V^{(r+1)}} \big\| F^{N_r}_{e'\cup\{v_{i_{r+1}}^{(r+1)}\}} \mathbb{h}^0_{x_{v_{i_{r+1}}^{(r+1)}}\in I_{r+1}} \big\|^{\frac{1}{N_r}}_{\mathrm{L}^1(\mathbb{R})} \Big)_{e'\in E_r} \Big]_{H_r,(\emptyset),Q_r} \\
&\le \prod_{e'\in E_r} \Big[\prod_{v_{i_{r+1}}^{(r+1)}\in V^{(r+1)}} \big\| F^{N_r}_{e'\cup\{v_{i_{r+1}}^{(r+1)}\}} \mathbb{h}^0_{x_{v_{i_{r+1}}^{(r+1)}}\in I_{r+1}} \big\|^{\frac{M_r}{N_r}}_{\mathrm{L}^1(\mathbb{R})} \Big]^{\frac{1}{M_r}}_{Q_r} \qquad (2.14) \\
&= \prod_{e'\in E_r} \Big\| \prod_{v_{i_{r+1}}^{(r+1)}\in V^{(r+1)}} \Big(\big\| F^{N_r}_{e'\cup\{v_{i_{r+1}}^{(r+1)}\}} \mathbb{h}^0_{x_{v_{i_{r+1}}^{(r+1)}}\in I_{r+1}} \big\|^{\frac{M_r}{N_r}}_{\mathrm{L}^1(\mathbb{R})} \big(\mathbb{h}^0_{V_r}\big)^{\frac{1}{n_{r+1}}} \Big) \Big\|^{\frac{1}{M_r}}_{\mathrm{L}^1(\mathbb{R}^{n-n_{r+1}})} \\
&\le \prod_{e'\in E_r} \prod_{v_{i_{r+1}}^{(r+1)}\in V^{(r+1)}} \Big\| \big\| F^{N_r}_{e'\cup\{v_{i_{r+1}}^{(r+1)}\}} \mathbb{h}^0_{x_{v_{i_{r+1}}^{(r+1)}}\in I_{r+1}} \big\|^{\frac{M_r n_{r+1}}{N_r}}_{\mathrm{L}^1(\mathbb{R})} \mathbb{h}^0_{V_r} \Big\|^{\frac{1}{M_r n_{r+1}}}_{\mathrm{L}^1(\mathbb{R}^{n-n_{r+1}})}. \qquad (2.15)
\end{aligned}
$$

Each of the L^1 or L^{N_r} norms are considered in those variables that appear in the expression; more precisely, in those variables appearing in all of the mentioned Haar functions.

The inequalities (2.13) and (2.15) follow by Theorem 1.1.3, while in (2.14) we used the assumption of the mathematical induction. In case that $M_r n_{r+1} \geq N_r$, using Theorem 1.1.1 for probabilistic measures and the convex function $x \longmapsto x^{\frac{M_r n_{r+1}}{N_r}}$, we can dominate the last term with

$$\prod_{e' \in E_r} \prod_{v^{(r+1)}_{i_{r+1}} \in V^{(r+1)}} \Big\| \big\| F^{M_r n_{r+1}}_{e' \cup \{v^{(r+1)}_{i_{r+1}}\}} \mathbb{h}^0_{x_{v^{(r+1)}_{i_{r+1}}} \in I_{r+1}} \big\|_{\mathrm{L}^1(\mathbb{R})} \mathbb{h}^0_{V_r} \Big\|^{\frac{1}{M_r n_{r+1}}}_{\mathrm{L}^1(\mathbb{R}^{n-n_{r+1}})}.$$

Otherwise, if $M_r n_{r+1} < N_r$, with Jensen's inequality for same function, which is concave in this case, the term is dominated with

$$\prod_{e' \in E_r} \prod_{v^{(r+1)}_{i_{r+1}} \in V^{(r+1)}} \Big\| \big\| F^{N_r}_{e' \cup \{v^{(r+1)}_{i_{r+1}}\}} \mathbb{h}^0_{x_{v^{(r+1)}_{i_{r+1}}} \in I_{r+1}} \big\|_{\mathrm{L}^1(\mathbb{R})} \mathbb{h}^0_{V_r} \Big\|^{\frac{1}{N_r}}_{\mathrm{L}^1(\mathbb{R}^{n-n_{r+1}})}.$$

Both expressions are equal to $\prod_{e \in E_{r+1}} \big[F_e^{M_{r+1}}\big]^{\frac{1}{M_{r+1}}}_{Q_{r+1}}$, so the desired inequality follows. ■

Without loss of generality, we will consider only those labeled hypergraphs for which the tuple $\big(\mathrm{card}(l_V^{-1}(L_1^{(i)})), \mathrm{card}(l_V^{-1}(L_2^{(i)})), \ldots, \mathrm{card}(l_V^{-1}(L_{m_i}^{(i)}))\big)$ is decreasing for each $i \in \{1, \ldots, r\}$; otherwise we could interchange the roles of the vertex labels (along with their copies) in a way that this becomes the decreasing tuple. That way we would operate with labeled hypergraphs with same set of vertices V and same set of vertex labels L_V, but with a different label function l_V. A family of such hypergraphs on the set of vertices $V = \cup_{i=1}^r V^{(i)}$ will be denoted by $\mathscr{H}_{(n_i)}$. We define

$$\mathscr{S} := \big\{ S = \big(S^{(i)}\big)_{1 \leq i \leq r} : S \neq (\emptyset) \text{ and, for all } i \in \{1, 2, \ldots, r\},\ S^{(i)} \subseteq V^{(i)} \\ \text{and } \mathrm{card}(S^{(i)}) \text{ is even} \big\}$$

Also, we define a binary relation $\preceq$ for hypergraphs $H, H' \in \mathscr{H}_{(n_i)}$ in the following way.

$$\begin{aligned} &H \preceq H' \iff \\ &\quad \big(\mathrm{card}(l_V^{-1}(L_1^{(1)})), \ldots, \mathrm{card}(l_V^{-1}(L_{m_1}^{(1)})); \ldots; \mathrm{card}(l_V^{-1}(L_1^{(r)})), \ldots, \mathrm{card}(l_V^{-1}(L_{m_r}^{(r)}))\big) \\ &\quad \geq \big(\mathrm{card}(l_V'^{-1}(L_1^{(1)})), \ldots, \mathrm{card}(l_V'^{-1}(L_{m_1}^{(1)})); \ldots; \mathrm{card}(l_V'^{-1}(L_1^{(r)})), \ldots, \mathrm{card}(l_V'^{-1}(L_{m_r}^{(r)}))\big), \end{aligned}$$

where we consider the latter relation on m-tuples to be a standard lexicographical order. We can notice that $(\mathscr{H}_{(n_i)}, \preceq)$ is a totally ordered finite set; therefore there exist minimal and maximal hypergraphs with respect to this relation.

In some of the proofs we will frequently change roles of vertices or switch labels of certain vertices. However, if the hypergraph that we are working with is not complete, it would be necessary to change the set of edges E, which might affect and change the family of functions $(F_e)_{e\in E}$ we are working with (by adding or removing certain functions) and therefore changing the forms. To avoid problems like these, we begin with a temporary additional assumption that the hypergraph must be complete, i.e. have all possible edges. Consequently, the numbers d_e are the same for all edges e and we write them simply as d. Later, while proving the proposition on boundedness on a finite convex tree, we will generalize the result for hypergraphs with any choice of edges. Moreover, we fix a finite convex tree $\mathscr{T}$. Any constants in the inequalities will be independent of the choice of that tree.

Finally, let us also, for a moment, assume that all functions constituting $\mathbf{F}$ are normalized so that

$$\max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})}[F_l^d]_Q^{1/d}=1.$$

for each $l\in L_E$. Later we will use homogeneity to remove this normalizing condition.

Lemma 2.2.5. For every complete r-partite r-regular hypergraph $H\in\mathscr{H}_{(n_i)}$ there exists an averaging paraproduct-type term $\mathscr{B}_{H,(\emptyset),Q}$ satisfying

$$\max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})}\mathscr{B}_{H,(\emptyset),Q}\lesssim_{(n_i)}1$$

and such that for every $\delta\in\langle 0,1\rangle$ and for every $Q\in\mathscr{C}_r$ the following inequality holds for some $C_{(n_i)}>0$:

$$|[\mathbf{F}]_{H,S,Q}|\leq\Box\mathscr{B}_{H,(\emptyset),Q}+C_{(n_i)}\delta^{-1}\sum_{\substack{H'\in\mathscr{H}_{(n_i)},H'\prec H\\ R\in\mathscr{S}}}|[\mathbf{F}]_{H',R,Q}|+C_{(n_i)}\delta\sum_{\substack{H'\in\mathscr{H}_{(n_i)},H'\succeq H\\ R\in\mathscr{S}}}|[\mathbf{F}]_{H',R,Q}|.$$

Proof. We will first cover the case for $k\in\{1,\dots,r\}$ and distinct $i,j\in\{1,\dots,n_k\}$ such that $l_V^{-1}(L_i^{(k)})\cap S^{(k)}\neq\emptyset$ and $l_V^{-1}(L_j^{(k)})\cap S^{(k)}\neq\emptyset$; without loss of generality, let $k=1$, $i=1$, $j=2$. Let $v_1\in l_V^{-1}(L_1^{(1)})\cap S^{(1)}$ and $v_2\in l_V^{-1}(L_2^{(1)})\cap S^{(1)}$. By separating products of functions depending on whether the edge $e\in E$ contains vertex v_1, vertex v_2 or none of them and then applying the inequality $|AB|\leq\frac{1}{2\delta}A^2+\frac{\delta}{2}B^2\leq\delta^{-1}A^2+\delta B^2$ for any $A,B\in\mathbb{R}$, we conclude that

$$|[\mathbf{F}]_{H,S,Q}|\leq\delta^{-1}[\mathbf{F}]_{H',R,Q}+\delta[\mathbf{F}]_{H'',R,Q}$$

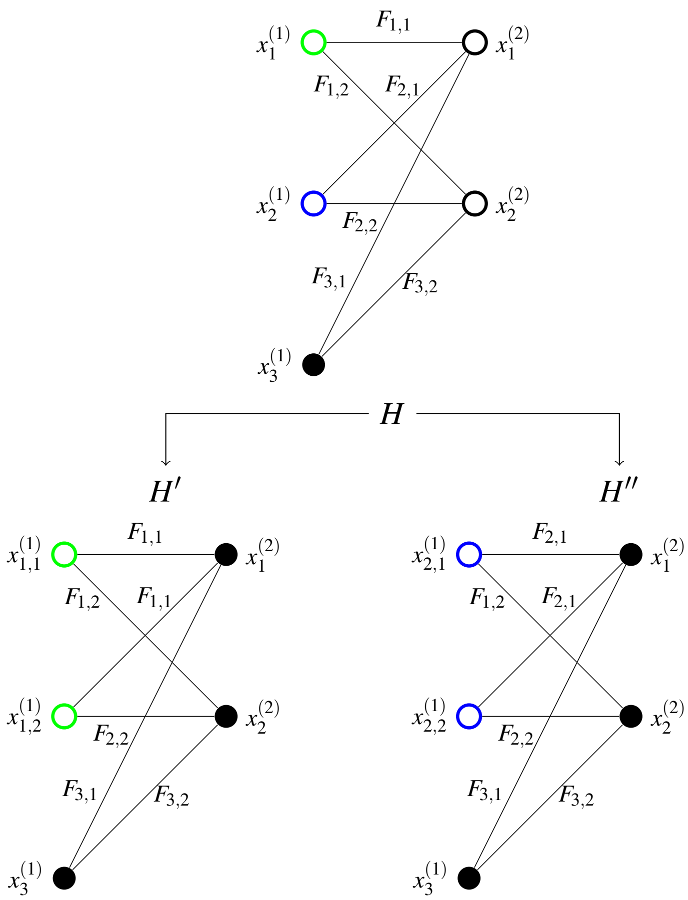

Figure 2.1: A construction of hypergraphs H' and H'' starting from the hypergraph H.

for labeled hypergraphs H', H'' and a tuple of subsets R defined in the following way. Starting with hypergraph H, let the label x_{v_2} be a copy of the label x_{v_1}, i.e. redefine $l_V(v_2)$ in a way that $l_V(v_2) \in L_1^{(1)} \backslash l_V(V^{(1)})$ (so that l_V remains an injective function). Also, remove all edges $e \in E$ for which $v_2 \in E$ and add edges $e' \in E$ which have $v_1 \in e'$, but with vertex v_2 instead of v_1. In analogous way we define labeled hypergraph H''. Intuitively, starting from the hypergraph H we removed one of vertices v_1 and v_2 and then we doubled the remaining vertex and its role. As for the sequence of subsets R, we take

$$R^{(1)} = \{v_1, v_2\}, \; R^{(k)} = \emptyset, k \geq 2.$$

This part of the proof is illustrated with the Figure 2.1.

Notice that $H' \prec H$ as the first different component from the definition of the relation $\prec$ got increased while constructing H'. On the other hand, it might happen that the tuple representing the number of times each vertex label appears for the hypergraph H'' did not decrease. In that case we will interchange the roles of the vertex labels according to our agreement before this lemma. With that agreement, it may still happen that $H'' \succeq H$ as well as $H'' \prec H$, in which case we use the inequality $\delta < \frac{1}{\delta}$, which is true for any $\delta \in \langle 0,1 \rangle$. The claim would then follow by adding the remaining terms $\delta^{-1}\,[\mathbf{F}]_{H',R,Q}$ or $\delta\,[\mathbf{F}]_{H',R,Q}$ to the whole expression. Note that $\mathscr{B}_{H,(\emptyset),Q} \equiv 0$ satisfies the first inequality required in the statement of the lemma.

The second case of possible hypergraphs H is when, for each $k \in \{1,\dots,r\}$, there exists at most one $i_k \in \{1,\dots,m_k\}$ such that $l_V^{-1}(L_{i_k}^{(k)}) \cap S^{(k)} \neq \emptyset$. Without loss of generality, let $l_V^{-1}(L_1^{(1)}) \cap S^{(1)} \neq \emptyset$; in that case, there exist distinct $v_1, v_2 \in l_V^{-1}(L_{i_1}^{(1)}) \cap S^{(1)}$. If we mark $S^{'(1)} := \{v_1, v_2\}$ and $S^{'(k)} = \emptyset$ for $k \geq 2$, we can notice that

$$|\,[\mathbf{F}]_{H,S,Q}\,| \leq [\mathbf{F}]_{H,S',Q}$$

Since it is enough to bound the expression for S', we will assume that S is already defined as S' above. Now, let $\mathscr{B}_{H,(\emptyset),Q} := [\mathbf{F}]_{H,(\emptyset),Q}$. Note that the first inequality in the statement of this lemma is satisfied by Lemma 2.2.4 and the normalization of the functions. By Theorem 2.2.2,

$$\Box\mathscr{B}_{H,(\emptyset),Q} = \sum_{R \in \mathscr{S}} [\mathbf{F}]_{H,R,Q}\,.$$

We will split the family $\mathscr{S}$ into three parts. For each $k \in \{1,\dots,r\}$ we define

$$\begin{aligned}
\mathscr{S}^{(1)} &:= \big\{S \in \mathscr{S} : (\exists! i_1 \in \mathbb{N}) l_V^{-1}(L_{i_1}^{(1)}) \cap S^{(1)} \neq \emptyset \wedge (\forall k \in \{2,\dots,r\})\, S^{(k)} = \emptyset\big\},\\
\mathscr{S}^{(2)} &:= \big\{S \in \mathscr{S} : (\exists k \in \{2,\dots,r\})(\exists! i_k \in \mathbb{N}) l_V^{-1}(L_{i_k}^{(k)}) \cap S^{(k)} \neq \emptyset\\
&\qquad \wedge \big(\forall k' \in \{k+1,\dots,r\}\big)\, S^{(k')} = \emptyset\big\},\\
\mathscr{S}^{(3)} &:= \big\{S \in \mathscr{S} : (\exists k \in \{1,\dots,r\})(\exists i_k, i'_k \in \mathbb{N}, i_k \neq i'_k)\; l_V^{-1}(L_{i_k}^{(k)}) \cap S^{(k)} \neq \emptyset\\
&\qquad \wedge l_V^{-1}(L_{i'_k}^{(k)}) \cap S^{(k)} \neq \emptyset \wedge (\forall k' \in \{k+1,\dots,r\}) S^{(k')} = \emptyset\big\}.
\end{aligned}$$

Notice that $\mathscr{S} = \dot\cup_{k=1}^{3} \mathscr{S}^{(k)}$ and that $S \in \mathscr{S}^{(1)}$. Take $R \in \mathscr{S}^{(1)}$; as each of the functions $F_e, e \in E$ is nonnegative, the only possible integration of negative function on a set of positive measure happens each time when the function $\mathbb{h}_I^1$ is involved, i.e. whenever we include the edge which consists a selected vertex. The only selected vertices appear in the set $S^{(1)}$ and all of them have the label of the form $x_{i_1,j_1}^{(1)}$ for even number of indices $j_1 \in \{1,\dots,n_1\}$. Notice that, no matter which of these variables we use to evaluate the integral expression, by the agreement in (1.20) and (1.21) we can separate the product

$\prod_{e\in E} F_e$ into equal products of the form $\prod_{v^{(1)}_{i_1,j_1}\in e\in E} F_e$. Therefore, by changing the order of the variables and separating the integral into more integrals, each of them having only one single variable of the form $x^{(1)}_{i_1,j_1}$, we get a product of same integral which appears an even amount of times. Having the same number to the power of the even natural number, we conclude that the whole expression is nonnegative. This works for any $R \in \mathscr{S}^{(1)}$, therefore

$$\sum_{R\in\mathscr{S}^{(1)}} [\mathbf{F}]_{H,R,Q} \geq [\mathbf{F}]_{H,S,Q}\,.$$

Now, let $k \in \{2,\dots,r\}$. For a moment, we will consider a labeled $(r-k+1)$-partite $(r-k+1)$-uniform hypergraph H_k on $\prod_{i=k}^r V^{(i)}$ obtained from H in a way that we keep all vertices from vertex components $V^{(k)},\dots,V^{(r)}$ with same vertex labels and along with edges which are reduced by removing its vertices from disregarded vertex components $V^{(1)},\dots,V^{(k-1)}$. Also, if $Q = \prod_{i=1}^r I_i$, define $Q_k := \prod_{i=k}^r I_i$. Along with $S'^{(i)} := \emptyset$ for $i \in \{k+1,\dots,r\}$ and for fixed real numbers $\big((x_{v_i^{(k')}})_{\substack{1\leq k'\leq k-2\\ 1\leq i\leq n_{k'}}}\big)$, we define

$$f_{k-1}\big((x_{v_i^{(k-1)}})_{1\leq i\leq n_{k-1}}\big) := \sum_{\substack{S'^{(k)}\subseteq l_V^{-1}(L^{(k)}_{i_k})\cap S^{(k)}\\ S'^{(k)}\neq\emptyset \text{ and } \mathrm{card}(S'^{(k)}) \text{ is even}}} [\mathbf{F}]_{H_k,(S'^{(i)})_{k\leq i\leq r},Q_k}\,.$$

The expression in the definition of this function is a sum of integral expressions containing the variables $x_{v_i^{(k')}}$ for each $k' \in \{1,\dots,r\}$ and $i \in \{1,\dots,n_{k'}\}$, integrating in each variable when $k' \geq k+1$. The function in though of as depending on the independent variables corresponding to $k' = k$ while the other variables for $k' \leq k-1$ are, at this moment, regarded as constants. Similarly as before, this function is nonnegative, so if we apply Lemma 2.2.3 to function f_{k-1}, we can conclude that the function

$$f_{k-2}\big((x_{v_i^{(k-2)}})_{1\leq i\leq n_{k-2}}\big) := \sum_{\substack{S'^{(k-1)}\subseteq V^{(k-1)}\\ \mathrm{card}(S'^{(k-1)}) \text{ is even}}} \sum_{\substack{S'^{(k)}\subseteq l_V^{-1}(L^{(k)}_{i_k})\cap S^{(k)}\\ S'^{(k)}\neq\emptyset \text{ and } \mathrm{card}(S'^{(k)}) \text{ is even}}} [\mathbf{F}]_{H_{k-1},(S'^{(i)})_{k-1\leq i\leq r},Q_{k-1}}$$

is also nonnegative, where H_{k-1} is a $(r-k+2)$-partite $(r-k+2)$-uniform hypergraph on $\prod_{i=k-1}^r V^{(i)}$ and $Q_{k_1} := \prod_{i=k-1}^r I_i$, defined analogously as H_k and Q_k before. Continuing to apply Lemma 2.2.3 to each class of variables until we reach last function f_2, in variables $x_{v_1^{(1)}},\dots,x_{v_{n_1}^{(1)}}$, we conclude that

$$\sum_{R\in\mathscr{S}^{(2)}} [\mathbf{F}]_{H,R,Q} \geq 0.$$

The case of $R \in \mathscr{S}^{(3)}$ is covered as the first case of this proof, from which follows that

$$|[\mathbf{F}]_{H,R,Q}| \le \delta^{-1} \sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \prec H \\ R' \in \mathscr{S}}} |[\mathbf{F}]_{H',R',Q}| + \delta \sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \succeq H \\ R' \in \mathscr{S}}} |[\mathbf{F}]_{H',R',Q}|.$$

Combining all these cases, we can conclude that

$$\begin{aligned}
[\mathbf{F}]_{H,S,Q} \le & \sum_{R \in \mathscr{S}^{(1)}} [\mathbf{F}]_{H,R,Q} + \sum_{R \in \mathscr{S}^{(2)}} [\mathbf{F}]_{H,R,Q} + \sum_{R \in \mathscr{S}^{(3)}} [\mathbf{F}]_{H,R,Q} - \sum_{R \in \mathscr{S}^{(3)}} [\mathbf{F}]_{H,R,Q} \\
\le & \Box \mathscr{B}_{H,(\emptyset),Q} + \sum_{R \in \mathscr{S}^{(3)}} \Bigg(\delta^{-1} \sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \prec H \\ R' \in \mathscr{S}}} |[\mathbf{F}]_{H',R',Q}| + \delta \sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \succeq H \\ R' \in \mathscr{S}}} |[\mathbf{F}]_{H',R',Q}| \Bigg) \\
= & \Box \mathscr{B}_{H,(\emptyset),Q} + C_{(n_i)} \delta^{-1} \sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \prec H \\ R' \in \mathscr{S}}} |[\mathbf{F}]_{H',R',Q}| + C_{(n_i)} \delta \sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \succeq H \\ R' \in \mathscr{S}}} |[\mathbf{F}]_{H',R',Q}|
\end{aligned}$$

for $C_{(n_i),\delta} := \operatorname{card}(\mathscr{S}^{(3)})$, which is the claim of this lemma. ■

Lemma 2.2.6. For every r-partite r-regular complete hypergraph H and for every $\varepsilon \in \langle 0,1 \rangle$ there exist an averaging paraproduct-type expression $\mathscr{B}^{\varepsilon}_{H,(\emptyset)}$ satisfying

$$\max_{Q \in \mathscr{T} \cup \mathscr{L}(\mathscr{T})} \mathscr{B}^{\varepsilon}_{H,(\emptyset),Q} \lesssim_{(n_i),\varepsilon} 1$$

and

$$\sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \preceq H \\ S \in \mathscr{S}}} |[\mathbf{F}]_{H',S,Q}| \le \Box \mathscr{B}^{\varepsilon}_{H,(\emptyset),Q} + \varepsilon \sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \succ H \\ S \in \mathscr{S}}} |[\mathbf{F}]_{H',S,Q}|.$$

Proof. As the totally ordered set $(\mathscr{H}_{(n_i)}, \preceq)$ is finite, we will prove this claim by induction over the hypergraphs from this family. Before we begin, let $H \in \mathscr{H}_{(n_i)}$ be arbitrary non-maximal hypergraph and let H_s be an immediate successor of H. Let $C_{(n_i)}$ be as in Lemma 2.2.5. Suppose that there exists an averaging paraproduct-term $\mathscr{B}^{\varepsilon'}_{H,(\emptyset),Q}$ such that

$$\begin{aligned}
\sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \preceq H \\ S \in \mathscr{S}}} |[\mathbf{F}]_{H',S,Q}| \le & \Box \mathscr{B}^{\varepsilon'}_{H,(\emptyset),Q} \\
& + \left(\frac{\varepsilon}{4C_{(n_i)} \operatorname{card}(\mathscr{H}_{(n_i)}) \operatorname{card}(V)} \right)^2 \sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \succ H \\ S \in \mathscr{S}}} |[\mathbf{F}]_{H',S,Q}|, \qquad (2.16)
\end{aligned}$$

where $\varepsilon' := \left(\frac{\varepsilon}{4C_{(n_i)} \operatorname{card}(\mathscr{H}_{(n_i)}) \operatorname{card}(V)} \right)^2$ and $\varepsilon \in \langle 0,1 \rangle$ is arbitrary. Applying Lemma 2.2.5 for every $H' \in \mathscr{H}_{(n_i)}, H' \le H_s$ and $\delta := \frac{\varepsilon}{4C_{(n_i)} \operatorname{card}(\mathscr{H}_{(n_i)}) \operatorname{card}(V)}$, we have

$$\sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \preceq H_s \\ S \in \mathscr{S}}} \left| [\mathbf{F}]_{H',S,Q} \right| \le \sum_{\substack{H' \in \mathscr{H}_{(n_i)}, H' \preceq H_s \\ S \in \mathscr{S}}} \Box \mathscr{B}_{H',(\emptyset),Q}$$

$$+\frac{4C_{(n_i)}^2\operatorname{card}(\mathscr{H}_{(n_i)})^2\operatorname{card}(V)^2}{\varepsilon}\sum_{\substack{H''\in\mathscr{H}_{(n_i)},H''\preceq H\\ R\in\mathscr{S}}}|[\mathbf{F}]_{H'',R,Q}|+\frac{\varepsilon}{4}\sum_{\substack{H''\in\mathscr{H}_{(n_i)}\\ R\in\mathscr{S}}}|[\mathbf{F}]_{H'',R,Q}|$$

$$\overset{(2.16)}{\leq}\sum_{\substack{H'\in\mathscr{H}_{(n_i)},H'\preceq H_s\\ S\in\mathscr{S}}}\Box\mathscr{B}_{H',(\emptyset),Q}+\frac{4C_{(n_i)}^2\operatorname{card}(\mathscr{H}_{(n_i)})^2\operatorname{card}(V)^2}{\varepsilon}\Box\mathscr{B}^{\varepsilon'}_{H,(\emptyset),Q}+\frac{\varepsilon}{2}\sum_{\substack{H'\in\mathscr{H}_{(n_i)}\\ S\in\mathscr{S}}}|[\mathbf{F}]_{H',S,Q}|$$

$$\leq\sum_{\substack{H'\in\mathscr{H}_{(n_i)},H'\preceq H_s\\ S\in\mathscr{S}}}\Box\mathscr{B}_{H',(\emptyset),Q}+\frac{4C_{(n_i)}^2\operatorname{card}(\mathscr{H}_{(n_i)})^2\operatorname{card}(V)^2}{\varepsilon}\Box\mathscr{B}^{\varepsilon'}_{H,(\emptyset),Q}$$

$$+\frac{\varepsilon}{2}\sum_{\substack{H'\in\mathscr{H}_{(n_i)},H'\succ H_s\\ S\in\mathscr{S}}}|[\mathbf{F}]_{H',S,Q}|+\frac{1}{2}\sum_{\substack{H'\in\mathscr{H}_{(n_i)},H'\preceq H_s\\ S\in\mathscr{S}}}|[\mathbf{F}]_{H',S,Q}|, \tag{2.17}$$

where we used $\operatorname{card}(\mathscr{S}_{H'})\leq\operatorname{card}(V)$ for any $H'\in\mathscr{H}_{(n_i)}$. Moving the last sum on the left side of the inequality and multiplying the inequality by 2, we get

$$\sum_{\substack{H'\in\mathscr{H}_{(n_i)},H'\preceq H_s\\ S\in\mathscr{S}}}|[\mathbf{F}]_{H',S,Q}|\leq\Box\mathscr{B}^{\varepsilon}_{H_s,(\emptyset),Q}+\varepsilon\sum_{\substack{H'\in\mathscr{H}_{(n_i)},H'\succ H_s\\ S\in\mathscr{S}}}|[\mathbf{F}]_{H',S,Q}|, \tag{2.18}$$

with additional notation

$$\mathscr{B}^{\varepsilon}_{H_s,(\emptyset),Q}:=2\sum_{\substack{H'\in\mathscr{H}_{(n_i)},H'\preceq H_s\\ S\in\mathscr{S}}}\mathscr{B}_{H',(\emptyset),Q}+\frac{8C_{(n_i)}^2\operatorname{card}(\mathscr{H}_{(n_i)})^2\operatorname{card}(V)^2}{\varepsilon}\mathscr{B}^{\varepsilon'}_{H,(\emptyset),Q},$$

which is an averaging paraproduct-type expression.

Now we proceed to the induction. The induction basis for the minimal hypergraph H_m is actually (2.18) with $H_s=H_m$ and it follows from (2.17), where, instead of (2.16) (we cannot refer to it as H_m does not have preceding elements), we use a trivial inequality

$$0\leq\Box\mathscr{B}^{\varepsilon'}_{H_m,(\emptyset),Q}+\left(\frac{\varepsilon}{4C_{(n_i)}\operatorname{card}(\mathscr{H}_{(n_i)})\operatorname{card}(V)}\right)^2\sum_{H'\in\mathscr{H}_{(n_i)}}|[\mathbf{F}]_{H',S,Q}|$$

for $\mathscr{B}^{\varepsilon'}_{H_m,(\emptyset),Q}:=0$, which trivially satisfies the required bound. Suppose that the claim of the lemma is satisfied for certain $H\in\mathscr{H}_{(n_i)}$, i.e. we have (2.16). Then the same claim follows from its successor H_s, which is actually (2.18), with $\mathscr{B}^{\varepsilon}_{H_s,(\emptyset),Q}$, which also satisfies the required bound by mathematical induction. With this, the required mathematical induction is complete. ■

Lemma 2.2.7. For every r-partite r-regular complete hypergraph H there exists an averaging paraproduct-type expression $\mathscr{B}_{H,(\emptyset),Q}$ satisfying

$$\max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})} \mathscr{B}_{H,(\emptyset),Q} \lesssim_{(n_i)} 1$$

and

$$\sum_{\substack{H'\in\mathscr{H}_{(n_i)}\\ S\in\mathscr{S}}} |\,[\mathbf{F}]_{H',S,Q}\,| \leq \square\mathscr{B}_{H,(\emptyset),Q}.$$

Proof. As discussed while defining the totally ordered set $(\mathscr{H}_{(n_i)},\preceq)$, there exists a maximal hypergraph H_M. The claim of this lemma follows from previous lemma by applying it for any fixed $\varepsilon\in\langle 0,1\rangle$ and then by using $\mathscr{B}_{H,(\emptyset),Q} := \mathscr{B}^{\varepsilon}_{H_M,(\emptyset),Q}$ ■

For each tuple of functions $\mathbf{F}$ and each finite convex tree $\mathscr{T}$ we define

$$\Lambda_{\mathscr{T}}(\mathbf{F}) := \sum_{Q\in\mathscr{T}} |Q|\,[\mathbf{F}]_{H,S,Q}$$

where $H=(V,E)$ is any r-partite r-uniform labeled hypergraph and $S=(S^{(i)})_{1\leq i\leq r}$ is a tuple such that $S^{(i)}\subseteq V^{(i)}$ for each $i\in\{1,\ldots,r\}$ and there exists $i_0\in\{1,\ldots,r\}$ such that $\mathrm{card}(S^{(i_0)})\geq 2$.

Lemma 2.2.8. Let $H=(V,E)$ be a r-regular r-uniform complete labeled hypergraph and let $\mathscr{T}$ be a finite convex tree. Suppose that for each $Q\in\mathscr{T}$ there exists an averaging paraproduct-type term $\mathscr{B}_{H,(\emptyset),Q}$ such that

$$|[\mathbf{F}]_{H,S,Q}| \leq \square\mathscr{B}_{H,(\emptyset),Q} \text{ and } \max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})} \mathscr{B}_{H,(\emptyset),Q} \lesssim_{(n_i)} 1.$$

Then,

$$|\Lambda_{\mathscr{T}}(\mathbf{F})| \lesssim_{(n_i)} |Q_{\mathscr{T}}|.$$

Proof. We have

$$\begin{aligned}
|\Lambda_{\mathscr{T}}(\mathbf{F})| &\leq \sum_{Q\in\mathscr{T}} |Q|\,|\,[\mathbf{F}]_{H,S,Q}\,| \leq \sum_{Q\in\mathscr{T}} |Q|\square\mathscr{B}_{H,(\emptyset),Q}\\
&= \sum_{Q\in\mathscr{T}}\Big(\sum_{Q'\in\mathscr{C}(Q)} |Q'|\mathscr{B}_{H,(\emptyset),Q'} - |Q|\mathscr{B}_{H,(\emptyset),Q}\Big)\\
&= \sum_{Q\in\mathscr{L}(\mathscr{T})} |Q|\mathscr{B}_{H,(\emptyset),Q} - |Q_{\mathscr{T}}|\mathscr{B}_{H,(\emptyset),Q_{\mathscr{T}}} \lesssim_{(n_i)} \sum_{Q\in\mathscr{L}(\mathscr{T})} |Q| \leq |Q_{\mathscr{T}}|,
\end{aligned}$$

where we also used that the averaging paraproduct-type term, given nonnegative functions $\mathbf{F}$, is also nonnegative. ■

Proposition 2.2.9. Let $H=(V,E)$ be a r-regular r-uniform labeled hypergraph such that its label function l_E is injective and, more explicitly, $l_E(e)=F_e$ for each $e\in E$. For any finite convex tree $\mathscr{T}$ with root $Q_{\mathscr{T}}$ we have

$$|\Lambda_{\mathscr{T}}(\mathbf{F})| \lesssim_{(n_i)} |Q_{\mathscr{T}}| \prod_{e\in E} \max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})} [F_e^{d_e}]_Q^{\frac{1}{d_e}}. \tag{2.19}$$

Proof. First, we will prove the proposition in the special case when $E=\prod_{i=1}^r\{v_1^{(i)}, \dots, v_{n_i}^{(i)}\}$, i.e. for a complete r-uniform hypergraph. In that case the hypergraph is connected and for every $e\in E$ the number d_e is same for each edge $e\in E$. First, notice that it will be enough to prove the claim of the proposition with additional assumptions

$$|Q_{\mathscr{T}}|=1 \text{ and } \max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})} [F_e^{d_e}]_Q^{\frac{1}{d_e}} = 1 \text{ for each } e\in E,$$

in which case we need to prove

$$\Lambda_{\mathscr{T}}(\mathbf{F}) \lesssim_{(n_i)} 1.$$

Starting with an arbitrary finite convex tree $\mathscr{T}$ and functions $(F_e)_{e\in E}$, let $l\in\mathbb{Z}$ be such that $|Q_{\mathscr{T}}|=2^l$. Notice that Λ is invariant under dyadic dilations, in a way that the following relation is satisfied:

$$\Lambda_{\mathscr{T}}\left((D_{2^l}F_e)_{e\in E}\right) = 2^{lr}\Lambda_{\mathscr{T}_l}(\mathbf{F}),$$

where, for any $l\in\mathbb{Z}$ and any function $f:\mathbb{R}^m\to\mathbb{R}$, we define $(D_{2^l}f)(x_1,\dots,x_m):= f\left(\frac{x_1}{2^l},\dots,\frac{x_m}{2^l}\right)$ for $(x_1,\dots,x_m)\in\mathbb{R}^m$ and where $\mathscr{T}_l$ is a finite convex tree such that each $Q\in\mathscr{T}$ is replaced with

$$Q_l = 2^{-l}Q = \{2^{-l}x : x\in Q\}.$$

We can also see that

$$|Q_{\mathscr{T}}|\prod_{e\in E}\max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})}[(D_{2^l}F_e)^{d_e}]_Q^{\frac{1}{d_e}} = 2^{lr}|Q_{\mathscr{T}_l}|\prod_{e\in E}\max_{Q\in\mathscr{T}_l\cup\mathscr{L}(\mathscr{T}_l)}[F_e^{d_e}]_Q^{\frac{1}{d_e}} = 2^{lr}.$$

It follows that

$$\begin{aligned}\Lambda_{\mathscr{T}}(\mathbf{F}) &= \Lambda_{\mathscr{T}}\left((D_{2^l}(D_{2^{-l}}F_e))_{e\in E}\right) = 2^{lr}\Lambda_{\mathscr{T}_l}\left((D_{2^{-l}}F_e)_{e\in E}\right) \lesssim_{(n_i)} 2^{lr} \\ &= |Q_{\mathscr{T}}|\prod_{e\in E}\max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})}\left[(D_{2^l}F_e)^{d_e}\right]_Q^{\frac{1}{d_e}}.\end{aligned}$$

This proves the claim for all finite convex trees $\mathscr{T}$, but still requiring the assumption

$$\max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})}[F_e^{d_e}]_Q^{\frac{1}{d_e}} = 1.$$

For arbitrary functions $(F_e)_{e\in E}$, either we have that $F_e \equiv 0$ for some $e \in E$, in which case the claim of the proposition trivially follows, or $M := \min_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})} \big[F_e^{d_e}\big]_Q^{\frac{1}{d_e}} > 0$. We can see that $\big(\frac{1}{M}F_e\big)_{e\in E}$ satisfies this assumption, therefore

$$\begin{aligned}\Lambda_{\mathscr{T}}((F_e)_{e\in E}) &= \Lambda_{\mathscr{T}}\big((M\cdot\frac{1}{M}F_e)_{e\in E}\big) = M^{|E|}\Lambda_{\mathscr{T}}\big((\frac{1}{M}F_e)_{e\in E}\big)\\ &\lesssim_{(n_i)} M^{|E|}\,|Q_{\mathscr{T}}|\prod_{e\in E}\max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})}\big[(\frac{1}{M}F_e)^{d_e}\big]_Q^{\frac{1}{d_e}} = |Q_{\mathscr{T}}|\prod_{e\in E}\max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})}\big[F_e^{d_e}\big]_Q^{\frac{1}{d_e}}.\end{aligned}$$

Now we start with the proof in the general case. We are required to dominate each term $\big[(F_e)_{e\in E}\big]_{H,S,Q}$, $Q\in\mathscr{T}$, from the definition of $\Lambda_{\mathscr{T}}$. First, notice that we do not necessarily have $S\in\mathscr{S}$. However, if we, without loss of generality, suppose that $\max_{i\in\mathbb{N}}\mathrm{card}(S^{(1)})\geq 2$ and take $v_{i_1},v_{i_2}\in S^{(1)}$ for some $i_1\neq i_2$, then using the Cauchy-Schwarz inequality we obtain

$$[\mathbf{F}]_{H,S,Q} \leq \frac{1}{2}[\mathbf{F}]_{H_1,S_1,Q}+\frac{1}{2}[\mathbf{F}]_{H_2,S_2,Q}$$

for hypergraphs H_1 and H_2 and tuples of selected vertices S_1 and S_2 defined in the following way. For each $j,j'\in\{1,2\}, j\neq j'$, a hypergraph H_j has the label function $l_V^j\big|_{V\setminus\{v_{i_{j'}}\}} := l_V\big|_{V\setminus\{v_{i_{j'}}\}}$ and $l_V^j(v_{i_{j'}})$ belongs to the same set $L_k^{(1)}$ as $l_V(v_{i_j})$, but does not take the exact same value. Also, $S_1^{(1)}=S_2^{(1)}:=\{v_1,v_2\}$ and $S_1^{(i)}=S_2^{(i)}:=\emptyset$ for $i\in\{2,\ldots,r\}$. We can see that $S_1,S_2\in\mathscr{S}$. For $Q\in\mathscr{C}_r$, let $\mathscr{B}_{H,(\emptyset),Q}$ be as in Lemma 2.2.7. Applying Lemma 2.2.8 and using the bound from Lemma 2.2.7, we have

$$\Lambda_{\mathscr{T}}(\mathbf{F})\leq\frac{1}{2}\sum_{Q\in\mathscr{T}}|Q|\big([\mathbf{F}]_{H_1,S_1,Q}+[\mathbf{F}]_{H_2,S_2,Q}\big)\lesssim_{(n_i)}|Q_{\mathscr{T}}|=1.$$

Now suppose that we are given an arbitrary set of edges E. It might happen that the hypergraph H contains isolated vertices, i.e. those that are not elements of any edge. If v is an isolated vertex, then, by the definition of $[\mathbf{F}]_{H,S,Q}$ and the injectivity of l_V, the variable $l_V(v)$ will appear either in the expression $\mathbb{h}_I^1(l_V(v))$ or in $\mathbb{h}_I^0(l_V(v))$ for some $I\in\mathscr{C}_1$. In first case, integrating by that variable we get $[\mathbf{F}]_{H,S,Q}=0$, while in the other case, since the integral of function $\mathbb{h}_{I_k}$ equals one, the expression remains the same if we leave out that variable (and the vertex) from the expression. Therefore, isolated vertices give no significant contribution to the expression to $\Lambda_{\mathscr{T}}$, so we may assume that there exist $k\in\mathbb{N}$ and connected components $\prod_{i=1}^r V_j^{(i)}$ for each $j\in\{1,\ldots,k\}$ with no isolated vertices. Notice that each number d_e depends on which of the components the edge e belongs to, so we will also denote that number as $d^{(j)}$, where $j \in \{1,\dots,k\}$ is such that $e \in \prod_{i=1}^r V_j^{(i)}$. We can also suppose that these k components form complete r-partite r-uniform graphs by adding missing edges from the set $\cup_{j=1}^k \prod_{i=1}^r V_j^{(i)}$ and, for those edges e, defining $F_e \equiv 1$. For each $j \in \{1,\dots,k\}$, let $H_1,\dots,H_k$ be the r-partite r-uniform complete hypergraphs representing connected components of the hypergraph H; also, let $S_j = (S_j^{(i)})_{1\le i\le r}$ be defined as $S_j^{(i)} := S^{(i)} \cap V_j^{(i)}$ for each $i \in \{1,\dots,r\}$ and $j \in \{1,\dots,k\}$. With the additional notation of $\mathbf{F}_E = (F_e)_{e\in E}$ and $\mathbf{F}_{E_j} = (F_e)_{e\in E_j}$ for each $j = 1,\dots,k$ we can notice that

$$\Lambda_{\mathscr{T}}(\mathbf{F}_E) = \sum_{Q\in\mathscr{T}} |Q| \prod_{j=1}^k [\mathbf{F}_{E_j}]_{H_j,S_j,Q}.$$

The first case is when there exists $j \in \{1,\dots,k\}$ such that $\mathrm{card}(S_j^{(1)}) \ge 2$. We can apply this proposition to the hypergraph H_j as it belongs to the first case that we already covered. Therefore

$$\sum_{Q\in\mathscr{T}} |Q|[\mathbf{F}_{E_j}]_{H_j,S_j,Q} \lesssim_{(n_i)} |Q_{\mathscr{T}}| \prod_{e\in E_j} \max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})} [F_e^{d^{(j)}}]_Q^{\frac{1}{d^{(j)}}} = 1.$$

As for each $Q \in \mathscr{T}$ and each $j' \in \{1,\dots,k\}\backslash\{j\}$, applying Lemma 2.2.4 we get

$$|[\mathbf{F}_{E_{j'}}]_{H_{j'},S_{j'},Q}| \le [\mathbf{F}_{E_{j'}}]_{H_{j'},(\emptyset),Q} \le \prod_{e\in E_{j'}} [F_e^{d^{(j')}}]_Q^{\frac{1}{d^{(j')}}} \le 1.$$

It follows that

$$\Lambda_{\mathscr{T}}(\mathbf{F}_E) = \sum_{Q\in\mathscr{T}} |Q|[\mathbf{F}_{E_j}]_{H_j,S_j,Q} \prod_{\substack{1\le j'\le k\\ j'\ne j}} [\mathbf{F}_{E_{j'}}]_{H_{j'},S_{j'},Q} \lesssim_{(n_i)} 1,$$

which proves the claim of this proposition. The second case is when there exist $j_1, j_2 \in \{1,\dots,k\}$ such that $S_{j_1}^{(1)} \ne \emptyset \ne S_{j_2}^{(1)}$; without loss of generality, let $j_1 = 1$ and $j_2 = 2$. Using Lemma 2.2.4 in similar way as above, we can observe that

$$|\,[\mathbf{F}_E]_{H,S,Q}\,| \le |[\mathbf{F}_{E_1}]_{H_1,S_1,Q}||[\mathbf{F}_{E_2}]_{H_2,S_2,Q}| \le \frac{1}{2}[\mathbf{F}_{E_1}]_{H_1,S_1,Q}^2 + \frac{1}{2}[\mathbf{F}_{E_2}]_{H_2,S_2,Q}^2.$$

By changing the roles of the vertices let us assume that $V_1^{(i)} = \{v_1^{(i)},\dots,v_{l_i}^{(i)}\}$ for each $i \in \{1,\dots,r\}$ and that $v_1^{(1)} \in S_1^{(1)}$. If $d^{(1)} = 1$, i.e. if $l_1 = \cdots = l_r = 1$, then, for $\mathscr{B}_{H_1,(\emptyset),Q} := [F_{(1,\dots,1)}]_{H_1,(\emptyset),Q}^2$ we have

$$\Box\mathscr{B}_{H_1,(\emptyset),Q} = \sum_{\substack{R^{(1)}\subseteq\{v_1^{(1)}\}\\ \cdots\\ R^{(r)}\subseteq\{v_1^{(r)}\}\\ R=(R^{(i)})\ne(\emptyset)}} [F_{(1,\dots,1)}]_{H_1,R,Q}^2 \ge [F_{(1,\dots,1)}]_{H_1,S,Q}^2.$$

Note as well that

$$\max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})}\mathscr{B}_{H_1,(\emptyset),Q}(F_{(1,\ldots,1)})=\big(\max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})}[F_{(1,\ldots,1)}]_{H_1,(\emptyset),Q}\big)^2=1.$$

Analogously, we construct $\mathscr{B}_{H_2,(\emptyset),Q}:=[F_{(1,\ldots,1)}]^2_{H_2,(\emptyset),Q}$. The proof of the proposition is complete in this case after we apply Lemma 2.2.8 with $\mathscr{B}_{H,(\emptyset),Q}:=\frac{1}{2}\mathscr{B}_{H_1,S_1,Q}+\frac{1}{2}\mathscr{B}_{H_2,S_2,Q}$. In the case $l_1=1$ and $l_2\geq 2$, using Theorem 1.1.1 for the convex function $x\longmapsto x^2$ and the integral of type $\int_Q\frac{1}{|Q|}d\mathbb{x}$, we have that

$$[\mathbf{F}_{E_1}]^2_{H_1,S_1,Q}\leq[\mathbf{F}_{E_1'}]_{H_1',S_1',Q}.$$

Here, H_1' is the r-partite r-uniform complete hypergraph with set of vertices $V':=V\cup\{v_2^{(1)}\}$, set of edges

$$E_1':=E_1\cup\{\{v_2^{(1)}\}\cup(e\backslash\{v_1^{(1)}\}):v_1^{(1)}\in e\in E_1\}$$

and the label function $l_{V'}$ given with $l_{V'}\big|_V:=l_V$, while the value $l_{V'}(v_2^{(1)})$ can be chosen as an arbitrary copy of $x_{v_1^{(1)}}$, as long as $l_{V'}$ is an injective function. Also, $S_1'^{(1)}:=\{v_1^{(1)},v_2^{(1)}\}$ and $S_1'^{(i)}:=\emptyset$ for $i\in\{2,\ldots,r\}$; in short, we copied the single vertex $v_1^{(1)}$ from the first part of the r-partition along with the edges that contain that vertex and selected only those two vertices ($v_1^{(1)}$ with its copy) out of all vertices in the hypergraph. Notice that $S_1'\in\mathscr{S}$; we can apply Lemma 2.2.8 with $\mathscr{B}_{H_1',(\emptyset),Q}$ that we get from Lemma 2.2.7. It is important to notice that the numbers of vertices in each of the partition sets of the hypergraphs H_1' and H_2' have changed, therefore affecting the exponents d_e and possibly changing the range of possible exponents p_e while applying Lemma 2.2.8. However, this is not the case as we only increased l_1 by one (when adding $v_2^{(1)}$) and $d_e>l_1$, so the maximum from the definition of that exponent remains the same.

The remaining case is when $l_1\geq 2$. First we can bound

$$[\mathbf{F}_{E_1}]^2_{H_1,S_1,Q}\leq[\mathbf{F}_{E_1}]^2_{H_1,S_1',Q},$$

in a way that $S_1'=(\{v_1^{(1)}\},\emptyset,\ldots,\emptyset)$. Then we can group the integral expression depending on whether any function F_e or any of the Haar functions appear to be evaluated in the cancellative variable $x_{v_1^{(1)}}$, the non-cancellative variable $x_{v_2^{(1)}}$ or if it has none of these two variables. Then, by the application of arithmetic-geometric inequality and also by bounding the complete non-cancellative integral expression with 1, we get

$$[\mathbf{F}_{E_1}]^2_{H_1,S'_1,Q} \leq [\mathbf{F}_{E''_1}]_{H''_1,S''_1,Q}.$$

This time, H''_1 is the r-partite r-uniform complete hypergraph with a set of vertices $V'' := (V \cup \{v_1^{'(1)}\}) \backslash \{v_2^{(1)}\}$, a set of edges E''_1 given with $E''_1 := \{e \in E_1 : v_2^{(1)} \notin e\} \cup \{\{v_1^{'(1)}\} \cup (e \backslash \{v_1^{(1)}\}) : v_1^{(1)} \in e \in E_1\}$ and the label function $l_{V''}$ such that $l_{V''}\big|_V := l_V$ and $l_{V''}(v_1^{'(1)})$ is a copy of $x_1^{(1)}$, in a way that $l_{V'}$ is still an injective function. With that, $S_1^{''(1)} := \{v_1^{(1)}, v_1^{'(1)}\}$ and $S_1^{''(i)} := \emptyset$ for $i \in \{2, \dots, r\}$. In this case we copied the vertex $v_1^{(1)}$ along with the edges that contain it and left off $v_2^{(1)}$ with each edge that might contain it. The selected vertices are only the first, already selected, vertex $v_1^{(1)}$ along with its new copy $v_1^{'(1)}$. Note that, again, $S''_1 \in \mathscr{S}$, so we use Lemma 2.2.7 to get $\mathscr{B}_{H''_1,(\emptyset),Q}$ and then the result of the proposition follows by applying Lemma 2.2.8 again. As in the previous case, we can notice that the lemma is applied for the same number d_e as the number of vertices in each of the partition sets remains unchanged. ■

Now we are ready to prove our simplest boundedness result on the entangled forms.

Proof of Theorem 2.2.1. For each $N \in \mathbb{N}$ we define

$$\mathscr{C}^N := \Big\{ \prod_{i=1}^{r} I_i \in \mathscr{C}_r : |I_1| = \cdots = |I_r| \geq 2^{-N} \Big\}. \tag{2.20}$$

First we will add an additional assumption

$$\|F_e\|_{\mathrm{L}^{p_e}(\mathbb{R}^n)} = 1 \text{ for each } e \in E.$$

Let H be the r-partite r-uniform hypergraph with set of edges E and set of vertices V, consisting of all vertices appearing in any edge from E, along with the injective label functions l_V and l_E. Given a card(E)-tuple of integers $\Bbbk = (k_e)_{e \in E}$, let

$$\mathscr{C}^N_{\Bbbk} := \{Q \in \mathscr{C}^N : 2^{k_e} \leq \sup_{\substack{Q' \in \mathscr{C}^N \\ Q' \supseteq Q}} [F_e^{d_e}]^{\frac{1}{d_e}}_{Q'} < 2^{k_e+1} \text{ for each } e \in E\}.$$

Let $Q \in \mathscr{C}^N_{\Bbbk}$ and $e \in E$. By the definition of supremum, there exists $Q' \in \mathscr{C}^N, Q' \supseteq Q$ such that $[F_e^{d_e}]^{\frac{1}{d_e}}_{Q'} > 2^{k_e-1}$. It follows that

$$2^{k_e-1} < [F_e^{d_e}]^{\frac{1}{d_e}}_{Q'} \overset{(*)}{\leq} [F_e^{p_e}]^{\frac{1}{p_e}}_{Q'} = |Q'|^{-\frac{1}{p_e}} \|F_e\|_{\mathrm{L}^{p_e}(\mathbb{R}^n)} = |Q'|^{-\frac{1}{p_e}}.$$

The inequality (*) follows from the fact that the L^p norms are increasing with respect to a probability measure, which follows from 1.1.4. From this, we have that

$$|Q| \leq |Q'| \leq 2^{-p_e(k_e-1)}.$$

As we can bound the volume of each cube $Q \in \mathscr{C}^N_{\Bbbk}$ from above (with $2^{-p_e(\min_{e\in E} k_e-1)}$), there exist maximal elements of that collection; denote a collection of such maximal elements as $\mathscr{M}^N_{\Bbbk}$. Now, notice that, if $Q \in \mathscr{M}^N_{\Bbbk}$, then

$$\mathscr{T}_Q := \{Q' \in \mathscr{C}^N_{\Bbbk} : Q' \subseteq Q\}$$

is a finite convex tree, which follows from the monotonicity of the function $(\mathscr{C}^N_{\Bbbk}, \subseteq) \longrightarrow (\mathbb{R}, \leq)$ given with $Q'' \longmapsto \sup_{\substack{Q' \in \mathscr{C}^N \\ Q' \supseteq Q''}} [F_e^{d_e}]_{Q'}^{\frac{1}{d_e}}$. If $Q' \in \mathscr{L}(\mathscr{T}_Q)$ and Q_P is a parent of Q', then

$$[F_e^{d_e}]_{Q'}^{\frac{1}{d_e}} \leq 2^r [F_e^{d_e}]_{Q_P}^{\frac{1}{d_e}} < 2^{k_e+r+1}.$$

Note that, if $Q' \in \mathscr{T}_Q$, by definition of $\mathscr{C}^N_{\Bbbk}$, we have $[F_e^{d_e}]_{Q'}^{\frac{1}{d_e}} < 2^{k_e+1}$. By Proposition 2.2.9,

$$\Lambda_{\mathscr{T}_Q}(\mathbf{F}) \lesssim_{(n_i)} |Q| \prod_{e\in E} 2^{k_e+r+1} = |Q| 2^{\sum_{e\in E} k_e + (r+1)\operatorname{card}(E)},$$

since $Q_{\mathscr{T}_Q} = Q$. Note that, if $Q \in \mathscr{C}^N \backslash (\cup_{\Bbbk \in \mathbb{Z}^{|E|}} \mathscr{C}^N_{\Bbbk})$, then there exists $e \in E$ so that supremum from the definition of $\mathscr{C}^N_{\Bbbk}$ equals zero, meaning that $F_e \equiv 0$ a.e. on Q and therefore $[\mathbf{F}]_{H,S,Q} = 0$. Also, for each $\Bbbk \in \mathbb{Z}^{|E|}$ and for each $Q_1, Q_2 \in \mathscr{M}_{\Bbbk}, Q_1 \neq Q_2$, since they are dyadic cubes and also maximal elements of $\mathscr{C}^N_{\Bbbk}$, we have $Q_1 \cap Q_2 = \emptyset$, so trees $\mathscr{T}_{Q_1}$ and $\mathscr{T}_{Q_2}$ also cover disjoint parts of space $\mathbb{R}^r$. We have

$$\sum_{Q\in\mathscr{C}^N} |Q| \, [\mathbf{F}]_{H,S,Q} = \sum_{\Bbbk\in\mathbb{Z}^{\operatorname{card}(E)}} \sum_{Q\in\mathscr{M}_{\Bbbk}} \Lambda_{\mathscr{T}_Q}(\mathbf{F}) \lesssim_{(n_i)} \sum_{\Bbbk\in\mathbb{Z}^{\operatorname{card}(E)}} 2^{\sum_{e\in E} k_e + (r+1)\operatorname{card}(E)} \sum_{Q\in\mathscr{M}_{\Bbbk}} |Q|.$$

In this proof we will use the fact that the operator, given as

$$M_d F(x_1, \ldots, x_m) := \sup_{\substack{Q\in\mathscr{C}_m \\ (x_1,\ldots,x_m)\in Q}} [|F|^d]_Q^{\frac{1}{d}}, \tag{2.21}$$

is bounded as an operator from $L^p(\mathbb{R}^m)$ to $L^p(\mathbb{R}^m)$, for each $m \in \mathbb{N}$, $p \in [d, \infty]$, $d \in [1, \infty]$. This follows from the relation $M_d = \left(M_1 F^d\right)^{\frac{1}{d}}$ and Theorem 1.5.1; it remains to recognize that $M_{\text{dyadic}} = M_1$. For each $e \in E$ denote

$$\mathscr{K}_e := \{\Bbbk \in \mathbb{Z}^{\operatorname{card}(E)} : p_e k_e \geq p_f k_f \text{ for each } f \in E\}.$$

Notice that $\mathbb{Z}^{\mathrm{card}(E)} = \cup_{e\in E}\mathscr{K}_e$ and that, for any $Q \in \mathscr{C}^N_{\Bbbk}$ and $(x^{(1)},\dots,x^{(r)}) \in Q$,

$$M_{d_e}F_e(x^{(1)},\dots,x^{(r)}) \geq \sup_{\substack{Q'\in\mathscr{C}^N \\ Q'\supseteq Q}} [|F|^{d_e}]_{Q'}^{\frac{1}{d_e}} \geq 2^{k_e},$$

which gives us, for any $e \in E$,

$$\sum_{Q\in\mathscr{M}_{\Bbbk}} |Q| = \Big|\bigcup_{Q\in\mathscr{M}_{\Bbbk}} Q\Big| \leq |\{M_{d_e}F_e \geq 2^{k_e}\}|.$$

It follows that

$$\begin{aligned}
\sum_{Q\in\mathscr{C}^N} |Q|\,[\mathbf{F}]_{H,S,Q} &\lesssim_{(n_i),r,\mathrm{card}(E)} \sum_{f\in E}\sum_{\Bbbk\in\mathscr{K}_f} 2^{p_f k_f(1-\sum_{e\in E\setminus\{f\}}\frac{1}{p_e})+\sum_{e\in E\setminus\{f\}}k_e}|\{M_{d_f}F_f \geq 2^{k_f}\}| \\
&= \sum_{f\in E}\sum_{k_f\in\mathbb{Z}} 2^{p_f k_f}|\{M_{d_f}F_f \geq 2^{k_f}\}| \prod_{e\in E\setminus\{f\}} \sum_{\substack{k_e\in\mathbb{Z} \\ k_e\leq \frac{p_f k_f}{p_e}}} 2^{k_e-\frac{p_f k_f}{p_e}} \\
&\lesssim_{\mathrm{card}(E)} \sum_{f\in E}\sum_{k_f\in\mathbb{Z}} 2^{p_f k_f}|\{M_{d_f}F_f \geq 2^{k_f}\}| \overset{(**)}{\lesssim}_{(p_f)} \sum_{f\in E} \|M_{d_f}F_f\|^{p_f}_{L^{p_f}(\mathbb{R}^r)} \\
&\lesssim_{(d_f),(p_f)} \sum_{f\in E} \|F_f\|^{p_f}_{L^{p_f}(\mathbb{R}^r)} = \mathrm{card}(E) \lesssim_{\mathrm{card}(E)} 1.
\end{aligned}$$

In (**) we used the following trick (which works for any measurable function F_f):

$$\begin{aligned}
\sum_{k_f\in\mathbb{Z}} 2^{p_f k_f}|\{M_{d_f}F_f \geq 2^{k_f}\}| &= \sum_{k_f\in\mathbb{Z}} \int_{2^{k_f}}^{2^{k_f+1}} 2^{(p_f-1)k_f}|\{M_{d_f}F_f \geq 2^{k_f}\}|dt \\
&\leq 2^{p_f-1}\sum_{k_f\in\mathbb{Z}} \int_{2^{k_f}}^{2^{k_f+1}} t^{p_f-1}|\{M_{d_f}F_f > t\}|dt = \frac{2^{p_f-1}}{p_f}\int_0^{+\infty} p_f t^{p_f-1}|\{M_{d_f}F_f \geq t\}|dt \\
&= \frac{2^{p_f-1}}{p_f}\|M_{d_f}F_f\|^{p_f}_{L^{p_f}(\mathbb{R}^r)}.
\end{aligned}$$

Note that the value $\mathrm{card}(E)$ is bounded from above with a constant depending only on the sequence $(n_e)_{e\in E}$. From this, it follows that

$$\sum_{Q\in\mathscr{C}^N} |Q|\,[\mathbf{F}]_{H,S,Q} \lesssim_{(n_i),r,(p_e)} 1.$$

If $\mathbf{F}$ is a tuple of arbitrary functions with $F_e \not\equiv 0$ a.e. for every $e \in E$ and if we apply this inequality to functions $\big(\frac{F_e}{\|F_e\|_{L^{p_f}(\mathbb{R}^r)}}\big)_{e\in E}$, we get the result

$$\sum_{Q\in\mathscr{C}^N} |Q|\,[\mathbf{F}]_{H,S,Q} \lesssim_{(n_i),r,(p_e)} \prod_{e\in E} \|F_e\|_{L^{p_f}(\mathbb{R}^r)}.$$

Note that this inequality trivially holds if $F_e \equiv 0$ a.e. for some $e \in E$. As this inequality holds for any $N \in \mathbb{N}$ and as the left-hand side is increasing in N, letting $N \to +\infty$ we obtain the same result with the sum over $\cup_{N\in\mathbb{N}}\mathscr{C}^N = \mathscr{C}_r$; therefore

$$\Lambda_{E,S}(\mathbf{F}) = \sum_{Q\in\mathscr{C}_r} |Q|\,[\mathbf{F}]_{H,S,Q} \lesssim_{(n_i),r,(p_e)} \prod_{e\in E} \|F_e\|_{L^{p_f}(\mathbb{R}^r)}.$$

This completes the proof of our theorem. ■

2.3. Estimates for localized paraproducts

Let us return to the form Λ_E^S defined in (2.7). This form will be called *cancellative* if either

(C1) $\max_{1\le i\le r} \operatorname{card}(S^{(i)}) \ge 2$, or

(C2) $\max_{1\le i\le r} \operatorname{card}(S^{(i)}) = 1$ and there does not exist $l \in \{1,\dots,k\}$ such that $\cup_{1\le i\le r} S^{(i)} \subseteq V_l$.

Otherwise, it is *non-cancellative*, which means that

(NC) $\max_{1\le i\le r} \operatorname{card}(S^{(i)}) = 1$ and there exists $l \in \{1,\dots,k\}$ such that $\cup_{1\le i\le r} S^{(i)} \subseteq V_l$.

We can consider the cancellative form as the one consisting of (at least) two different variables that bring cancellation to the whole expression, but are not entangled in any way (so that those cancellations do not depend on or influence each other).

While trying to obtain certain estimate for Λ_E^S, again, first we are going to prove a certain bound locally, by taking the sum only over the dyadic cubes belonging to the certain finite convex tree $\mathscr{T}$. Therefore we define the localized version of the form Λ_E^S as

$$\Lambda_{E,\mathscr{T}}^S(\mathbf{F}) := \sum_{Q\in\mathscr{T}} |Q|\lambda_Q [\mathbf{F}]_{H,S,Q}. \tag{2.22}$$

Strictly speaking, we are slightly abusing the notation λ_Q, as it this coefficient is sometimes associated with a dyadic cube in $\mathbb{R}^r$ and sometimes with the corresponding "diagonal" dyadic cube in $\mathbb{R}^n$.

Proposition 2.3.1. Let Λ_E^S be a cancellative entangled dyadic paraproduct.

(a) If (2.8) holds, then for the corresponding coefficients $\lambda = (\lambda_Q)_{Q\in\mathscr{C}_r}$ we have

$$\|\lambda\|_{\ell^\infty(\mathscr{C}_r)} \lesssim 1.$$

(b) For a finite convex tree $\mathscr{T}$ and a localized cancellative entangled dyadic paraproduct $\Lambda_{E,\mathscr{T}}^S$ we have

$$|\Lambda_{E,\mathscr{T}}^S(\mathbf{F})| \lesssim \|\lambda\|_{\ell^\infty(\mathscr{C}_r)} |Q_{\mathscr{T}}| \prod_{e\in E} \max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})} [F_e^{d_e}]_Q^{\frac{1}{d_e}}.$$

Proof. (a) Let $Q = \prod_{i=1}^{r} I^{(i)} \in \mathscr{C}_r$. If we take $F_e := \mathbb{1}_Q = \otimes_{i=1}^{r} \mathbb{1}_{I^{(i)}}$ for each $e \in E$, then our form Λ_E takes the form

$$|\left\langle K, \otimes_{i=1}^{r} \otimes_{j=1}^{n_i} \mathbb{1}_{I^{(i)}} \right\rangle_{\mathrm{L}^2(\mathbb{R}^n)}| = |\Lambda_E(\mathbf{F})| \overset{(2.8)}{\lesssim} |Q|. \tag{2.23}$$

Notice that both the cancellative and the non-cancellative Haar function can be written in the form $\frac{1}{|I|}(\mathbb{1}_{I_L} \pm \mathbb{1}_{I_R})$ and, as left and right halves of each dyadic interval are mutually disjoint, we can bound λ_Q as

$$|\lambda_Q| \le |Q|^{-1} \sum_{\substack{1\le i\le r\\ 1\le j\le n_i}} \sum_{I_j^{(i)} \in \{I_L^{(i)}, I_R^{(i)}\}} |\langle K, \otimes_{i=1}^{r} \otimes_{j=1}^{n_i} \mathbb{1}_{I_j^{(i)}} \rangle_{\mathrm{L}^2(\mathbb{R}^n)}|.$$

In the 2^r cases when $I_1^{(i)} = \cdots = I_{n_i}^{(i)}$ for each $i \in \{1,\ldots,r\}$ we can apply (2.23) to obtain boundedness of each summand by a constant. To show the same bound for the remaining cases we can without loss of generality assume that, for a certain $k \in \{1,\ldots,r\}$, we have $I_1^{(i)} = I_L^{(i)}$ and $I_2^{(i)} = I_R^{(i)}$ for each $i \in \{1,\ldots,k\}$ and $I_1^{(i)} = \cdots = I_{n_i}^{(i)}$ for each $i \in \{k+1,\ldots,r\}$. Let $x_0^{(i)}$ be a common endpoint of $I_L^{(i)}$ and $I_R^{(i)}$ (i.e. a midpoint of $I^{(i)}$) for $i \in \{1,\ldots,k-1\}$ and let $x_j^{(i)} \in I^{(i)}$ for each $(i,j) \in \{1,\ldots,r\} \times \{1,\ldots,n_i\}$, $(i,j) \notin \{1,\ldots,k-1\} \times \{1,2\}$. Then for each $i \in \{1,\ldots,k-1\}$ we have

$$|x_1^{(i)} - x_2^{(i)}| = |x_1^{(i)} - x_0^{(i)}| + |x_0^{(i)} - x_2^{(i)}|,$$
$$|x_j^{(i)} - x_1^{(i)}| + |x_j^{(i)} - x_2^{(i)}| \ge |x_j^{(i)} - x_0^{(i)}| \text{ for each } j \in \{3,\ldots,n_i\}.$$

We can use this to bound the expression under the brackets on the right hand side of (2.2) from below with

$$\begin{aligned}\sum_{i=1}^{r} \sum_{1\le j_1<j_2\le n_i} |x_{j_1}^{(i)} - x_{j_2}^{(i)}| &\ge \sum_{i=1}^{k-1}\sum_{j=1}^{n_i} |x_j^{(i)} - x_0^{(i)}| + \sum_{j=2}^{n_k} |x_j^{(k)} - x_1^{(k)}| \\ &\ge \Big(\sum_{i=1}^{k-1}\sum_{j=1}^{n_i} |x_j^{(i)} - x_0^{(i)}|^2 + \sum_{j=2}^{n_k} |x_j^{(k)} - x_1^{(k)}|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Let $\mathbb{x}_0 := (\underbrace{x_0^{(1)},\ldots,x_0^{(1)}}_{n_1 \text{ times}},\ldots,\underbrace{x_0^{(k-1)},\ldots,x_0^{(k-1)}}_{n_{k-1} \text{ times}},\underbrace{x_1^{(k)},\ldots,x_1^{(k)}}_{n_k-1 \text{ times}})$. Note that

$$\Big(\prod_{i=1}^{k-1}\prod_{j=1}^{n_i} I_j^{(i)}\Big) \times \Big(\prod_{j=2}^{n_k} I_j^{(k)}\Big) \subseteq B(\mathbb{x}_0, n|I^{(1)}|),$$

where the latter set is a $(n_1+\cdots+n_k-1)$-dimensional ball with the center $\mathbb{x}_0$ and a radius $n|I_1^{(1)}|$. Using this, the inequality from above that we showed earlier and the integration in spherical coordinates, for all possible choices of $I_j^{(i)} \in \{I_L^{(i)}, I_R^{(i)}\}$, where $(i,j) \notin \{1,\dots,k-1\} \times \{1,2\}$, we get

$$\begin{aligned}
|Q|^{-1}\left|\int_{\mathbb{R}^n} K(\mathbb{x}) \otimes_{i=1}^r \otimes_{j=1}^{n_i} \mathbb{1}_{I_j^{(i)}}(\mathbb{x}) d\mathbb{x}\right| &\le |Q|^{-1} \int_{\prod_{i=1}^r \prod_{j=1}^{n_i} I_j^{(i)}} |K(\mathbb{x})| d\mathbb{x} \\
&\le |Q|^{-1} \int_{\prod_{i=k}^r I^{(i)}} \int_{B(\mathbb{x}_0, n|I^{(1)}|)} |K(\mathbb{x})| \Big(\prod_{i=1}^{k-1} \prod_{j=1}^{n_i} dx_j^{(i)} \cdot \prod_{j=2}^{n_k} dx_j^{(i)} \Big) \prod_{i=k}^r dx_1^{(i)} \\
&\le |I^{(1)}|^{-r} \int_{\prod_{i=k}^r I^{(i)}} \int_0^{n|I^{(1)}|} t^{r-n} \cdot t^{\sum_{i=1}^k n_i - 2} dt \prod_{i=k}^r dx_1^{(i)} \\
&\lesssim |I^{(1)}|^{-r} \cdot |I^{(1)}|^{k-1} \cdot |I^{(1)}|^{r-k+1} = 1.
\end{aligned}$$

Since the choice of $Q \in \mathscr{C}_r$ was arbitrary, we conclude $\|\lambda\|_{\ell^\infty(\mathscr{C}_r)} \lesssim 1$.

(b) Just as we showed at the beginning of the proof of Proposition 2.2.9, we can, without loss of generality, assume $|Q_{\mathscr{T}}| = 1$ and $\max_{Q \in \mathscr{T} \cup \mathscr{L}(\mathscr{T})} \big[F_e^{d_e}\big]_Q^{\frac{1}{d_e}} = 1$ for each $e \in E$. Also, notice that the result for the case (C1) already follows from Proposition 2.2.9, also using $|\lambda_Q| \le \|\lambda\|_{\ell^\infty(\mathscr{C}_r)}$ for each $Q \in \mathscr{T}$.

As for the case (C2), let H_1 and H_2 be the connected components of H such that each of them has at least one selected vertex. If there are k connected components altogether, we can estimate

$$\begin{aligned}
|\big[\mathbf{F}_E\big]_{H,S,Q}| = \prod_{l=1}^k |\big[\mathbf{F}_{E_l}\big]_{H_l,S_l,Q}| &\le |\big[\mathbf{F}_{E_1}\big]_{H_1,S_1,Q}| \cdot |\big[\mathbf{F}_{E_2}\big]_{H_2,S_2,Q}| \\
&\le \frac{1}{2}(|\big[\mathbf{F}_{E_1}\big]_{H_1,S_1,Q}|^2 + |\big[\mathbf{F}_{E_2}\big]_{H_2,S_2,Q}|^2),
\end{aligned}$$

where we used Lemma 2.2.4 applied to the hypergraphs $H_3, \dots, H_k$. We can rewrite this inequality as

$$|\big[\mathbf{F}_E\big]_{H,S,Q}| \le \frac{1}{2}(|\big[\mathbf{F}_{E_1}\big]_{H_1',S_1',Q}| + |\big[\mathbf{F}_{E_2}\big]_{H_2',S_2',Q}|),$$

where H_l' is a new hypergraph consisting of two copies of the hypergraph H_l and, similarly, S_l' has same vertices as S_l along with its analogous copies, for $l = 1,2$. Formally, we construct the hypergraph $H_l' = (V_l', E_l')$ such that, for each vertex $v^{(i)} \in V_l$ we add both $v^{(i)}$ and a new vertex $v'^{(i)}$, also keeping the agreement that, for each newly constructed vertices $v_1'^{(i)}$ and $v_2'^{(i)}$, the label $x_{v_1'^{(i)}}$ is the copy of the label $x_{v_2'^{(i)}}$ if and only if the label $x_{v_1^{(i)}}$ is the copy of the label $x_{v_2^{(i)}}$; also, no label of the newly constructed vertex is a copy of the label of any vertex from V_l. Analogously, we define

$$E'_l := E_l \cup \{(v'^{(1)},\dots,v'^{(r)}) : (v^{(1)},\dots,v^{(r)}) \in E\} \text{ and } S'_l := S_l \cup \{v'^{(i)} : v^{(i)} \in S_l\}.$$

Note that both of the hypergraphs H'_1 and H'_2 belong to the case (C1), therefore for each $l = 1,2$ we define

$$\Lambda^{S'_l}_{E'_1,\mathscr{T}}(\mathbf{F}_{E'_l}) := \sum_{Q\in\mathscr{T}} |Q| [\mathbf{F}_{E'_1}]_{H'_l,S'_l,Q}.$$

By Proposition 2.2.9,

$$|\Lambda^{S}_{E,\mathscr{T}}(\mathbf{F}_E)| \le \frac{1}{2}\big(\Lambda^{S'_1}_{E'_1,\mathscr{T}}(\mathbf{F}_{E'_1}) + \Lambda^{S'_2}_{E'_2,\mathscr{T}}(\mathbf{F}_{E'_2})\big) \lesssim 1.$$

Notice that the thresholds d_e required for this result are those thresholds that we get while applying the Proposition 2.2.9 on the modified hypergraphs. However, with this construction the thresholds cannot increase and are still at most equal the quantity defined in (2.1). This completes the proof of the proposition. ■

Proposition 2.3.2. Let Λ^S_E be a non-cancellative entangled dyadic paraproduct.

(a) If (2.9) holds, then for the corresponding coefficients $\lambda^S = (\lambda^S_Q)_{Q\in\mathscr{C}_r}$ we have

$$\|\lambda^S\|_{\mathrm{bmo}} := \sup_{Q_0\in\mathscr{C}_r} \Bigg(\frac{1}{|Q_0|} \sum_{\substack{Q\in\mathscr{C}_r\\ Q\subseteq Q_0}} |Q||\lambda^S_Q|^2\Bigg)^{\frac{1}{2}} \lesssim 1.$$

(b) For a finite convex tree $\mathscr{T}$ and a localized non-cancellative entangled dyadic paraproduct $\Lambda^S_{E,\mathscr{T}}$ we have

$$|\Lambda^S_{E,\mathscr{T}}(\mathbf{F})| \lesssim \|\lambda^S\|_{\mathrm{bmo}} |Q_{\mathscr{T}}| \prod_{e\in E} \max_{Q\in\mathscr{T}\cup\mathscr{L}(\mathscr{T})} [F_e^{d_e}]_Q^{\frac{1}{d_e}}.$$

Proof. (a) Let us see what we can conclude with the assumption of (2.9). Fix $e_0 = (v^{(1)},\dots,v^{(r)}) \in E$. By the definition of the operator T_{e_0} given in (2.4), in this case with kernel defined as in (2.5), we have

$$T_{e_0}\Big(\mathbf{F}_{E\setminus\{e_0\}}\Big)(\mathbb{x}_{e_0}) = \sum_{\substack{S=(S^{(i)})_{i=1}^r\\ (\forall i\in\{1,\dots,r\})S^{(i)}\subseteq V^{(i)}\\ (\exists i_0\in\{1,\dots,r\})S^{(i_0)}\neq\emptyset}} \sum_{Q=\prod_{i=1}^r (I^{(i)})^{n_i}\in\mathscr{C}_n} |Q|^{\frac{1}{2}}\lambda^S_Q$$
$$\int_{\mathbb{R}^{n-r}} \Big(\prod_{e\in E\setminus\{e_0\}} F_e(\mathbb{x}_e)\Big) \mathbb{h}^S_Q(\mathbb{x}) \prod_{v\in V\setminus e_0} dx_v.$$

We turn our attention to the case when $F_e = \mathbb{1}_{\mathbb{R}^r}$ for each $e \in E \backslash \{e_0\}$. The function appearing under the integral sign in that case is $\mathbb{h}_Q^S$ which, up to the constant $|Q|^{\frac{1}{2}}$, equals the product of functions of one variable $\mathbb{h}_{I_i}^1$ and $\mathbb{h}_{I_i}^0$ for each $i \in \{1, \dots, r\}$, where $Q = \prod_{i=1}^r (I^{(i)})^{n_i} \in \mathscr{C}_n$. Depending on whether the cancellation appears or not, the function $T_{e_0}(\mathbb{1}_{\mathbb{R}^r}, \dots, \mathbb{1}_{\mathbb{R}^r})$ can either be identically equal to zero or, if $v^{(1)}, \dots, v^{(s)}$ are all the selected vertices for $s \in \mathbb{N}$, it can be given as

$$T_{e_0}(\mathbb{1}_{\mathbb{R}^r}, \dots, \mathbb{1}_{\mathbb{R}^r}) = \sum_{\substack{S=(S^{(1)},\dots,S^{(s)},\emptyset,\dots,\emptyset) \\ (\forall i \in \{1,\dots,s\}) S^{(i)} \subseteq \{v^{(i)}\} \\ (\exists i_0 \in \{1,\dots,s\}) S^{(i_0)} \neq \emptyset}} \sum_{Q=\prod_{i=1}^r (I^{(i)})^{n_i} \in \mathscr{C}_n} |Q| \lambda_Q^S |I^{(1)}|^{r-n} \bigotimes_{i=1}^r \mathbb{h}_{I^{(i)}}^{v^{(i)}},$$

where we define $\mathbb{h}_{I^{(i)}}^{v^{(i)}}$ as $\mathbb{h}_{I^{(i)}}^1$ if $1 \leq i \leq s$ or as $\mathbb{h}_{I^{(i)}}^0$ otherwise. From the definition of the dyadic BMO-seminorm, taking care of the cancellation again (which happens to appear in at least one variable of each summand of the above expression), we have

$$\begin{aligned} &\|T_{e_0}(\mathbb{1}_{\mathbb{R}^r}, \dots, \mathbb{1}_{\mathbb{R}^r})\|_{\mathrm{BMO}(\mathbb{R}^r)} \\ &\qquad = \sup_{Q_0 \in \mathscr{C}_r} \Bigg(\frac{1}{|Q_0|} \sum_{\substack{S=(S^{(1)},\dots,S^{(s)},\emptyset,\dots,\emptyset) \\ (\forall i \in \{1,\dots,s\}) S^{(i)} \subseteq \{v^{(i)}\} \\ (\exists i_0 \in \{1,\dots,s\}) S^{(i_0)} \neq \emptyset}} \sum_{\substack{Q=\prod_{i=1}^r (I^{(i)})^{n_i} \in \mathscr{C}_n \\ \prod_{i=1}^r I^{(i)} \subseteq Q_0}} |I^{(1)}|^r |\lambda_Q^S|^2 \Bigg)^{\frac{1}{2}}. \end{aligned}$$

From this, recognizing the expression inside the brackets as the BMO-norms of the coefficients, it follows that from each such choice of S we have

$$\|\lambda^S\|_{\mathrm{bmo}} \leq \|T_{e_0}(\mathbb{1}_{\mathbb{R}^r}, \dots, \mathbb{1}_{\mathbb{R}^r})\|_{\mathrm{BMO}(\mathbb{R}^r)} \lesssim 1.$$

Notice that for the preceding proof we were required to have an edge $e_0 \in E$ that contains all of the selected vertices from the starting hypergraph, which is precisely the condition (NC) together with completeness of the corresponding hypergraph component.

(b) Without loss of generality we can assume that

$$|Q_{\mathscr{T}}| = 1 \text{ and } \max_{Q \in \mathscr{T} \cup \mathscr{L}(\mathscr{T})} [F_e^{d_e}]_Q^{\frac{1}{d_e}} = 1 \text{ for each } e \in E.$$

By the Cauchy-Schwarz inequality we have

$$|\Lambda_{E,\mathscr{T}}^S(\mathbf{F})| = \sum_{Q \in \mathscr{T}} |Q| |\lambda_Q| [\mathbf{F}]_{H,S,Q} \leq \Big(\sum_{Q \in \mathscr{T}} |Q| |\lambda_Q|^2 \Big)^{\frac{1}{2}} \Big(\sum_{Q \in \mathscr{T}} |Q| [\mathbf{F}]_{H,S,Q}^2 \Big)^{\frac{1}{2}}.$$

We can notice that

$$\sum_{Q\in\mathscr{T}} |Q||\lambda_Q|^2 \leq \sum_{\substack{Q\in\mathscr{C}_r\\ Q\subseteq Q_{\mathscr{T}}}} |Q||\lambda_Q|^2 \leq |Q_{\mathscr{T}}|\left\|\lambda^S\right\|_{\mathrm{bmo}}^2 = \left\|\lambda^S\right\|_{\mathrm{bmo}}^2.$$

Let H' be a hypergraph consisting of two copies of the hypergraph H (up to the labels of vertices and edges) and let S' be an r-tuple consisting of the vertices from S and their corresponding copies. This hypergraph belongs to the case (C1), for which we already have

$$\sum_{Q\in\mathscr{T}} |Q|\,[\mathbf{F}]_{H,S,Q}^2 = \sum_{Q\in\mathscr{T}} |Q|\,[\mathbf{F}]_{H',S',Q} \lesssim 1.$$

All together, we achieve the desired claim: $\Lambda^S_{E,\mathscr{T}}(\mathbf{F}) \lesssim \left\|\lambda^S\right\|_{\mathrm{bmo}}^2$. ■

2.4. PROOF OF THEOREM 2.1.1

Finally, we are ready to prove the main theorem of this chapter. We can notice that the statement (c) trivially implies statement (d). Also, a tuple $\mathbf{w} = (w_e)_{e\in E}$ given with $w_e := \mathbb{1}_{\mathbb{R}^r}$ for each $e \in E$ satisfies (1.3), so assuming that (f) is valid and checking that $[\mathbf{w}]_{\mathbf{p},\mathbf{d}} = 1$ we obtain statement (c). In the following sections we are going to present proof of other adequate implications so that we will form a cycle of implications of all six statements, showing that they are all equivalent.

2.4.1. Statement (a) implies (e)

For each $Q_0 \in \mathscr{C}_r$ denote $\mathscr{D}(Q_0) := \{Q \in \mathscr{C}_r : Q \subseteq Q_0\}$ and $M := \frac{\log_2(2\mathrm{card}(E))}{\min_{e\in E} d_e}$. For a fixed $e \in E$ let us define

$$\mathscr{I}^e_{Q_0} := \{Q \in \mathscr{D}(Q_0) : [F_e^{d_e}]_Q^{\frac{1}{d_e}} > 2^M [F_e^{d_e}]_{Q_0}^{\frac{1}{d_e}}\}.$$

Then define $\mathscr{M}_{Q_0}$ to be the collection of maximal cubes in $\cup_{e\in E}\mathscr{I}^e_{Q_0}$ and finally set $\mathscr{M}^e_{Q_0} := \mathscr{M}_{Q_0} \cap \mathscr{I}^e_{Q_0}$. Consequently, $\mathscr{M}_{Q_0} = \cup_{e\in E}\mathscr{M}^e_{Q_0}$, but the union does not have to be disjoint. From these definitions we have

$$\begin{aligned}\sum_{Q\in\mathscr{M}^e_{Q_0}} |Q| &\leq \sum_{Q\in\mathscr{M}^e_{Q_0}} 2^{-Md_e}\left[F_e^{d_e}\right]^{-1}_{Q_0}\int_Q F_e(\mathbb{x}_e)^{d_e}d\mathbb{x}_e \\ &\leq (2\mathrm{card}(E))^{-1}\left[F_e^{d_e}\right]^{-1}_{Q_0}\int_{Q_0} F_e(\mathbb{x}_e)^{d_e}d\mathbb{x}_e = \frac{|Q_0|}{2\mathrm{card}(E)}.\end{aligned}$$

In the second inequality we used the fact that the elements of $\mathscr{M}^e_{Q_0}$ are mutually disjoint (by maximality), allowing us to increase the sum to the integral over the largest cube Q_0. This gives us

$$\sum_{Q\in\mathscr{M}_{Q_0}} |Q| \leq \sum_{e\in E}\sum_{Q\in\mathscr{M}^e_{Q_0}} |Q| \leq \frac{|Q_0|}{2}. \tag{2.24}$$

Now, choose $Q_1,\ldots,Q_{2^r} \in \mathscr{C}_r$ such that $\cup_{i=1}^{2^r} Q_i \supset \cup_{e\in E}\operatorname{supp} F_e$. Indeed, if the supports of functions F_e are contained in more than one quadrant of the space $\mathbb{R}^r$, we may need at most 2^r dyadic cubes that cover their supports. For each $i \in \{1,\ldots,2^r\}$ we inductively define

$$\mathscr{S}_{\mathscr{D},i,0} := \{Q_i\}, \quad \mathscr{S}_{\mathscr{D},i,n} := \cup_{Q \in \mathscr{S}_{\mathscr{D},i,n-1}} \mathscr{M}_Q, n \in \mathbb{N},$$
$$\mathscr{S}_{\mathscr{D},i} := \cup_{n=0}^{\infty} \mathscr{S}_{\mathscr{D},i,n}, i \in \{1, \ldots, 2^r\}, \quad \mathscr{S}_{\mathscr{D}} := \cup_{i=1}^{2^r} \mathscr{S}_{\mathscr{D},i}.$$

Let us notice that $\mathscr{S}_{\mathscr{D}}$ is a sparse family of dyadic cubes. Indeed, for any $Q \in \mathscr{S}_{\mathscr{D}}$ let $E_Q := Q \backslash \big(\cup_{Q' \in \mathscr{M}_Q} Q' \big)$. For each two dyadic cubes $Q_1, Q_2 \in \mathscr{S}_{\mathscr{D}}$, $Q_1 \neq Q_2$, we have that they are either mutually disjoint, therefore E_{Q_1} and E_{Q_2} are mutually disjoint as well, or, without loss of generality, $Q_2 \subseteq Q_1$, in which case, by construction, $Q_2 \subseteq Q_1' \in \mathscr{M}_{Q_1}$, so $Q_2 \cap E_{Q_1} = \emptyset$, therefore E_{Q_1} and E_{Q_2} are again mutually disjoint. Also, for each $Q \in \mathscr{S}_{\mathscr{D}}$, by (2.24) we have

$$|E_Q| = |Q| - \sum_{Q' \in \mathscr{M}_Q} |Q'| \geq \frac{1}{2}|Q|.$$

Now, for each $Q \in \mathscr{S}_{\mathscr{D}}$ and a fixed $N \in \mathbb{N}$ let us define

$$\mathscr{T}_Q^N := \mathscr{C}^N \cap \mathscr{D}(Q) \backslash \big(\cup_{Q' \in \mathscr{M}_Q} \mathscr{D}(Q') \big),$$

where $\mathscr{C}^N$ is given in (2.20). Notice that $\mathscr{T}_Q^N$ is a finite convex tree where the set of leaves $\mathscr{L}(\mathscr{T}_Q^N)$ are either elements of $\mathscr{M}_Q$ or they are dyadic cubes with length of each side equal to 2^{-N-1}. An application of Propositions 2.3.1 and 2.3.2 gives us

$$|\Lambda^S_{E,\mathscr{T}_Q^N}|(\mathbf{F}) \lesssim |Q| \prod_{e \in E} \max_{Q' \in \mathscr{T}_Q^N \cup \mathscr{L}(\mathscr{T}_Q^N)} [F_e^{d_e}]_{Q'}^{\frac{1}{d_e}}.$$

If $Q' \in \mathscr{T}_Q^N$ then $Q' \notin \mathscr{M}_Q^e$, which means that $[F_e^{d_e}]_{Q'}^{\frac{1}{d_e}} \leq 2^M [F_e^{d_e}]_Q^{\frac{1}{d_e}}$. If $Q' \in \mathscr{L}(\mathscr{T}_Q^N) \cap \mathscr{S}_{\mathscr{D}}$ and Q'_P is a parent of Q', then by maximality we have

$$[F_e^{d_e}]_{Q'}^{\frac{1}{d_e}} \leq 2^{\frac{r}{d_e}} [F_e^{d_e}]_{Q'_P}^{\frac{1}{d_e}} \leq 2^{r+M} [F_e^{d_e}]_Q^{\frac{1}{d_e}}.$$

The remaining option is if each side of Q' has length equal to 2^{-N-1}. But even then its parent Q'_P satisfies $Q'_P \notin \mathscr{M}_Q^e$, so that the above inequality is valid again. Altogether,

$$|\Lambda^S_{E,\mathscr{T}_Q^N}|(\mathbf{F}) \lesssim 2^{\mathrm{card}(E)M} |Q| \prod_{e \in E} [F_e^{d_e}]_Q^{\frac{1}{d_e}}.$$

This holds for any $Q \in \mathscr{S}_{\mathscr{D}}$. Note that the trees $\mathscr{T}_Q^N$, $Q \in \mathscr{S}_{\mathscr{D}}$ form a partition of $\big(\cup_{i=1}^{2^r} \mathscr{D}(Q_i) \big) \cap \mathscr{C}^N$, therefore

$$|\Lambda^S_{E,(\cup_{i=1}^{2^r} \mathscr{D}(Q_i)) \cap \mathscr{C}^N}(\mathbf{F})| \lesssim 2^{\mathrm{card}(E)M} \sum_{Q \in \mathscr{S}_{\mathscr{D}}} |Q| \prod_{e \in E} [F_e^{d_e}]_Q^{\frac{1}{d_e}}.$$

As the right side of the inequality and the inequality itself does not depend on N, with $N \to \infty$ we get

$$|\Lambda^S_{E,\cup_{i=1}^{2^r}\mathscr{D}(Q_i)}(\mathbf{F})| \lesssim 2^{\mathrm{card}(E)M} \sum_{Q\in\mathscr{S}_{\mathscr{D}}} |Q| \prod_{e\in E} [F_e^{d_e}]_Q^{\frac{1}{d_e}}.$$

Note that for $Q \in \mathscr{C}_r, Q \notin \cup_{i=1}^{2^r}\mathscr{D}(Q_i)$ we have that the summand in (2.22) equals zero, so by summing this inequality over each S satisfying conditions of the first sum in (2.6) we get

$$|\Lambda_E(\mathbf{F})| \lesssim \Theta_{\mathscr{S}_{\mathscr{D}}}(\mathbf{F}),$$

with a sparse form given associated with the sparse family $\mathscr{S}_{\mathscr{D}}$, which completes the proof.

2.4.2. Statement (e) implies (f)

Assume that the statement (e) from Theorem 2.1.1 is valid and let $\Theta_{\mathscr{S}}$ be the sparse form that bounds the form Λ_E. It will be enough to prove the analogous inequality for $\Theta_{\mathscr{S}}$. Once again, it is sufficient to work with nonnegative functions F_e. For each $e \in E$ let $h_e := w_e^{\frac{-d_e}{p_e-d_e}}$ and let G_e be a function such that $F_e = G_e h_e^{\frac{1}{d_e}}$. Note that we have $\|F_e\|_{\mathrm{L}^{p_e}(w_e)} = \|G_e\|_{\mathrm{L}^{p_e}(h_e)}$. Let us rewrite the form $\Theta_{\mathscr{S}}$ in the following way:

$$\Theta_{\mathscr{S}}(\mathbf{F}) = \sum_{Q\in\mathscr{S}} \Big(\prod_{e\in E}[h_e]_Q^{\frac{1}{d_e}-\frac{1}{p_e}}\Big)\Big(|Q|\prod_{e\in E}\Big(\frac{[h_e]_Q}{|E_Q|[h_e]_{E_Q}}\Big)^{\frac{1}{p_e}}\Big)$$
$$\Big(\prod_{e\in E}(|E_Q|[h_e]_{E_Q})^{\frac{1}{p_e}}\Big(\frac{[G_e^{d_e}h_e]_Q}{[h_e]_Q}\Big)^{\frac{1}{d_e}}\Big). \tag{2.25}$$

We can see directly from the definition of the Muckenhoupt constant that $\prod_{e\in E}[h_e]_Q^{\frac{1}{d_e}-\frac{1}{p_e}} \leq [\mathbf{w}]_{\mathbf{p},\mathbf{d}}$. To bound the expression inside the second pair of parentheses, first notice that, by the Hölder inequality, by (1.3) and along with with $r_e := \frac{p_e-d_e}{p_e d_e}$ for each $e \in E$, $r := \sum_{e\in E} r_e$ and the constant c from Definition 1.1.9 for the given family $\mathscr{S}$ we can see that

$$\prod_{e\in E}(|E_Q|[h_e]_{E_Q})^{\frac{r_e}{r}} = \prod_{e\in E}\Big(\int_{E_Q} h_e(x)dx\Big)^{\frac{r_e}{r}} \geq \int_{E_Q}\prod_{e\in E} h_e(x)^{\frac{r_e}{r}}dx = |E_Q| \geq c|Q|.$$

for each $Q \in \mathscr{S}$. Denote $m := \max_{e\in E}\frac{1}{r_e p_e}$. This gives us

$$\prod_{e\in E}\Big(\frac{|Q|[h_e]_Q}{|E_Q|[h_e]_{E_Q}}\Big)^{\frac{1}{p_e}} \leq \prod_{e\in E}\Big(\frac{|Q|}{|E_Q|[h_e]_{E_Q}}\Big)^{r_e m}[h_e]_Q^{r_e m} \leq c^{-rm}[\mathbf{w}]^m_{\mathbf{p},\mathbf{d}}.$$

It remains to note that we have already obtained one power of the Muckenhoupt constant and observe that

$$1+m = 1+\max_{e\in E}\frac{d_e}{p_e-d_e} = \max_{e\in E}\frac{p_e}{p_e-d_e}.$$

We have

$$\Big(\prod_{e\in E}[h_e]_Q^{\frac{1}{d_e}-\frac{1}{p_e}}\Big)|Q|\prod_{e\in E}\Big(\frac{[h_e]_Q}{|E_Q|[h_e]_{E_Q}}\Big)^{\frac{1}{p_e}} \lesssim [\mathbf{w}]_{\mathbf{p},\mathbf{d}}^{\max_{e\in E}\frac{p_e}{p_e-d_e}},$$

with the implicit constant depending only on c. Note that the expressions in the first two parentheses of (2.25) are bounded uniformly in $Q\in\mathscr{S}$. Since the first two terms in (2.25) are bounded independently of Q, we turn to the sum of the third terms over $Q\in\mathscr{S}$:

$$\begin{aligned}\sum_{Q\in\mathscr{S}}\prod_{e\in E}\Big(\int_{E_Q}h_e(x)dx\Big)^{\frac{1}{p_e}}\Big(\frac{[G_e^{d_e}h_e]_Q}{[h_e]_Q}\Big)^{\frac{1}{d_e}} &= \sum_{Q\in\mathscr{S}}\prod_{e\in E}\Big(\int_{E_Q}\Big(\frac{[G_e^{d_e}h_e]_Q}{[h_e]_Q}\Big)^{\frac{p_e}{d_e}}h_e(x)dx\Big)^{\frac{1}{p_e}}\\ &\le \sum_{Q\in\mathscr{S}}\prod_{e\in E}\Big(\int_{E_Q}(M_{d_e,h_e}G_e)(x)^{p_e}h_e(x)dx\Big)^{\frac{1}{p_e}}.\end{aligned}$$

In the last expression we introduced the notation $M_{d,w}$, for $d\in[1,\infty\rangle$ and a strictly positive measurable function w, for the operator given as

$$M_{d,w}F(x) := \sup_{\substack{Q\in\mathscr{C}_r\\ x\in Q}}\Big(\frac{[|F|^d w]_Q}{[w]_Q}\Big)^{\frac{1}{d}} \quad \text{for } x\in\mathbb{R}^r.$$

This operator is bounded on the weighted space $\mathrm{L}^p(w)$ for each $p>d$, by Theorem 1.5.2 and the identity $M_{d,w}F = \big(M_wF^d\big)^{\frac{1}{d}}$. By Hölder's inequality for the summation in Q, the disjointness of E_Q and boundedness of M_{d_e,h_e} the last expression is at most

$$\begin{aligned}\prod_{e\in E}\Big(\sum_{Q\in\mathscr{S}}\int_{E_Q}(M_{d_e,h_e}G_e)(x)^{p_e}h_e(x)dx\Big)^{\frac{1}{p_e}} &\le \prod_{e\in E}\|M_{d_e,h_e}G_e\|_{\mathrm{L}^{p_e}(h_e)} \lesssim \prod_{e\in E}\|G_e\|_{\mathrm{L}^{p_e}(h_e)}\\ &= \prod_{e\in E}\|F_e\|_{\mathrm{L}^{p_e}(w_e)}\end{aligned}$$

which gives the desired weighted estimate.

2.4.3. Statement (d) implies (b)

Let $p_e\in\langle d_e,\infty],e\in E$ be the exponents which satisfy the statement (d) from 2.1.1 and let $e_0\in E$ and $Q\in\mathscr{C}_r$ be arbitrary. Specially, if we take $F_e=\mathbb{1}_Q$ for each $e\in E\backslash\{e_0\}$, we have

$$\left|\int_{\mathbb{R}^r} T_{e_0}(\mathbb{1}_Q)_{e\in E\setminus\{e_0\}} F_{e_0}(\mathbb{x}_{e_0})(\mathbb{x}_{e_0}) d\mathbb{x}_{e_0}\right| = |\Lambda_E(\mathbf{F})| \lesssim \|F_{e_0}\|_{\mathrm{L}^{p_{e_0}}(\mathbb{R}^r)} \prod_{e\in E\setminus\{e_0\}} \|\mathbb{1}_Q\|_{\mathrm{L}^{p_e}(\mathbb{R}^r)}$$
$$= \|F_{e_0}\|_{\mathrm{L}^{p_{e_0}}(\mathbb{R}^r)} |Q|^{\sum_{e\in E\setminus\{e_0\}} \frac{1}{p_e}} = \|F_{e_0}\|_{\mathrm{L}^{p_{e_0}}(\mathbb{R}^r)} |Q|^{\frac{1}{q_{e_0}}},$$

where q_{e_0} is the conjugated exponent of p_{e_0}. This gives us

$$\|T_{e_0}(\mathbb{1}_Q)_{e\in E\setminus\{e_0\}}\|_{\mathrm{L}^{q_{e_0}}(Q)} \lesssim |Q|^{\frac{1}{q_{e_0}}}.$$

Combining this with Jensen's inequality,

$$\frac{1}{|Q|}\int_Q |T_{e_0}(\mathbb{1}_Q)_{e\in E\setminus\{e_0\}}(\mathbb{x}_{e_0})| d\mathbb{x}_{e_0} \leq \left(\frac{1}{|Q|}\int_Q |T_{e_0}(\mathbb{1}_Q)_{e\in E\setminus\{e_0\}}(\mathbb{x}_{e_0})|^{q_{e_0}} d\mathbb{x}_{e_0}\right)^{\frac{1}{q_{e_0}}} \lesssim 1.$$

This shows that condition (2.10) is valid.

2.4.4. Statement (b) implies (a)

Note that from the inequality (2.10) for any $Q \in \mathscr{C}_r$ we have

$$|\Lambda_E((\mathbb{1}_Q)_{e\in E})| = \left|\int_{\mathbb{R}^r} T_{e_0}\left((\mathbb{1}_Q)_{e\in E\setminus\{e_0\}}\right)(\mathbb{x}_{e_0}) \mathbb{1}_Q(\mathbb{x}_{e_0}) d\mathbb{x}_{e_0}\right|$$
$$\leq \|T_{e_0}\left((\mathbb{1}_Q)_{e\in E\setminus\{e_0\}}\right)\|_{\mathrm{L}^1(Q)} \lesssim |Q|.$$

This shows us (2.8) from the statement of Theorem 2.1.1. Take $r > 0$ such that the support of the kernel K is contained in $[-r,r]^n$. Let $e_0 \in E$ and $Q_{e_0} \in \mathscr{C}_r$ be arbitrary. Define

$$\mathscr{S}(Q_{e_0}) := \{Q' \in \mathscr{C}_r : |Q'| = |Q_{e_0}| \text{ and } |Q' \cap [-r,r]^n| > 0\}.$$

Note that

$$T_{e_0}((\mathbb{1}_{\mathbb{R}^r})_{e\in E\setminus\{e_0\}})(\mathbb{x}_{e_0}) \mathbb{1}_{Q_{e_0}}(\mathbb{x}_{e_0}) = \sum_{e\in E\setminus\{e_0\}} \sum_{Q_e\in\mathscr{S}(Q_{e_0})} T_{e_0}((\mathbb{1}_{Q_{e'}})_{e'\in E\setminus\{e_0\}})(\mathbb{x}_{e_0}) \mathbb{1}_{Q_{e_0}}(\mathbb{x}_{e_0})$$
$$= \sum_{e\in E\setminus\{e_0\}} \sum_{Q_e\in\mathscr{S}(Q_{e_0})} \int_{\mathbb{R}^{n-r}} \left(\prod_{e'\in E} \mathbb{1}_{Q_{e'}}(\mathbb{x}_{e'})\right) K(\mathbb{x}) \prod_{v\in V\setminus e_0} dx_v.$$

As the cubes $Q_{e'}, e' \in E$ all have equal Lebesgue measure, they are either identical or disjoint, which means that each integral expression is of the form

$$\int_{\mathbb{R}^{n-r}} \left(\prod_{i=1}^{r} \prod_{v^{(i)}\in V^{(i)}} \mathbb{1}_{I_{v^{(i)}}}(x_{v^{(i)}})\right) K(\mathbb{x}) \prod_{v\in V\setminus e_0} dx_v,$$

for dyadic intervals $I_{v^{(i)}}, v^{(i)} \in V^{(i)}, i = 1, \dots, r$ such that $\prod_{i=1}^r \prod_{v^{(i)} \in V^{(i)}} I_{v^{(i)}} = Q_{e_0}$. As K is constant on dyadic cubes $\prod_{i=1}^r \prod_{v^{(i)} \in V^{(i)}} I_{v^{(i)}}$ for which $I_{v^{(i_1)}} \neq I_{v^{(i_2)}}$ for some $v^{(i_1)}, v^{(i_2)} \in V^{(i)}$ and some $i \in \{1, \dots, r\}$, i.e. on those cubes that do not intersect the diagonal, the above expression is the constant that coincides with its average over the same cube (the integral over the same cube divided by its Lebesgue measure). In case that for certain dyadic intervals $I_1, \dots, I_r$, we have $I_{v^{(i_1)}} = I_{v^{(i_2)}} = I_i$ for every $v^{(i_1)}, v^{(i_2)} \in V^{(i)}$ and $i \in \{1, \dots, r\}$, we can realize that $Q_{e_0} = I_1^{n_1} \times \cdots \times I_r^{n_r} = Q_e$ for each $e \in E$, therefore the above expression takes the form

$$\begin{aligned} T_{e_0}((\mathbb{1}_{Q_{e'}})_{e' \in E \setminus \{e_0\}})(\mathbb{x}_{e_0}) \mathbb{1}_{Q_{e_0}}(\mathbb{x}_{e_0}) &= \int_{\mathbb{R}^{n-r}} \Big(\prod_{i=1}^r \prod_{v^{(i)} \in V^{(i)}} \mathbb{1}_{I_i}(x_{v^{(i)}}) \Big) K(\mathbb{x}) \prod_{v \in V \setminus e_0} dx_v \\ &= \int_{\mathbb{R}^{n-r}} \Big(\prod_{e' \in E} \mathbb{1}_{Q_{e_0}}(\mathbb{x}_{e'}) \Big) K(\mathbb{x}) \prod_{v \in V \setminus e_0} dx_v \\ &= T_{e_0}((\mathbb{1}_{Q_{e_0}})_{e' \in E \setminus \{e_0\}})(\mathbb{x}_{e_0}) \mathbb{1}_{Q_{e_0}}(\mathbb{x}_{e_0}). \end{aligned}$$

Combining both cases, we get, for each $\mathbb{x}_{e_0} \in Q_{e_0}$,

$$\begin{aligned} & T_{e_0}((\mathbb{1}_{\mathbb{R}^r})_{e \in E \setminus \{e_0\}})(\mathbb{x}_{e_0}) - \frac{1}{|Q_{e_0}|} \int_{Q_{e_0}} T_{e_0}((\mathbb{1}_{\mathbb{R}^r})_{e \in E \setminus \{e_0\}})(\mathbb{y}_{e_0}) d\mathbb{y}_{e_0} \\ & \qquad = T_{e_0}((\mathbb{1}_{Q_{e_0}})_{e \in E \setminus \{e_0\}})(\mathbb{x}_{e_0}) - \frac{1}{|Q_{e_0}|} \int_{Q_{e_0}} T_{e_0}((\mathbb{1}_{Q_{e_0}})_{e \in E \setminus \{e_0\}})(\mathbb{y}_{e_0}) d\mathbb{y}_{e_0}. \end{aligned}$$

This gives us

$$\begin{aligned} & \frac{1}{|Q_{e_0}|} \int_{Q_{e_0}} \Big| T_{e_0}((\mathbb{1}_{\mathbb{R}^r})_{e \in E \setminus \{e_0\}})(\mathbb{x}_{e_0}) - \frac{1}{|Q_{e_0}|} \int_{Q_{e_0}} T_{e_0}((\mathbb{1}_{\mathbb{R}^r})_{e \in E \setminus \{e_0\}})(\mathbb{y}_{e_0}) d\mathbb{y}_{e_0} \Big| d\mathbb{x}_{e_0} \\ & \qquad \le \frac{2}{|Q_{e_0}|} \int_{Q_{e_0}} |T_{e_0}((\mathbb{1}_{Q_{e_0}})_{e \in E \setminus \{e_0\}})(\mathbb{x}_{e_0})| d\mathbb{x}_{e_0} \lesssim 1, \end{aligned}$$

where we applied (2.10). By Theorem 1.1.8, the expression

$$\sup_{Q_{e_0} \in \mathscr{C}_r} \frac{1}{|Q_{e_0}|} \int_{Q_{e_0}} \Big| T_{e_0}((\mathbb{1}_{\mathbb{R}^r})_{e \in E \setminus \{e_0\}})(\mathbb{x}_{e_0}) - \frac{1}{|Q_{e_0}|} \int_{Q_{e_0}} T_{e_0}((\mathbb{1}_{\mathbb{R}^r})_{e \in E \setminus \{e_0\}})(\mathbb{y}_{e_0}) d\mathbb{y}_{e_0} \Big| d\mathbb{x}_{e_0}$$

is comparable with $\|T_{e_0}((\mathbb{1}_{\mathbb{R}^r})_{e \in E \setminus \{e_0\}})\|_{\mathrm{BMO}(\mathbb{R}^r)}$. This shows us that (2.9) is valid, which, by applying the Theorem 2.1.1 (a), in turn establishes this theorem.

3. Convergence of ergodic-martingale paraproducts

In this chapter we are going to define paraproducts of our interest and give a proof of their convergence in the Lebesgue spaces, depending on the range of exponents that we will state.

3.1. Definition and statement

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. The paraproduct we are about to define will appear to be a combination of two simpler, more common sequences. One sequence is a standard backward martingale consisting of conditional expectations given a backward filtration, while the other one consists of *Cesàro averages* A_n with respect to the iterates of an $(\mathscr{F}, \mathscr{F})$-measurable and measure-$\mathbb{P}$-preserving transformation $T : \Omega \to \Omega$, as defined in (1.9). In order to obtain the main result we will require a commutativity condition of operators $f \mapsto f \circ T$ and $f \mapsto \mathbb{E}(f|\mathscr{G}_n)$ for each $n \in \mathbb{N}_0$. Although this condition may sound a bit too restricing, it has already appeared in some other papers related to probability, for example in Podvigin's works in [46] and [47]. To see that this condition is not too restricting, we can see that, in the case of T being bijective, it is equivalent to T and T^{-1} being $(\mathscr{G}_n, \mathscr{G}_n)$-measurable for each $n \in \mathbb{N}_0$; see Lemma 3.1.1 below. The commutativity requirement that we just imposed is only slightly more general than the conditions of the following lemma.

Lemma 3.1.1. Suppose that $T\colon \Omega \to \Omega$ is a bijective transformation such that T and T^{-1} are both $(\mathscr{F},\mathscr{F})$-measurable. The following two conditions are equivalent.

(1) T^j is $(\mathscr{G}_n,\mathscr{G}_n)$-measurable for each $j \in \mathbb{Z}$ and $n \in \mathbb{N}_0$.

(2) $\mathbb{E}(f \circ T^j|\mathscr{G}_n) = \mathbb{E}(f|\mathscr{G}_n) \circ T^j$ for each $f \in \mathrm{L}^1(\Omega)$, $j \in \mathbb{Z}$ and $n \in \mathbb{N}_0$.

Proof. $(1) \Rightarrow (2)$ Fix $j \in \mathbb{Z}$, $n \in \mathbb{N}_0$ and take any $G \in \mathscr{G}_n$. We have

$$\begin{aligned}\int_\Omega (f \circ T^j)\mathbb{1}_G \, d\mathbb{P} &= \int_\Omega (f\mathbb{1}_{T^j(G)}) \circ T^j \, d\mathbb{P} \stackrel{(*)}{=} \int_\Omega f\mathbb{1}_{T^j(G)} \, d\mathbb{P} \stackrel{(**)}{=} \int_\Omega \mathbb{E}(f|\mathscr{G}_n)\mathbb{1}_{T^j(G)} \, d\mathbb{P} \\ &\stackrel{(*)}{=} \int_\Omega (\mathbb{E}(f|\mathscr{G}_n)\mathbb{1}_{T^j(G)}) \circ T^j \, d\mathbb{P} = \int_\Omega (\mathbb{E}(f|\mathscr{G}_n) \circ T^j)\mathbb{1}_G \, d\mathbb{P}.\end{aligned}$$

In equalities marked as $(*)$ we used that $\mathbb{P}$ is invariant under T^j, while in equality $(**)$ we used the definition of the conditional expectation, along with $T^j(G) \in \mathscr{G}_n$ which follows from the assumption applied to T^{-j}. By the same definition we conclude that $\mathbb{E}(f \circ T^j|\mathscr{G}_n) = \mathbb{E}(f|\mathscr{G}_n) \circ T^j$.

$(2) \Rightarrow (1)$ Again, fix $j \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. For $G \in \mathscr{G}_n$ let $f = \mathbb{1}_G$. By (2) we have

$$\mathbb{E}(\mathbb{1}_G \circ T^j|\mathscr{G}_n) = \mathbb{E}(\mathbb{1}_G|\mathscr{G}_n) \circ T^j = \mathbb{1}_G \circ T^j.$$

This means that $\mathbb{1}_G \circ T^j$ is $(\mathscr{G}_n, \mathscr{B}(\mathbb{C}))$-measurable, from which it follows that

$$(\mathbb{1}_G \circ T^j)^{-1}(\{1\}) = (T^j)^{-1}(G) \in \mathscr{G}_n.$$

Therefore, T^j is $(\mathscr{G}_n,\mathscr{G}_n)$-measurable. ■

Let us fix $a \in \langle 1, \infty\rangle$. We are interested in the *ergodic-martingale paraproduct* (with respect to T and $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$) which is a sequence $(\Pi_n^{\mathrm{em}})_{n\in\mathbb{N}}$ of bilinear operators defined as

$$\Pi_n^{\mathrm{em}}(f,g) := \sum_{i=0}^{n-1} (A_{\lfloor a^i \rfloor} f)(\mathbb{E}(g|\mathscr{G}_{i+1}) - \mathbb{E}(g|\mathscr{G}_i)) \tag{3.1}$$

for each $n \in \mathbb{N}$ and complex $\mathscr{F}$-measurable functions f and g. We are interested in the convergence of this sequence in $\mathrm{L}^r(\Omega)$ space for $r \in [1,\infty\rangle$.

Similar to the above sequence of operators is the *martingale-ergodic paraproduct* (with respect to T and $(\mathscr{G}_n)_{n\in\mathbb{N}_0}$), a sequence $(\Pi_n^{\mathrm{me}})_{n\in\mathbb{N}}$ of bilinear operators which, for each $n \in \mathbb{N}$ and complex $\mathscr{F}$-measurable functions f and g, are defined as

$$\Pi_n^{\mathrm{me}}(f,g) := \sum_{i=0}^{n-1} (A_{\lfloor a^{i+1} \rfloor} f - A_{\lfloor a^i \rfloor} f))\mathbb{E}(g|\mathscr{G}_{i+1}). \tag{3.2}$$

Note that, by telescoping,

$$\begin{aligned}\Pi_n^{\mathrm{em}}(f,g)+\Pi_n^{\mathrm{me}}(g,f)=&\sum_{i=0}^{n-1}\Big((A_{\lfloor a^i\rfloor}f)\mathbb{E}(g|\mathscr{G}_{i+1})-(A_{\lfloor a^i\rfloor}f)\mathbb{E}(g|\mathscr{G}_i)\\&+\mathbb{E}(g|\mathscr{G}_{i+1})(A_{\lfloor a^{i+1}\rfloor}f)-\mathbb{E}(g|\mathscr{G}_{i+1})(A_{\lfloor a^i\rfloor}f)\Big)\\=&(A_{\lfloor a^n\rfloor}f)\mathbb{E}(g|\mathscr{G}_n)-fg.\end{aligned}\tag{3.3}$$

We already know that sequences $(A_{\lfloor a^n\rfloor}f)_{n\in\mathbb{N}_0}$ and $(\mathbb{E}(g|\mathscr{G}_n))_{n\in\mathbb{N}_0}$ converge in both $\mathrm{L}^r(\Omega)$ and $\mathbb{P}$-almost surely. The convergence of Cesáro averages follows from Corrolaries 1.3.2 and 1.3.3, while the convergence of backward martingales is a consequence of Doob's martingale convergence theorem from [9]. In conclusion, by (3.3) the convergence of $(\Pi_n^{\mathrm{em}}(f,g))_{n\in\mathbb{N}}$ is equivalent to the convergence of same type of $(\Pi_n^{\mathrm{me}}(g,f))_{n\in\mathbb{N}}$.

We are going to show the following result.

Theorem 3.1.2. Take $a\in\langle 1,\infty\rangle$ and suppose that $p,q\in\left[\frac{4}{3},4\right]$, $r\in\left[1,\frac{4}{3}\right]$ satisfy the Hölder scaling $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. For any functions $f\in\mathrm{L}^p(\Omega,\mathscr{F},\mathbb{P})$ and $g\in\mathrm{L}^q(\Omega,\mathscr{F},\mathbb{P})$ the sequences $(\Pi_n^{\mathrm{em}}(f,g))_{n\in\mathbb{N}}$ and $(\Pi_n^{\mathrm{me}}(f,g))_{n\in\mathbb{N}}$ given by (3.1) and (3.2), respectively, converge in the L^r-norm.

It is good to point out that this range is far from the largest possible one. For example, since the measure $\mathbb{P}$ is finite, with Proposition 1.1.4 we can expand the range by taking the triple (p,q,r) that satisfies the Hölder scaling and then by replacing r with any lower value that is still larger or equal than one. However, even if we decide to work with Hölder triplets only, it is still very likely that the range of exponents stated in the theorem could be enlarged.

3.2. Proof of Theorem 3.1.2

3.2.1. Reducing to L^p estimates for the paraproduct

The convergence stated in this theorem will follow from the estimate

$$\|\Pi_n^{\mathrm{em}}(f,g)\|_{\mathrm{L}^r(\Omega)} \lesssim \|f\|_{\mathrm{L}^p(\Omega)}\|g\|_{\mathrm{L}^q(\Omega)}, \tag{3.4}$$

where the constant in this inequality depends on constant a and the exponents p,q and r, meaning that we will avoid the dependence on the sequence $(\mathscr{G}_i)_{i\in\mathbb{N}}$, operator T, n and functions f and g. Once we have this inequality, for fixed $m,n\in\mathbb{N}$ such that $m<n$ by inserting $\mathbb{E}(g|\mathscr{G}_n)-\mathbb{E}(g|\mathscr{G}_m)$ in the place of g, we have

$$\begin{aligned}
\|\Pi_n^{\mathrm{em}}(f,g)-\Pi_m^{\mathrm{em}}(f,g)\|_{\mathrm{L}^r(\Omega)} &= \Big\|\sum_{i=m}^{n-1}((A_{\lfloor a^i\rfloor}f)\mathbb{E}(g|\mathscr{G}_i)-(A_{\lfloor a^i\rfloor}f)\mathbb{E}(g|\mathscr{G}_{i+1}))\Big\|_{\mathrm{L}^r(\Omega)} \\
&= \Big\|\sum_{i=0}^{n-1}(A_{\lfloor a^i\rfloor}f)(\mathbb{E}(\mathbb{E}(g|\mathscr{G}_n)-\mathbb{E}(g|\mathscr{G}_m))|\mathscr{G}_{i+1})-\mathbb{E}(\mathbb{E}(g|\mathscr{G}_n)-\mathbb{E}(g|\mathscr{G}_m))|\mathscr{G}_i))\Big\|_{\mathrm{L}^r(\Omega)} \\
&= \|\Pi_n^{\mathrm{em}}(f,\mathbb{E}(g|\mathscr{G}_n)-\mathbb{E}(g|\mathscr{G}_m))\|_{\mathrm{L}^r(\Omega)} \lesssim \|f\|_{\mathrm{L}^p(\Omega)}\|\mathbb{E}(g|\mathscr{G}_n)-\mathbb{E}(g|\mathscr{G}_m)\|_{\mathrm{L}^q(\Omega)}.
\end{aligned}$$

Note that we have used the fact that $(\mathscr{G}_i)_{i\in\mathbb{N}}$ is a decreasing sequence of σ-algebras, so by Proposition 1.2.5 we have $\mathbb{E}(\mathbb{E}(g|\mathscr{G}_n)|\mathscr{G}_{i+1}))=\mathbb{E}(\mathbb{E}(g|\mathscr{G}_n)|\mathscr{G}_i))=\mathbb{E}(g|\mathscr{G}_n)$ for each $i\in\{0,\dots,n-1\}$. Also, $\mathbb{E}(\mathbb{E}(g|\mathscr{G}_m)|\mathscr{G}_{i+1}))=\mathbb{E}(g|\mathscr{G}_m)$ when $i<m$ and $\mathbb{E}(\mathbb{E}(g|\mathscr{G}_m)|\mathscr{G}_{i+1}))=\mathbb{E}(g|\mathscr{G}_{i+1})$ otherwise; similar conclusion follows for the expression $\mathbb{E}(\mathbb{E}(g|\mathscr{G}_m)|\mathscr{G}_i))$. Since $(\mathbb{E}(g|\mathscr{G}_n))_{n\in\mathbb{N}}$ converges in $\mathrm{L}^q(\Omega)$ by Theorem 1.2.9, it is also a Cauchy sequence in the same space, so $(\Pi_n^{\mathrm{em}}(f,g))_{n\in\mathbb{N}}$ is also a Cauchy sequence in complete space $\mathrm{L}^r(\Omega)$ and therefore convergent.

While trying to bound L^p norms of the expressions to follow, it is good to note that Cesáro averages and conditional expectations as operators on each Lebesgue space are also bounded. This follows from

$$\|A_nf\|_{\mathrm{L}^p(\Omega)} \le \frac{1}{n}\sum_{i=0}^{n-1}\|f\circ T^i\|_{\mathrm{L}^p(\Omega)} = \|f\|_{\mathrm{L}^p(\Omega)}$$

by the $\mathbb{P}$-measure invariance of T, and by Theorem 1.2.6.

First of all, let us assume that this inequality is valid for nonnegative simple functions f and g. If f and g are arbitrary nonnegative functions, by Lemma 1.1.5 there exist sequences $(f_m)_{m\in\mathbb{N}}$ and $(g_m)_{m\in\mathbb{N}}$ of simple functions such that $f_m \le f_{m+1} \le f$ and $g_m \le g_{m+1} \le g$ $\mathbb{P}$-almost surely for each $m \in \mathbb{N}$ and that they converge to, respectively, f and g $\mathbb{P}$-almost surely as well as in each Lebesgue space. We can notice that $(\Pi_n^{\mathrm{em}}(f_m, g_m))_{m\in\mathbb{N}}$ converges to $\Pi_n^{\mathrm{em}}(f,g)$ $\mathbb{P}$-almost surely (by using Theorem 1.2.8), so by Fatou lemma

$$\begin{aligned}\|\Pi_n^{\mathrm{em}}(f,g)\|_{\mathrm{L}^r(\Omega)} &\le \liminf_{m\in\mathbb{N}} \|\Pi_n^{\mathrm{em}}(f_m,g_m)\|_{\mathrm{L}^r(\Omega)} \lesssim \liminf_{m\in\mathbb{N}} \|f_m\|_{\mathrm{L}^p(\Omega)}\|g_m\|_{\mathrm{L}^q(\Omega)} \\ &= \|f\|_{\mathrm{L}^p(\Omega)}\|g\|_{\mathrm{L}^q(\Omega)}.\end{aligned}$$

In the general case we can split f and g into its positive and negative parts and obtain (3.4). This is why we are going to work with nonnegative simple functions f and g in this paper.

3.2.2. Reducing to times that are powers of 2

The next step in this proof is to show that it is enough to prove the required bound for $a = 2$. For $i \in \mathbb{N}_0$ let $K(i) := \min\{k \in \mathbb{N}_0 : \lfloor a^k \rfloor \ge 2^i\} = \lceil \log_a 2^i \rceil$. We are going to estimate the $\mathrm{L}^r(\Omega)$ norm of the difference

$$\begin{aligned}&\sum_{i=0}^{n-1} (A_{\lfloor a^i \rfloor} f)(\mathbb{E}(g|\mathscr{G}_{i+1}) - \mathbb{E}(g|\mathscr{G}_i)) - \sum_{i=0}^{\lfloor (n-1)\log_2 a \rfloor} (A_{2^i} f)(\mathbb{E}(g|\mathscr{G}_{K(i+1)}) - \mathbb{E}(g|\mathscr{G}_{K(i)})) \\ &= \sum_{i=0}^{n-1} (A_{\lfloor a^i \rfloor} f)(\mathbb{E}(g|\mathscr{G}_{i+1}) - \mathbb{E}(g|\mathscr{G}_i)) - \sum_{i=0}^{\lfloor (n-1)\log_2 a \rfloor} \sum_{j=K(i)}^{K(i+1)-1} (A_{2^i} f)(\mathbb{E}(g|\mathscr{G}_{j+1}) - \mathbb{E}(g|\mathscr{G}_j)).\end{aligned} \tag{3.5}$$

First, we need to notice that this expression can be written as

$$\sum_i (A_{\lfloor a^i \rfloor} f - A_{2^{M_i}} f)(\mathbb{E}(g|\mathscr{G}_{i+1}) - \mathbb{E}(g|\mathscr{G}_i)) \pm \sum_i (A_{\lfloor a^i \rfloor} f)(\mathbb{E}(g|\mathscr{G}_{i+1}) - \mathbb{E}(g|\mathscr{G}_i)). \tag{3.6}$$

Here, M_i is defined as nonnegative integer k for which $K(k) \le i < K(k+1)$. In words, we joined sums over i in (3.5) and factored out $\mathbb{E}(g|\mathscr{G}_{i+1}) - \mathbb{E}(g|\mathscr{G}_i)$; as it is possible that these two sums do not have the same amount of summands, we can have an extra part in form of the first sum, where the sign in front of it depends on whether the first or the second sum in (3.5) had more summands (that actually depends on whether $a < 2$ or $a \ge 2$). At this point it is useful to notice that

$$K(\lfloor (n-1)\log_2 a \rfloor) = \lceil \log_a 2^{\lfloor (n-1)\log_2 a \rfloor} \rceil < \log_a 2^{\lfloor \log_2 a^{n-1} \rfloor} + 1 \le \log_a 2^{\log_2 a^{n-1}} + 1 = n.$$

Therefore, $K(\lfloor (n-1)\log_2 a \rfloor) \leq n-1$. This means that the factor $\mathbb{E}(g|\mathscr{G}_n) - \mathbb{E}(g|\mathscr{G}_{n-1})$ that appears in the first sum of (3.5) has to appear in the second sum as well, more precisely in the last iteration over i, when $i = \lfloor (n-1)\log_2 a \rfloor$. Additionally, we may notice that, when $a \geq 2$, then either $K(i+1) = K(i)$ or $K(i+1) = K(i)+1$ for each $i \in \mathbb{N}_0$, since $a^{k+1} \geq 2a^k$ implies $\lfloor a^{k+1} \rfloor \geq \lfloor 2a^k \rfloor \geq 2\lfloor a^k \rfloor$ for each $k \in \mathbb{N}_0$. This shows that there is no additional part in form of the second sum in (3.6), so we can actually write the minus sign in front to cover the case $a < 2$.

Let us observe the absolute value of the first sum in (3.6).

$$
\begin{aligned}
&\left| \sum_{i=0}^{\lfloor (n-1)\log_2 a \rfloor} (A_{\lfloor a^i \rfloor} f - A_{2^{M_i}} f)(\mathbb{E}(g|\mathscr{G}_{i+1}) - \mathbb{E}(g|\mathscr{G}_i)) \right| \\
\leq & \sum_{i=0}^{\lfloor (n-1)\log_2 a \rfloor} \sum_{j=K(i)}^{K(i+1)-1} |A_{\lfloor a^j \rfloor} f - A_{2^i} f| |\mathbb{E}(g|\mathscr{G}_{j+1}) - \mathbb{E}(g|\mathscr{G}_j)| \\
\leq & \sum_{i=0}^{\lfloor (n-1)\log_2 a \rfloor} \sum_{j=K(i)}^{K(i+1)-1} \Bigg(\sum_{k=K(i)}^{j-1} |A_{\lfloor a^{k+1} \rfloor} f - A_{\lfloor a^k \rfloor} f| \\
& \qquad + |A_{\lfloor a^{K(i)} \rfloor} f - A_{2^i} f| \Bigg) |\mathbb{E}(g|\mathscr{G}_{j+1}) - \mathbb{E}(g|\mathscr{G}_j)| \\
\leq & \sum_{i=0}^{\lfloor (n-1)\log_2 a \rfloor} \sum_{j=K(i)}^{K(i+1)-1} \Bigg(|A_{2^{i+1}} f - A_{\lfloor a^{K(i+1)-1} \rfloor} f| + \sum_{k=K(i)}^{K(i+1)-2} |A_{\lfloor a^{k+1} \rfloor} f - A_{\lfloor a^k \rfloor} f| \\
& \qquad + |A_{\lfloor a^{K(i)} \rfloor} f - A_{2^i} f| \Bigg) |\mathbb{E}(g|\mathscr{G}_{j+1}) - \mathbb{E}(g|\mathscr{G}_j)| \\
\leq & \Bigg(\sum_{i=0}^{\lfloor (n-1)\log_2 a \rfloor} \sum_{j=K(i)}^{K(i+1)-1} \Bigg(|A_{2^{i+1}} f - A_{\lfloor a^{K(i+1)-1} \rfloor} f|^2 + \sum_{k=K(i)}^{K(i+1)-2} |A_{\lfloor a^{k+1} \rfloor} f - A_{\lfloor a^k \rfloor} f|^2 \\
& \qquad + |A_{\lfloor a^{K(i)} \rfloor} f - A_{2^i} f|^2 \Bigg) \Bigg)^{\frac{1}{2}} \Bigg(\sum_{i=0}^{\lfloor (n-1)\log_2 a \rfloor} \sum_{j=K(i)}^{K(i+1)-1} \sum_{k=K(i)}^{K(i+1)} |\mathbb{E}(g|\mathscr{G}_{j+1}) - \mathbb{E}(g|\mathscr{G}_j)|^2 \Bigg)^{\frac{1}{2}} \\
< & (\log_a 2 + 2) \Bigg(\sum_{i=0}^{\lfloor (n-1)\log_2 a \rfloor} \Bigg(|A_{2^{i+1}} f - A_{\lfloor a^{K(i+1)-1} \rfloor} f|^2 + \sum_{k=K(i)}^{K(i+1)-2} |A_{\lfloor a^{k+1} \rfloor} f - A_{\lfloor a^k \rfloor} f|^2 \\
& \qquad + |A_{\lfloor a^{K(i)} \rfloor} f - A_{2^i} f|^2 \Bigg) \Bigg)^{\frac{1}{2}} \Bigg(\sum_{i=0}^{\lfloor (n-1)\log_2 a \rfloor} \sum_{j=K(i)}^{K(i+1)-1} |\mathbb{E}(g|\mathscr{G}_{j+1}) - \mathbb{E}(g|\mathscr{G}_j)|^2 \Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Note that in case of $K(i) > K(i+1) - 2$ the sum over parameter k will be considered as zero; similarly with the sum over j if we even have $K(i) > K(i+1) - 1$. The value $\log_a 2 + 2$ appearing after the application of the last inequality comes from the estimation of the number of integers from $K(i)$ to $K(i+1)$:

$$K(i+1)-K(i)=\lceil\log_a 2^{i+1}\rceil-\lceil\log_a 2^{i}\rceil<\log_a 2^{i+1}+1-\log_a 2^{i}=\log_a 2+1.$$

The $\mathrm{L}^r(\Omega)$ norm of the first sum in (3.6) by Hölder's inequality can be bounded by

$$\sup_{\substack{N_i\in\mathbb{N}\text{ for }i\in\mathbb{N}\\ (N_i)_{i\in\mathbb{N}}\text{ is strictly}\\ \text{increasing}}}\Big\|\Big(\sum_{i=1}^{\infty}|A_{N_{i+1}}f-A_{N_i}f|^2\Big)^{1/2}\Big\|_{\mathrm{L}^p(\Omega)}\Big\|\Big(\sum_{i=0}^{\infty}|\mathbb{E}(g|\mathscr{G}_{i+1})-\mathbb{E}(g|\mathscr{G}_i)|^2\Big)^{1/2}\Big\|_{\mathrm{L}^q(\Omega)},$$

which, thanks to Theorems 1.5.3 and 1.5.5, is bounded by $\|f\|_{\mathrm{L}^p(\Omega)}\|g\|_{\mathrm{L}^q(\Omega)}$, along with multiplicative constant depending on p and q only. As for the bound of the second sum in (3.6), notice that, thanks to the observation above, it may have at most $K(\lfloor(n-1)\log_2 a\rfloor+1)-K(\lfloor(n-1)\log_2 a\rfloor)$ summands. However, we already estimated that this number cannot be larger than $\log_a 2+1$. This allows us to simply apply Hölder's inequlity and the boundedness of operators of Cesáro averages and conditional expectations to bound the corresponding sum with

$$(\log_a 2+1)\|f\|_{\mathrm{L}^p(\Omega)}\|g\|_{\mathrm{L}^q(\Omega)},$$

as we allowed for the constant to depend on the parameter a. Altogether, what helps us focus on the $\Pi_n^{\mathrm{em}}(f,g)$ for $a=2$ is a simple use of a triangle inequality for norms:

$$\begin{aligned}\Big\|\sum_{i=0}^{n-1}(A_{\lfloor a^i\rfloor}f)(\mathbb{E}(g|\mathscr{G}_{i+1})-\mathbb{E}(g|\mathscr{G}_i))\Big\|_{\mathrm{L}^r(\Omega)}&\le\Big\|\sum_{i=0}^{n-1}(A_{\lfloor a^i\rfloor}f)(\mathbb{E}(g|\mathscr{G}_{i+1})-\mathbb{E}(g|\mathscr{G}_i))\\&\quad-\sum_{i=0}^{\lfloor(n-1)\log_2 a\rfloor}(A_{2^i}f)(\mathbb{E}(g|\mathscr{G}_{K(i+1)})-\mathbb{E}(g|\mathscr{G}_{K(i)}))\Big\|_{\mathrm{L}^r(\Omega)}\\&\quad+\Big\|\sum_{i=0}^{\lfloor(n-1)\log_2 a\rfloor}(A_{2^i}f)(\mathbb{E}(g|\mathscr{G}_{K(i+1)})-\mathbb{E}(g|\mathscr{G}_{K(i)}))\Big\|_{\mathrm{L}^r(\Omega)}\end{aligned}$$

So, in order to show (3.4) it will be enough to show the following bound for each $n\in\mathbb{N}$:

$$\Big\|\sum_{i=0}^{n-1}(A_{2^i}f)(\mathbb{E}(g|\mathscr{G}_{i+1})-\mathbb{E}(g|\mathscr{G}_i))\Big\|_{\mathrm{L}^r(\Omega)}\lesssim\|f\|_{\mathrm{L}^p(\Omega)}\|g\|_{\mathrm{L}^q(\Omega)}.$$

Note that the replacement of the sequence $(\mathscr{G}_{K(i)})_{i\in\mathbb{N}_0}$ with $(\mathscr{G}_i)_{i\in\mathbb{N}_0}$ does not represent a problem as the replaced sequence is also a general backward filtration with $\mathscr{G}_{K(0)}=\mathscr{G}_0=\mathscr{F}$, so the result still follows from the newly stated estimate.

3.2.3. Calderón's transference principle

Let $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$ be arbitrary. Notice that, by the compatibility condition (2) from Lemma 3.1.1,

$$\begin{aligned}\Pi_n^{\mathrm{em}}(f,g)\circ T^k &= \sum_{i=0}^{n-1}(A_{2^i}f\circ T^k)(\mathbb{E}(g|\mathscr{G}_{i+1})\circ T^k - \mathbb{E}(g|\mathscr{G}_i)\circ T^k)\\ &= \sum_{i=0}^{n-1}\Big(\frac{1}{2^i}\sum_{j=0}^{2^i-1} f\circ T^{j+k}\Big)(\mathbb{E}(g\circ T^k|\mathscr{G}_{i+1}) - \mathbb{E}(g\circ T^k|\mathscr{G}_i)).\end{aligned}$$

Take $\omega\in\Omega$ and let

$$\tilde{F}(k,\omega) := \begin{cases} f(T^k\omega), & 0\le k\le 2^{n+1}-1,\\ 0, & \text{otherwise},\end{cases} \quad \tilde{G}(k,\omega) := \begin{cases} g(T^k\omega), & 0\le k\le 2^{n+1}-1,\\ 0, & \text{otherwise}.\end{cases}$$

By the measure invariance of $\mathbb{P}$ under T we have

$$\begin{aligned}\|\Pi_n^{\mathrm{em}}(f,g)\|_{\mathrm{L}^r(\Omega)}^r = &\frac{1}{2^n}\sum_{k=0}^{2^n-1}\|\Pi_n^{\mathrm{em}}(f,g)\circ T^k\|_{\mathrm{L}^r(\Omega)}^r \le \frac{1}{2^n}\int_{\mathbb{Z}\times\Omega}\Big|\sum_{i=0}^{n-1}\Big(\frac{1}{2^i}\sum_{j=0}^{2^i-1}\tilde{F}(j+k,\omega)\Big)\\ &\cdot(\mathbb{E}(\tilde{G}(k,\cdot)|\mathscr{G}_{i+1})(\omega) - \mathbb{E}(\tilde{G}(k,\cdot)|\mathscr{G}_i)(\omega))\Big|^r d(\nu\times\mathbb{P})(k,\omega),\end{aligned}$$

where the conditional expectations are taken in the second variable of $\tilde{G}$. In analogous way we can deduce that

$$\|f\|_{\mathrm{L}^p(\Omega)}^p = \frac{1}{2^{n+1}}\sum_{k=0}^{2^{n+1}-1}\|f\circ T^k\|_{\mathrm{L}^p(\Omega)}^p = \frac{1}{2^{n+1}}\int_{\mathbb{Z}\times\Omega}|\tilde{F}(k,\omega)|^p d(\nu\times\mathbb{P})(k,\omega),$$

$$\|g\|_{\mathrm{L}^q(\Omega)}^q = \frac{1}{2^{n+1}}\sum_{k=0}^{2^{n+1}-1}\|g\circ T^k\|_{\mathrm{L}^q(\Omega)}^q = \frac{1}{2^{n+1}}\int_{\mathbb{Z}\times\Omega}|\tilde{G}(k,\omega)|^q d(\nu\times\mathbb{P})(k,\omega).$$

Therefore, it is enough to prove the inequality

$$\begin{aligned}\Big\|\sum_{i=0}^{n-1}\Big(\frac{1}{2^i}\sum_{j=0}^{2^i-1}\tilde{F}(k+j,\omega)\Big)(\mathbb{E}(\tilde{G}(k,\omega)|\mathscr{G}_{i+1}) - \mathbb{E}(\tilde{G}(k,\omega)|\mathscr{G}_i))\Big\|_{\mathrm{L}^r_{(k,\omega)}(\mathbb{Z}\times\Omega)}&\\ \lesssim \|\tilde{F}\|_{\mathrm{L}^p(\mathbb{Z}\times\Omega)}\|\tilde{G}\|_{\mathrm{L}^q(\mathbb{Z}\times\Omega)}.& \qquad (3.7)\end{aligned}$$

The notation $\|\cdot\|_{\mathrm{L}^r_{(k,\omega)}}$ stands for the L^r norm taken in the pair of variables (k,ω). Also, from this context it should be clear that the conditional expectation is taken in the second variable ω as the only probability space that we are working at the moment has Ω as the sample space. Now, for $(x,\omega)\in\mathbb{R}\times\Omega$, let

$$F(x,\omega) := \sum_{k\in\mathbb{Z}}\tilde{F}(k,\omega)\mathbb{1}_{[k,k+1\rangle}(x) \text{ and } G(x,\omega) := \sum_{k\in\mathbb{Z}}\tilde{G}(k,\omega)\mathbb{1}_{[k,k+1\rangle}(x).$$

Note that

$$
\begin{aligned}
\|\tilde{F}\|^p_{\mathrm{L}^p(\mathbb{Z}\times\Omega)} &= \int_\Omega \sum_{k\in\mathbb{Z}} |\tilde{F}(k,\omega)|^p d\mathbb{P}(\omega) \\
&= \int_\Omega \Big(\sum_{k\in\mathbb{Z}} \int_{[k,k+1\rangle} |\tilde{F}(k,\omega)|^p \mathbb{1}_{[k,k+1\rangle}(x) dx \Big) d\mathbb{P}(\omega) \\
&= \int_\Omega \int_\mathbb{R} \Big(\sum_{k\in\mathbb{Z}} |\tilde{F}(k,\omega)|^p \mathbb{1}_{[k,k+1\rangle}(x) dx \Big) d\mathbb{P}(\omega) = \int_\Omega \int_\mathbb{R} |F(x,\omega)|^p d\mathbb{P}(\omega) \\
&= \|F\|^p_{\mathrm{L}^p(\mathbb{R}\times\Omega)}.
\end{aligned}
$$

At the beginning of the second row we applied the (standard) monotone convergence theorem. Similarly we can show that $\|\tilde{G}\|^q_{\mathrm{L}^q(\mathbb{Z}\times\Omega)} = \|G\|^q_{\mathrm{L}^q(\mathbb{R}\times\Omega)}$. This will allow us to replace the norms on the right side of the inequality, however we will have to approximate the left one. Let us for a moment fix $\omega \in \Omega$ and, for the simplicity, denote $\tilde{F}_\omega := \tilde{F}$, $\tilde{G}_\omega := \tilde{G}, F_\omega := F$ and $G_\omega := G$. Take $x \in \mathbb{R}$ and $m \in \mathbb{Z}$ and define

$$
\begin{aligned}
\tilde{A}_n(\tilde{F}_\omega, \tilde{G}_\omega)(m) &:= \sum_{i=0}^{n-1} \Big(\frac{1}{2^i} \sum_{j=0}^{2^i-1} \tilde{F}_\omega(j+m)(\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1}) - \mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)) \Big), \\
A_n(F_\omega, G_\omega)(x) &:= \sum_{i=0}^{n-1} \Big(\frac{1}{2^i} \int_0^{2^i} F_\omega(s+x) ds \Big) \Big(\mathbb{E}(G_\omega(x)|\mathscr{G}_{i+1}) - \mathbb{E}(G_\omega(x)|\mathscr{G}_i) \Big).
\end{aligned}
$$

Fix $\alpha \in [0,1\rangle$. Observe that, for $m \in \mathbb{Z}$,

$$
\begin{aligned}
&A_n(F_\omega, G_\omega)(m+\alpha) \\
&\quad = \sum_{i=0}^{n-1} \Big(\frac{1}{2^i} \int_0^{2^i} F_\omega(s+m+\alpha) ds \Big) \Big(\mathbb{E}(G_\omega(m+\alpha)|\mathscr{G}_{i+1}) - \mathbb{E}(G_\omega(m+\alpha)|\mathscr{G}_i) \Big) \\
&\quad = \sum_{i=0}^{n-1} \Big(\frac{1}{2^i} \int_0^{2^i} \sum_{k\in\mathbb{Z}} \tilde{F}_\omega(k) \mathbb{1}_{[k,k+1\rangle}(s+m+\alpha) ds \Big) \\
&\qquad\qquad \Big(\mathbb{E}(\sum_{l\in\mathbb{Z}} \tilde{G}_\omega(l) \mathbb{1}_{[l,l+1\rangle}(m+\alpha)|\mathscr{G}_{i+1}) - \mathbb{E}(\sum_{l\in\mathbb{Z}} \tilde{G}_\omega(l) \mathbb{1}_{[l,l+1\rangle}(m+\alpha)|\mathscr{G}_i) \Big) \\
&\quad = \sum_{i=0}^{n-1} \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{2^i} \tilde{F}_\omega(k)(\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1}) - \mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)).
\end{aligned}
$$

Notice that the last equality follows from the previous one as $m+\alpha \in [l, l+1\rangle$ only when $l = m$. Here,

$$\lambda_k := \lambda([0,2^i\rangle \cap [k-m-\alpha, k+1-m-\alpha\rangle) = \begin{cases} 0, & k \le m-1 \text{ or } k \ge m+2^i+1, \\ 1, & m+1 \le k \le m+2^i-1, \\ \alpha, & k = m+2^i, \\ 1-\alpha, & k = m. \end{cases}$$

for each $k \in \mathbb{Z}$. This gives us

$$\begin{aligned} &A_n(F_\omega, G_\omega)(m+\alpha) - \tilde{A}_n(\tilde{F}_\omega, \tilde{G}_\omega)(m) \\ &\qquad \le \sum_{i=0}^{n-1} \frac{1}{2^i} \big(\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1}) - \mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)\big)\big(-\alpha\tilde{F}_\omega(m) + \alpha\tilde{F}_\omega(m+2^i)\big). \end{aligned}$$

By applying the absolute value, taking to the power of r and integrating over $\alpha \in [0,1\rangle$ we get

$$\begin{aligned} &\int_0^1 \big|A_n(F_\omega, G_\omega)(m+\alpha) - \tilde{A}_n(\tilde{F}_\omega, \tilde{G}_\omega)(m)\big|^r d\alpha \\ &\qquad = \int_0^1 \Big|\sum_{i=0}^{n-1} \frac{1}{2^i} \big(\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1}) - \mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)\big)\big(-\tilde{F}_\omega(m) + \tilde{F}_\omega(m+2^i)\big)\Big|^r \alpha^r d\alpha \\ &\qquad \le \Big|\sum_{i=0}^{n-1} \frac{1}{2^i} \big(\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1}) - \mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)\big)\big(-\tilde{F}_\omega(m) + \tilde{F}_\omega(m+2^i)\big)\Big|^r. \end{aligned}$$

From this, by triangle inequality for the norm of $\mathrm{L}^r([0,1\rangle)$ space,

$$\begin{aligned} &\bigg|\Big(\int_0^1 |A_n(F_\omega, G_\omega)(m+\alpha)|^r d\alpha\Big)^{\frac{1}{r}} - \Big(\int_0^1 |\tilde{A}_n(\tilde{F}_\omega, \tilde{G}_\omega)(m)|^r d\alpha\Big)^{\frac{1}{r}}\bigg| \\ &\qquad \le \Big(\int_0^1 |A_n(F_\omega, G_\omega)(m+\alpha) - \tilde{A}_n(\tilde{F}_\omega, \tilde{G}_\omega)(m)|^r d\alpha\Big)^{\frac{1}{r}} \\ &\qquad \le \Big|\sum_{i=0}^{n-1} \frac{1}{2^i} \big(\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1}) - \mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i\big)\big(-\tilde{F}_\omega(m) + \tilde{F}_\omega(m+2^i)\big)\Big| \\ &\qquad \le \sum_{i=0}^{n-1} \frac{1}{2^i} \big(|\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1})| + |\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)|\big)\big(|\tilde{F}_\omega(m)| + |\tilde{F}_\omega(m+2^i)|\big). \end{aligned}$$

Finally, if we observe the $\mathrm{L}^r(\mathbb{Z}\times\Omega)$ norm of $(m,\omega) \mapsto |\tilde{A}_n(\tilde{F}_\omega, \tilde{G}_\omega)(m)|$,

$$\begin{aligned} &\Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\tilde{A}_n(\tilde{F}_\omega, \tilde{G}_\omega)(m)|^r d\mathbb{P}(\omega)\Big)^{\frac{1}{r}} \\ \le &\bigg(\int_\Omega \sum_{m\in\mathbb{Z}} \bigg|\Big(\int_0^1 |A_n(F_\omega, G_\omega)(m+\alpha)|^r d\alpha\Big)^{\frac{1}{r}} - |\tilde{A}_n(\tilde{F}_\omega, \tilde{G}_\omega)(m)|\bigg|^r d\mathbb{P}(\omega)\bigg)^{\frac{1}{r}} \end{aligned}$$

$$+\Big(\int_\Omega \sum_{m\in\mathbb{Z}} \int_0^1 |A_n(F_\omega,G_\omega)(m+\alpha)|^r d\alpha d\mathbb{P}(\omega)\Big)^{\frac{1}{r}}$$

$$\leq \Big(\int_\Omega \sum_{m\in\mathbb{Z}} \Big(\sum_{i=0}^{n-1} \frac{1}{2^i} \big(|\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1})| + |\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)|\big)$$

$$\cdot \big(|\tilde{F}_\omega(m)| + |\tilde{F}_\omega(m+2^i)|\big)\Big)^r d\mathbb{P}(\omega)\Big)^{\frac{1}{r}}$$

$$+\Big(\int_\Omega \int_\mathbb{R} |A_n(F_\omega,G_\omega)(x)|^r dx d\mathbb{P}(\omega)\Big)^{\frac{1}{r}}$$

$$\leq \sum_{i=0}^{n-1} \frac{1}{2^i}\Big(\int_\Omega \sum_{m\in\mathbb{Z}} \Big(|\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1})||\tilde{F}_\omega(m)| + |\mathbb{E}_2(\tilde{G}_\omega(m)|\mathscr{G}_{i+1})||\tilde{F}_\omega(m+2^i)|$$

$$+|\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)||\tilde{F}_\omega(m)| + |\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)||\tilde{F}_\omega(m+2^i)|\Big)^r d\mathbb{P}(\omega)\Big)^{\frac{1}{r}}$$

$$+\Big(\int_\Omega \int_\mathbb{R} |A_n(F_\omega,G_\omega)(x)|^r dx d\mathbb{P}(\omega)\Big)^{\frac{1}{r}}$$

$$\leq \sum_{i=0}^{n-1} \frac{1}{2^i}\Big(4^{1-\frac{1}{r}} \int_\Omega \sum_{m\in\mathbb{Z}} \Big(|\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1})|^r|\tilde{F}_\omega(m)|^r + |\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1})|^r|\tilde{F}_\omega(m+2^i)|^r$$

$$+|\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)|^r|\tilde{F}_\omega(m)|^r + |\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)|^r|\tilde{F}_\omega(m+2^i)|^r\Big) d\mathbb{P}(\omega)\Big)^{\frac{1}{r}}$$

$$+\Big(\int_\Omega \int_\mathbb{R} |A_n(F_\omega,G_\omega)(x)|^r dx d\mathbb{P}(\omega)\Big)^{\frac{1}{r}}$$

$$\leq \sum_{i=0}^{n-1} \frac{1}{2^{i-2+\frac{2}{r}}}\Big(\Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1})|^q d\mathbb{P}(\omega)\Big)^{\frac{1}{q}} \Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\tilde{F}_\omega(m)|^p d\mathbb{P}(\omega)\Big)^{\frac{1}{p}}$$

$$+\Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_{i+1})|^q d\mathbb{P}(\omega)\Big)^{\frac{1}{q}} \Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\tilde{F}_\omega(m+2^i)|^p d\mathbb{P}(\omega)\Big)^{\frac{1}{p}}$$

$$+\Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)|^q d\mathbb{P}(\omega)\Big)^{\frac{1}{q}} \Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\tilde{F}_\omega(m)|^p d\mathbb{P}(\omega)\Big)^{\frac{1}{p}}$$

$$+\Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\mathbb{E}(\tilde{G}_\omega(m)|\mathscr{G}_i)|^q d\mathbb{P}(\omega)\Big)^{\frac{1}{q}} \Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\tilde{F}_\omega(m+2^i)|^p d\mathbb{P}(\omega)\Big)^{\frac{1}{p}}\Big)$$

$$+\Big(\int_\Omega \int_\mathbb{R} |A_n(F_\omega,G_\omega)(x)|^r dx d\mathbb{P}(\omega)\Big)^{\frac{1}{r}}$$

$$\leq \sum_{i=0}^{n-1} \frac{1}{2^{i-4+\frac{2}{r}}}\Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\tilde{G}_\omega(m)|^q d\mathbb{P}(\omega)\Big)^{\frac{1}{q}} \Big(\int_\Omega \sum_{m\in\mathbb{Z}} |\tilde{F}_\omega(m)|^p d\mathbb{P}(\omega)\Big)^{\frac{1}{p}}$$

$$+\Big(\int_\Omega \int_\mathbb{R} |A_n(F_\omega,G_\omega)(x)|^r dx d\mathbb{P}(\omega)\Big)^{\frac{1}{r}}.$$

First, third and fifth inequality follow from the triangle inequality for the norm of $\mathrm{L}^r(\mathbb{Z}\times\Omega)$. In the fifth inequality we also used Hölder's inequality and Theorem 1.2.6. Comparing this with (3.7), we conclude that

$$\begin{aligned}
&\Big\|\sum_{i=0}^{n-1}\Big(\frac{1}{2^i}\sum_{j=0}^{2^i-1}\tilde{F}(k+j,\omega)\Big)(\mathbb{E}(\tilde{G}(k,\omega)|\mathscr{G}_{i+1})-\mathbb{E}(\tilde{G}(k,\omega)|\mathscr{G}_i))\Big\|_{\mathrm{L}^r_{(k,\omega)}(\mathbb{Z}\times\Omega)}\\
&\quad\le 32\|\tilde{F}\|_{\mathrm{L}^p(\mathbb{Z}\times\Omega)}\|\tilde{G}\|_{\mathrm{L}^q(\mathbb{Z}\times\Omega)}\\
&\qquad+\Big\|\sum_{i=0}^{n-1}\Big(\frac{1}{2^i}\int_0^{2^i}F(x+y,\omega)dy\Big)(\mathbb{E}(G(x,\omega)|\mathscr{G}_{i+1})-\mathbb{E}(G(x,\omega)|\mathscr{G}_i))\Big\|_{\mathrm{L}^r_{(x,\omega)}(\mathbb{R}\times\Omega)}.
\end{aligned}$$

In conclusion, to prove (3.7) it will be enough to prove the following inequality:

$$\begin{aligned}
&\Big\|\sum_{i=0}^{n-1}\Big(\frac{1}{2^i}\int_0^{2^i}F(x+y,\omega)dy\Big)(\mathbb{E}(G(x,\omega)|\mathscr{G}_{i+1})-\mathbb{E}(G(x,\omega)|\mathscr{G}_i))\Big\|_{\mathrm{L}^r_{(x,\omega)}(\mathbb{R}\times\Omega)}\\
&\qquad\lesssim\|F\|_{\mathrm{L}^p(\mathbb{R}\times\Omega)}\|G\|_{\mathrm{L}^q(\mathbb{R}\times\Omega)}.
\end{aligned}\tag{3.8}$$

3.2.4. Comparing the integral with dyadic martingales

If we consider the left side of this inequality for the rescaled functions $x\mapsto F(2^{-n-1}x,\omega)$ and $x\mapsto G(2^{-n-1}x,\omega)$ instead of $x\mapsto F(x,\omega)$ and $x\mapsto G(x,\omega)$, we get

$$\begin{aligned}
&\Big\|\sum_{i=0}^{n-1}\Big(\frac{1}{2^i}\int_0^{2^i}F(2^{-n-1}x+2^{-n-1}y,\omega)dy\Big)\\
&\qquad\cdot(\mathbb{E}(G(2^{-n-1}x,\omega)|\mathscr{G}_{i+1})-\mathbb{E}(G(2^{-n-1}x,\omega)|\mathscr{G}_i)))\Big\|_{\mathrm{L}^r_{(x,\omega)}(\mathbb{R}\times\Omega)}\\
&=2^{\frac{n+1}{r}}\Big\|\sum_{i=0}^{n-1}\Big(\frac{1}{2^{i-n-1}}\int_0^{2^{i-n-1}}F(x+y,\omega)dy\Big)\\
&\qquad\cdot(\mathbb{E}(G(x,\omega)|\mathscr{G}_{i+1})-\mathbb{E}(G(x,\omega)|\mathscr{G}_i)))\Big\|_{\mathrm{L}^r_{(x,\omega)}(\mathbb{R}\times\Omega)}.
\end{aligned}$$

The right side of the inequality turns to $2^{(n+1)\left(\frac{1}{p}+\frac{1}{q}\right)}\|F\|_{\mathrm{L}^p(\mathbb{R}\times\Omega)}\|G\|_{\mathrm{L}^q(\mathbb{R}\times\Omega)}$. As we have $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, the powers of 2 on both sides of inequalities are actually equal. Furthermore, note that

$$\begin{aligned}
&\Big|\sum_{i=0}^{n-1}\Big(\frac{1}{2^{i-n-1}}\int_0^{2^{i-n-1}}F(x+y,\omega)dy\Big)(\mathbb{E}(G(x,\omega)|\mathscr{G}_{i+1})-\mathbb{E}(G(x,\omega)|\mathscr{G}_i))\Big|\\
&\qquad\le\Big|\sum_{i=0}^{n-1}(\mathbb{E}(F|\mathscr{D}_{n+1-i})(x,\omega))(\mathbb{E}(G(x,\omega)|\mathscr{G}_{i+1})-\mathbb{E}(G(x,\omega)|\mathscr{G}_i))\Big|\\
&\qquad\quad+\sum_{i=0}^{n-1}\Big|\Big(\Big(\frac{1}{2^{i-n-1}}\int_0^{2^{i-n-1}}F(x+y,\omega)dy\Big)-\mathbb{E}(F|\mathscr{D}_{n+1-i})(x,\omega)\Big)\\
&\qquad\quad\cdot(\mathbb{E}(G(x,\omega)|\mathscr{G}_{i+1})-\mathbb{E}(G(x,\omega)|\mathscr{G}_i))\Big|,
\end{aligned}\tag{3.9}$$

where $(\mathscr{D}_i)_{i\in\mathbb{N}_0}$ is a dyadic filtration of $[0,1\rangle$. This time the conditional expectation is taken over variable x. To avoid further confusion, from this point on we are going to use notations $\mathbb{E}_1$ and $\mathbb{E}_2$ for conditional expectations in, respectively, first and second variable. Cauchy-Schwarz inequality gives us

$$\begin{aligned}
&\sum_{i=0}^{n-1}\Big|\Big(\Big(\frac{1}{2^{i-n-1}}\int_0^{2^{i-n-1}}F(x+y,\omega)dy\Big)-\mathbb{E}_1(F|\mathscr{D}_{n+1-i})(x,\omega)\Big)\\
&\qquad\cdot(\mathbb{E}_2(G|\mathscr{G}_{i+1})(x,\omega)-\mathbb{E}_2(G|\mathscr{G}_i))(x,\omega)\Big|\\
&\quad\le\Big(\sum_{i=0}^{n-1}\Big|\Big(\frac{1}{2^{i-n-1}}\int_0^{2^{i-n-1}}F(x+y,\omega)dy\Big)-\mathbb{E}_1(F|\mathscr{D}_{n+1-i})(x,\omega)\Big|^2\Big)^{\frac12}\\
&\qquad\cdot\Big(\sum_{i=0}^{n-1}|\mathbb{E}_2(G|\mathscr{G}_{i+1})(x,\omega)-\mathbb{E}_2(G|\mathscr{G}_i)(x,\omega)|^2\Big)^{\frac12}.
\end{aligned}$$

By Hölder's inequality we can bound the L^r norm of the expression of left side of this inequality with $\|\mathscr{S}_1F\|_{\mathrm{L}^p(\mathbb{R}\times\Omega)}\|\mathscr{S}_2G\|_{\mathrm{L}^q(\mathbb{R}\times\Omega)}$, where

$$(\mathscr{S}_1F)(x,\omega):=\Big(\sum_{i=0}^{n-1}\Big|\Big(\frac{1}{2^{i-n-1}}\int_0^{2^{i-n-1}}F(x+y,\omega)dy\Big)-\mathbb{E}_1(F|\mathscr{D}_{n+1-i})(x,\omega)\Big|^2\Big)^{\frac12}\ \text{and}$$
$$(\mathscr{S}_2G)(x,\omega):=\Big(\sum_{i=0}^{n-1}|\mathbb{E}_2(G|\mathscr{G}_{i+1})(x,\omega)-\mathbb{E}_2(G|\mathscr{G}_i)(x,\omega)|^2\Big)^{\frac12}.$$

Let us point out that both of the operators $\mathscr{S}_1$ and $\mathscr{S}_2$ are bounded. Indeed, the boundedness of the second operator follows from Theorem 1.5.5. As for the first operator, by Theorem 1.5.6, $\|\mathscr{S}_1 F\|^p_{\mathrm{L}^p(\mathbb{R}\times\Omega)}$ equals

$$\begin{aligned}
&\int_\Omega \Big[\int_{\mathbb{R}} \Big|\sum_{i=0}^{n-1}\Big|\Big(\frac{1}{2^{i-n-1}}\int_0^{2^{i-n-1}} F(x+y,\omega)dy\Big) - \mathbb{E}_1(F|\mathscr{D}_{n+1-i})(x,\omega)\Big|^2\Big|^{\frac{p}{2}} dx\Big] d\mathbb{P}(\omega)\\
&\qquad \le \int_\Omega \Big\|\Big(\sum_{i=0}^{\infty}\Big|\Big(2^i\int_0^{2^{-i}} F(x+y,\omega)dy\Big) - \mathbb{E}_1(F|\mathscr{D}_i)(x,\omega)\Big|^2\Big)^{\frac{1}{2}}\Big\|^p_{\mathrm{L}^p_x(\mathbb{R})} d\mathbb{P}(\omega)\\
&\qquad \le \int_\Omega C_p^p \|F(x,\omega)\|^p_{\mathrm{L}^p_x(\mathbb{R})} d\mathbb{P}(\omega) = C_p^p\|F\|^p_{\mathrm{L}^p(\mathbb{R}\times\Omega)}.
\end{aligned}$$

Combining this with (3.9), we can see that, in order to show (3.8), it will be enough to prove

$$\Big\|\sum_{i=0}^{n-1}\mathbb{E}_1(F|\mathscr{D}_{n+1-i})(\mathbb{E}_2(G|\mathscr{G}_{i+1}) - \mathbb{E}_2(G|\mathscr{G}_i))\Big\|_{\mathrm{L}^r(\mathbb{R}\times\Omega)} \lesssim \|F\|_{\mathrm{L}^p(\mathbb{R}\times\Omega)}\|G\|_{\mathrm{L}^q(\mathbb{R}\times\Omega)}. \tag{3.10}$$

3.2.5. Reducing the filtrations to a specific case

Since the sum under the norm is finite, we can interchange the order of summation so that both of sequences of σ-algebras are increasing, rather than decreasing. Furthermore, we are going to replace the dyadic σ-algebra with a more general one; precisely, we will show that

$$\Big\|\sum_{i=0}^{n-1}\mathbb{E}_1(F|\mathscr{F}_i)(\mathbb{E}_2(G|\mathscr{G}_{i+1}) - \mathbb{E}_2(G|\mathscr{G}_i))\Big\|_{\mathrm{L}^r(\Omega_1\times\Omega_2)} \lesssim \|F\|_{\mathrm{L}^p(\Omega_1\times\Omega_2)}\|G\|_{\mathrm{L}^q(\Omega_1\times\Omega_2)} \tag{3.11}$$

where $(\mathscr{F}_i)_{i\in\mathbb{N}}$ and $(\mathscr{G}_i)_{i\in\mathbb{N}}$ are forward filtrations on probability spaces, respectively, $(\Omega_1,\mathscr{F},\mathbb{P}_1)$ and $(\Omega_2,\mathscr{G},\mathbb{P}_2)$ and, additionally, each $\mathscr{F}_i$, $i\in\mathbb{N}$, is finitely generated.

Interestingly, it is possible to replace a filtration $(\mathscr{G}_i)_{i\in\mathbb{N}_0}$ with the one where each of its σ-algebras are also finitely generated. For that, let us notice that, if we follow the construction from function g appearing in (3.4) to this function, for fixed $\omega_1\in\Omega_1$ the function $G(\omega_1,\cdot)$ still remains a simple function. There exist $m_{\omega_1}\in\mathbb{N}$, $\alpha_1^{\omega_1},\dots,\alpha_{m_{\omega_1}}^{\omega_1}\in\mathbb{R}$ and $G_1^{\omega_1},\dots,G_{m_{\omega_1}}^{\omega_1}\in\mathscr{G}$ such that $G(\omega_1,\cdot)=\sum_{j=1}^{m_{\omega_1}}\alpha_j^{\omega_1}\mathbb{1}_{G_j^{\omega_1}}$. Moreover, following the construction of G from function g we can see that there are only finitely many different fibers $G(\omega_1,\cdot)$, therefore we can denote all scalars and sets previously mentioned with $\alpha_1,\dots,\alpha_m$ and $G_1,\dots,G_m$ for $m\in\mathbb{N}$. Let

$$\mathscr{H}_i := \sigma(\{\{\mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{G}_l)>\alpha\} : l\in\{0,\dots,i\}, j\in\{1,\dots,m\},\alpha\in\mathbb{Q}\}).$$

We can notice that $(\mathscr{H}_l)_{l\in\mathbb{N}_0}$ is a filtration. Also, as for any $l \in \{0,\dots,i\}$, $\alpha \in \mathbb{Q}$ and $j \in \{1,\dots,m\}$ we have

$$\{\mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{G}_l) > \alpha\} = (\mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{G}_l))^{-1}(\langle\alpha,+\infty\rangle) \in \mathscr{G}_l \subseteq \mathscr{G}_i$$

and as those sets generate $\mathscr{H}_i$, we conclude that $\mathscr{H}_i \subseteq \mathscr{G}_i$. Also, $\mathscr{H}_i$ is a countably generated σ-algebra. We can also show that

$$\mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{H}_i) = \mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{G}_i) \;\; \mathbb{P}_2 - \text{almost surely.}$$

To show this, assume the opposite. First, let us assume that there exists $\alpha \in \mathbb{Q}$ such that a set

$$B := \{\mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{H}_i) \le \alpha < \mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{G}_i)\}$$

has a $\mathbb{P}_2$-probability strictly larger than zero. Notice that $B = \{\mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{H}_i) \le \alpha\} \cap \{\alpha < \mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{G}_i)\} \in \mathscr{H}_i$. That is why we can write

$$\mathbb{E}_2[\mathbb{1}_{G_j}\mathbb{1}_B] = \mathbb{E}_2[\mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{G}_i)\mathbb{1}_B] > \alpha\mathbb{P}(B) \ge \mathbb{E}_2[\mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{H}_i)\mathbb{1}_B] = \mathbb{E}_2[\mathbb{1}_{G_j}\mathbb{1}_B],$$

which is impossible. The case of $\mathbb{P}_2(\{\mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{G}_i) \le \alpha < \mathbb{E}_2(\mathbb{1}_{G_j}|\mathscr{H}_i)\}) > 0$ is completely analogous. This obviously gives us

$$\mathbb{E}_2(G(\omega_1,\cdot)|\mathscr{G}_{i+1}) - \mathbb{E}_2(G(\omega_1,\cdot)|\mathscr{G}_i) = \mathbb{E}_2(G(\omega_1,\cdot)|\mathscr{H}_{i+1}) - \mathbb{E}_2(G(\omega_1,\cdot)|\mathscr{H}_i),$$

meaning that, without loss of generality, we can assume that every $\mathscr{G}_i$, $i \in \mathbb{N}_0$, from (3.10), is countably generated σ-algebra. Additionally, by application of Theorem 1.2.7 we can work with finitely generated $\mathscr{G}_i$, $i \in \mathbb{N}_0$.

Let us for a moment assume that, for each $i \in \mathbb{N}_0$, $\mathscr{G}_i$ is a σ-algebra generated with a partition $\mathscr{A}_i = \{G_1^{(i)},\dots,G_{m_i}^{(i)}\}$ of a set Ω such that each element has a $\mathbb{P}_2$-measure strictly larger than zero. As $(\mathscr{G}_i)_{i\in\mathbb{N}_0}$ is a filtration, let us furthermore assume that $\mathscr{A}_{i+1}$ and $\mathscr{A}_i$ have $k_i < m_i$ same elements, while the remaining $m_i - k_i$ elements of $\mathscr{A}_i$ are a disjoint union of two elements of $\mathscr{A}_{i+1}$; let us fix $i \in \mathbb{N}_0$ and, without loss of generality, assume that $G_1^{(i+1)} = G_1^{(i)},\dots,G_{k_i}^{(i+1)} = G_{k_i}^{(i)}$; additionally, let $G_k^{(i)} = G_{k,1}^{(i+1)} \cup G_{k,2}^{(i+1)}$ where $k \in \{k_i+1,\dots,m_i\}$ and $G_{k,1}^{(i+1)}, G_{k,2}^{(i+1)} \in \mathscr{A}^{(i+1)}$. We have

$$\mathbb{E}_2(G(\omega_1,\cdot)|\mathscr{G}_{i+1}) - \mathbb{E}_2(G(\omega_1,\cdot)|\mathscr{G}_i)$$

$$
\begin{aligned}
&= \sum_{k=0}^{m_{i+1}} \frac{\mathbb{E}_2[\mathbb{1}_{G_k^{(i+1)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_k^{(i+1)})} \mathbb{1}_{G_k^{(i+1)}} - \sum_{k=0}^{m_i} \frac{\mathbb{E}_2[\mathbb{1}_{G_k^{(i)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_k^{(i)})} \mathbb{1}_{G_k^{(i)}} \\
&= \sum_{k=k_i+1}^{m_i} \Bigg(\Big(\frac{\mathbb{E}_2[\mathbb{1}_{G_{k,1}^{(i+1)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_{k,1}^{(i+1)})} \mathbb{1}_{G_{k,1}^{(i+1)}} + \frac{\mathbb{E}_2[\mathbb{1}_{G_{k,2}^{(i+1)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_{k,2}^{(i+1)})} \mathbb{1}_{G_{k,2}^{(i+1)}} \\
&\qquad - \frac{\mathbb{E}_2[\mathbb{1}_{G_k^{(i)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_k^{(i)})} \mathbb{1}_{G_k^{(i)}} \Big) \Bigg) \\
&= \sum_{k=k_i+1}^{m_i} \Bigg(\Big(\frac{\mathbb{E}_2[\mathbb{1}_{G_{k,1}^{(i+1)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_{k,1}^{(i+1)})} - \frac{\mathbb{E}_2[\mathbb{1}_{G_k^{(i)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_k^{(i)})} \Big) \mathbb{1}_{G_{k,1}^{(i+1)}} \\
&\qquad + \Big(\frac{\mathbb{E}_2[\mathbb{1}_{G_{k,2}^{(i+1)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_{k,2}^{(i+1)})} - \frac{\mathbb{E}_2[\mathbb{1}_{G_k^{(i)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_k^{(i)})} \Big) \mathbb{1}_{G_{k,2}^{(i+1)}} \Bigg) \\
&= \sum_{k=k_i+1}^{m_i} \Bigg(\Big(\mathbb{E}_2[\mathbb{1}_{G_{k,1}^{(i+1)}} G(\omega_1,\cdot)] \Big(\frac{1}{\mathbb{P}_2(G_{k,1}^{(i+1)})} - \frac{1}{\mathbb{P}_2(G_k^{(i)})} \Big) - \frac{\mathbb{E}_2[\mathbb{1}_{G_{k,2}^{(i+1)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_k^{(i)})} \Big) \mathbb{1}_{G_{k,1}^{(i+1)}} \\
&\qquad + \Big(\mathbb{E}_2[\mathbb{1}_{G_{k,2}^{(i+1)}} G(\omega_1,\cdot)] \Big(\frac{1}{\mathbb{P}_2(G_{k,2}^{(i+1)})} - \frac{1}{\mathbb{P}_2(G_k^{(i)})} \Big) - \frac{\mathbb{E}_2[\mathbb{1}_{G_{k,1}^{(i+1)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_k^{(i)})} \Big) \mathbb{1}_{G_{k,2}^{(i+1)}} \Bigg) \\
&= \sum_{k=k_i+1}^{m_i} \Bigg(\Big(\mathbb{E}_2[\mathbb{1}_{G_{k,1}^{(i+1)}} G(\omega_1,\cdot)] \frac{\mathbb{P}_2(G_{k,2}^{(i+1)})}{\mathbb{P}_2(G_{k,1}^{(i+1)})\mathbb{P}_2(G_k^{(i)})} - \frac{\mathbb{E}_2[\mathbb{1}_{G_{k,2}^{(i+1)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_k^{(i)})} \Big) \mathbb{1}_{G_{k,1}^{(i+1)}} \\
&\qquad + \Big(\mathbb{E}_2[\mathbb{1}_{G_{k,2}^{(i+1)}} G(\omega_1,\cdot)] \frac{\mathbb{P}_2(G_{k,1}^{(i+1)})}{\mathbb{P}_2(G_{k,2}^{(i+1)})\mathbb{P}_2(G_k^{(i)})} - \frac{\mathbb{E}_2[\mathbb{1}_{G_{k,1}^{(i+1)}} G(\omega_1,\cdot)]}{\mathbb{P}_2(G_k^{(i)})} \Big) \mathbb{1}_{G_{k,2}^{(i+1)}} \Bigg).
\end{aligned}
$$

Motivated by this, let us define *adapted Haar functions* as

$$
\mathbb{h}_{G_1^{(i)}} = \cdots = \mathbb{h}_{G_{k_i}^{(i)}} \equiv 0,
$$

$$
\mathbb{h}_{G_k^{(i)}} := \sqrt{\frac{\mathbb{P}_2(G_{k,2}^{(i+1)})}{\mathbb{P}_2(G_{k,1}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}} \mathbb{1}_{G_{k,1}^{(i+1)}} - \sqrt{\frac{\mathbb{P}_2(G_{k,1}^{(i+1)})}{\mathbb{P}_2(G_{k,2}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}} \mathbb{1}_{G_{k,2}^{(i+1)}} \tag{3.12}
$$

for each $k = k_i + 1, \ldots, m_i$. Notice that the non-zero functions defined like this are supported on the sets appearing in their respective indices and that they are constant on each of two subsets from their definition formula. Also, they are cancellative and mutually orthogonal. The cancellation is easy to notice:

$$
\mathbb{E}\mathbb{h}_{G_k^{(i)}} = \sqrt{\frac{\mathbb{P}_2(G_{k,2}^{(i+1)})}{\mathbb{P}_2(G_k^{(i)})}} \sqrt{\mathbb{P}_2(G_{k,1}^{(i+1)})} - \sqrt{\frac{\mathbb{P}_2(G_{k,1}^{(i+1)})}{\mathbb{P}_2(G_k^{(i)})}} \sqrt{\mathbb{P}_2(G_{k,2}^{(i+1)})} = 0.
$$

To show the orthogonality, take any $i,j=1,\dots,n$ and $k=k_i+1,\dots,m_i$, $l=k_j+1,\dots,m_j$ such that $i\neq j$ or $k\neq l$. Sets $G_k^{(i)}$ and $G_l^{(j)}$ are either disjoint, so $\mathbb{E}\big(\mathbb{h}_{G_k^{(i)}}\mathbb{h}_{G_l^{(j)}}\big)=0$ because of disjoint support of functions, or $G_l^{(j)}\subsetneq G_k^{(i)}$; without loss of generality, $G_l^{(j)}\subseteq G_{k,1}^{(i+1)}$. As $\mathbb{h}_{G_k^{(i)}}$ is constant of the support of $\mathbb{h}_{G_l^{(j)}}$,

$$\mathbb{E}\big(\mathbb{h}_{G_k^{(i)}}\mathbb{h}_{G_l^{(j)}}\big)=\sqrt{\frac{\mathbb{P}_2(G_{k,2}^{(i+1)})}{\mathbb{P}_2(G_{k,1}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}}\mathbb{E}\mathbb{h}_{G_l^{(j)}}=0.$$

Such definition of functions was motivated so that we would have

$$\begin{aligned}&\sum_{k=1}^{m_i}\mathbb{E}_2(G(\omega_1,\cdot)\mathbb{h}_{G_k^{(i)}})\mathbb{h}_{G_k^{(i)}}=\sum_{k=k_i+1}^{m_i}\mathbb{E}_2(G(\omega_1,\cdot)\mathbb{h}_{G_k^{(i)}})\mathbb{h}_{G_k^{(i)}}\\
=&\sum_{k=k_i+1}^{m_i}\Bigg(\Bigg(\frac{\mathbb{P}_2(G_{k,2}^{(i+1)})}{\mathbb{P}_2(G_{k,1}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}\mathbb{E}_2[\mathbb{1}_{G_{k,1}^{(i+1)}}G(\omega_1,\cdot)]-\frac{1}{\mathbb{P}_2(G_k^{(i)})}\mathbb{E}_2[\mathbb{1}_{G_{k,2}^{(i+1)}}G(\omega_1,\cdot)]\Bigg)\mathbb{1}_{G_{k,1}^{(i+1)}}\\
&+\Bigg(-\frac{1}{\mathbb{P}_2(G_k^{(i)})}\mathbb{E}_2[\mathbb{1}_{G_{k,1}^{(i+1)}}G(\omega_1,\cdot)]+\frac{\mathbb{P}_2(G_{k,1}^{(i+1)})}{\mathbb{P}_2(G_{k,2}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}\mathbb{E}_2[\mathbb{1}_{G_{k,2}^{(i+1)}}G(\omega_1,\cdot)]\Bigg)\mathbb{1}_{G_{k,2}^{(i+1)}}\Bigg)\\
=&\mathbb{E}_2(G(\omega_1,\cdot)|\mathscr{G}_{i+1})-\mathbb{E}_2(G(\omega_1,\cdot)|\mathscr{G}_i).\end{aligned}$$

Now we can see an analogy with the standard Haar function basis, as its functions are also supported on disjoint sets which form a partition of a whole space and also their supports split in two subsets which are elements of the succeeding σ-algebra of a standard dyadic filtration. Returning to (3.10), we would now like to show that

$$\Big\|\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\mathbb{E}_1(F|\mathscr{F}_i)\mathbb{E}_2(G\mathbb{h}_{G_k^{(i)}})\mathbb{h}_{G_k^{(i)}}\Big\|_{\mathrm{L}^r(\Omega_1\times\Omega_2)}\lesssim\|F\|_{\mathrm{L}^p(\Omega_1\times\Omega_2)}\|G\|_{\mathrm{L}^q(\Omega_1\times\Omega_2)}. \quad (3.13)$$

Note that this was possible with the assumption that elements of $\mathscr{A}_i$ either remain the same or they can be separated in two mutually disjoint subsets of non-zero $\mathbb{P}_2$-measure in order to obtain $\mathscr{A}_{i+1}$. Let us consider more general case when there exists $i_0\in\mathbb{N}_0$ such that, unless $\mathscr{A}_{i_0}=\mathscr{A}_{i_0+1}$ (and then we have a trivial case $\mathscr{G}_{i_0}=\mathscr{G}_{i_0+1}$), there are $G_1^{(i_0)},\dots,G_k^{(i_0)}\in\mathscr{A}_{i_0}$ such that, for each $j\in\{1,\dots,k\}$, there exist mutually different $G_{j,1}^{(i_0+1)},\dots,G_{j,k_j}^{(i_0+1)}\in\mathscr{A}_{i_0+1}$ such that $G_j^{(i_0)}=\cup_{l=1}^{k_j}G_{j,l}^{(i_0+1)}$. Let $K:=\max_{1\leq j\leq k}k_j>2$. Let us construct $\mathscr{A}_1'$ from $\mathscr{A}_{i_0}$ in a way that, for each $j=1,\dots,k$, we remove sets $G_1^{(i_0)},\dots,G_k^{(i_0)}$ and add $G_{1,1}^{(i_0+1)},\dots,G_{k,1}^{(i_0+1)}$ and $G_1^{(i_0)}\backslash G_{1,1}^{(i_0+1)},\dots,G_k^{(i_0)}\backslash G_{k,1}^{(i_0+1)}$. Inductively, we construct each $\mathscr{A}_l'$, $l\in\{2,\dots,K-2\}$ in a way that, for each $j\in\{1,\dots,k\}$ such that $l<k_j$, we remove

$G_j^{(i_0)}\backslash(\cup_{p=1}^{l-1}G_{p,j}^{(i_0+1)})$ and then add $G_{l,j}^{(i_0+1)}$ and $G_j^{(i_0)}\backslash(\cup_{p=1}^{l}G_{p,j}^{(i_0+1)})$. Now, let

$$\mathscr{G}_l' := \begin{cases} \mathscr{G}_l, & 0 \le l \le i_0, \\ \sigma(\mathscr{A}_{l-i_0}'), & i_0+1 \le l \le i_0+K-2, \\ \mathscr{G}_{l-K+2}, & l \ge i_0+K-1, \end{cases} \quad \mathscr{F}_l' := \begin{cases} \mathscr{F}_l, & 0 \le l \le i_0, \\ \mathscr{F}_{i_0}, & i_0+1 \le l \le i_0+K-2, \\ \mathscr{F}_{l-K+2}, & l \ge i_0+K-1. \end{cases}$$

This construction helped us obtain the following. As the transition from $\mathscr{G}_{i_0}$ to $\mathscr{G}_{i_0+1}$ can happen by breaking more atoms into more than two parts, we added σ-algebras in between so that the transition from $\mathscr{G}_{i_0}'$ from $\mathscr{G}_{i_0+K}'$ (which are same σ-algebras as those that we started with) actually happens by breaking atoms into exactly two parts, step by step. If we replace $(\mathscr{G}_m)_{m\in\mathbb{N}_0}$ with $(\mathscr{G}_m')_{m\in\mathbb{N}_0}$ and, additionally, $(\mathscr{F}_m)_{m\in\mathbb{N}_0}$ with $(\mathscr{F}_m')_{m\in\mathbb{N}_0}$, we can notice that

$$\begin{aligned}\sum_{i=0}^{n+K-3} \mathbb{E}_1(F|\mathscr{F}_i')(\mathbb{E}_2(G|\mathscr{G}_{i+1}') - \mathbb{E}_2(G|\mathscr{G}_i')) &= \sum_{i=0}^{i_0-1} \mathbb{E}_1(F|\mathscr{F}_i)(\mathbb{E}_2(G|\mathscr{G}_{i+1}) - \mathbb{E}_2(G|\mathscr{G}_i)) \\ &\quad + \sum_{i=i_0}^{i_0+K-2} \mathbb{E}_1(F|\mathscr{F}_{i_0})(\mathbb{E}_2(G|\mathscr{G}_{i+1}') - \mathbb{E}_2(G|\mathscr{G}_i')) \\ &\quad + \sum_{i=i_0+K+1}^{n-1} \mathbb{E}_1(F|\mathscr{F}_i)(\mathbb{E}_2(G|\mathscr{G}_{i+1}) - \mathbb{E}_2(G|\mathscr{G}_i)) \\ &= \sum_{i=0}^{n-1} \mathbb{E}_1(F|\mathscr{F}_i)(\mathbb{E}_2(G|\mathscr{G}_{i+1}) - \mathbb{E}_2(G|\mathscr{G}_i)),\end{aligned}$$

as the second sum is actually a telescoping sum:

$$\begin{aligned}\sum_{i=i_0}^{i_0+K-2} &\mathbb{E}_1(F|\mathscr{F}_{i_0})(\mathbb{E}_2(G|\mathscr{G}_{i+1}') - \mathbb{E}_2(G|\mathscr{G}_i')) \\ &= \mathbb{E}_1(F|\mathscr{F}_{i_0}) \sum_{i=i_0}^{i_0+K-2} (\mathbb{E}_2(G|\mathscr{G}_{i+1}') - \mathbb{E}_2(G|\mathscr{G}_i')) \\ &= \mathbb{E}_1(F|\mathscr{F}_{i_0})(\mathbb{E}_2(G|\mathscr{G}_{i_0+K-1}') - \mathbb{E}_2(G|\mathscr{G}_{i_0}')) \\ &= \mathbb{E}_1(F|\mathscr{F}_{i_0})(\mathbb{E}_2(G|\mathscr{G}_{i_0+1}) - \mathbb{E}_2(G|\mathscr{G}_{i_0})).\end{aligned}$$

We can notice that, by shifting both σ-algebras and also copying $\mathscr{F}_{i_0}$ and adding intermediate σ-algebras between $\mathscr{G}_{i_0}$ and $\mathscr{G}_{i_0+1}$, we still keep the property of filtration that we wanted, but also the paraproduct remains unchanged, so we are still trying to bound the exact same expression. As this construction required adding only finitely many new σ-subalgebras, what we have obtained here is that not only we reduced to the case when

σ-algebras $\mathscr{G}_i$, $i \in \mathbb{N}_0$ are finitely generated, but also that each atom from $\mathscr{G}_i$ either remains an atom in $\mathscr{G}_{i+1}$ or can be presented as a disjoint union of exactly two atoms from $\mathscr{G}_{i+1}$.

Once both filtrations have finitely generated σ-algebras, we are able to show a result that will turn out to be helpful later, during the attempt of extending the range of exponents for our main result. We can see that (3.10) will not only give us the boundedness for the specific triple (p,q,r), but also a range that will vary for exponents q and r.

Lemma 3.2.1. Let $(X,\mathscr{F},\mu)$ and $(Y,\mathscr{G},\nu)$ be probability spaces and let $(\mathscr{F}_m)_{m\in\mathbb{N}_0}$ and $(\mathscr{G}_m)_{m\in\mathbb{N}_0}$ be filtrations on, respectively, $\mathscr{F}$ and $\mathscr{G}$ such that $\mathscr{F}_m$ and $\mathscr{G}_m$ are finitely generated for each $m \in \mathbb{N}_0$. If we have

$$\Big\|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(\mathbb{E}_\nu(G|\mathscr{G}_{i+1})-\mathbb{E}_\nu(G|\mathscr{G}_i))\Big\|_{\mathrm{L}^r(X\times Y)} \lesssim \|F\|_{\mathrm{L}^p(X\times Y)}\|G\|_{\mathrm{L}^q(X\times Y)} \tag{3.14}$$

for some simple $(\mathscr{F}\times\mathscr{G})$-measurable functions $F,G : X\times Y \to \mathbb{R}$ and p,q,r such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, then we have

$$\Big\|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(\mathbb{E}_\nu(G|\mathscr{G}_{i+1})-\mathbb{E}_\nu(G|\mathscr{G}_i))\Big\|_{\mathrm{L}^{r_t}(X\times Y)} \lesssim \|F\|_{\mathrm{L}^p(X\times Y)}\|G\|_{\mathrm{L}^{q_t}(X\times Y)}$$

for each $r_t = \big(\frac{t}{r}+(1-t)\big(1+\frac{1}{p}\big)\big)^{-1}$ and $q_t = \big(\frac{t}{q}+1-t\big)^{-1}$ where $t \in \langle 0,1\rangle$ such that $r_t, q_t \in [1,\infty]$.

Proof. The key of the proof lies in Theorem 1.1.6 and the weak boundedness

$$\Big\|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(\mathbb{E}_\nu(G|\mathscr{G}_{i+1})-\mathbb{E}_\nu(G|\mathscr{G}_i))\Big\|_{\mathrm{L}^{\frac{p}{p+1},\infty}(X\times Y)} \lesssim \|F\|_{\mathrm{L}^p(X\times Y)}\|G\|_{\mathrm{L}^1(X\times Y)}. \tag{3.15}$$

A specific case of one of filtrations being a dyadic filtration can be found in Kovač's article [33]. However, unlike his approach of cutting dyadic intervals in halves of equal length by Lebesgue measure, we will have to approach this problem in different way, for general martingales. Let us assume that $\|F\|_{\mathrm{L}^p(X\times Y)} = \|G\|_{\mathrm{L}^1(X\times Y)} = 1$. Recall that F and G are finite linear combination of characteristic functions over sets from $\mathscr{F}\times\mathscr{G}$ so it is not difficult to notice that each of the conditional expectations $\mathbb{E}_\mu(F|\mathscr{F}_i)$ and $\mathbb{E}_\nu(F|\mathscr{G}_i)$ are also $(\mathscr{F}\times\mathscr{G})$-measurable for each $i=0,\dots,n$. Following the proof from [39], we are going to fix $x \in X$ and use Theorem 1.2.13 for $\alpha = s^{\frac{p}{p+1}}$ and where $s \in \langle 0,\infty\rangle$ is arbitrary, for splitting martingale $(\mathbb{E}_\nu(G(x,\cdot)|\mathscr{G}_m))_{m\in\mathbb{N}_0}$ into three parts:

good part $G^g(x,\cdot)=(G^g_m(x,\cdot))_{m\in\mathbb{N}_0}$, bad part $G^b(x,\cdot)=(G^b_m(x,\cdot))_{m\in\mathbb{N}_0}$ and harmless part $G^h(x,\cdot)=(G^h_m(x,\cdot))_{m\in\mathbb{N}_0}$. They satisfy the following identities:

$$\mathbb{E}_\nu(G(x,\cdot)|\mathscr{G}_m)=G^g_m(x,\cdot)+G^b_m(x,\cdot)+G^h_m(x,\cdot) \text{ for each } m\in\mathbb{N}_0,$$
$$G^g_0(x,\cdot)=\mathbb{E}_\nu(G(x,\cdot)|\mathscr{G}_0), G^b_0(x,\cdot)=G^h_0(x,\cdot)\equiv 0,$$
$$\|G^g_n(x,\cdot)\|_{\mathrm{L}^\infty(Y)}\le 2s^{\frac{p}{p+1}}, \|G^g_n(x,\cdot)\|_{\mathrm{L}^1(Y)}\le 4\|\mathbb{E}_\nu(G(x,\cdot)|\mathscr{G}_n)\|_{\mathrm{L}^1(Y)},$$
$$\nu\big(\max_{0\le m\le n}|G^b_m(x,\cdot)|>0\big)\le 3s^{-\frac{p}{p+1}}\|\mathbb{E}_\nu(G(x,\cdot)|\mathscr{G}_n)\|_{\mathrm{L}^1(Y)},$$
$$\sum_{m=0}^{n-1}\|G^h_{m+1}(x,\cdot)-G^h_m(x,\cdot)\|_{\mathrm{L}^1(Y)}\le 4\|\mathbb{E}_\nu(G(x,\cdot)|\mathscr{G}_n)\|_{\mathrm{L}^1(Y)}.$$

With eventual integration over $x\in X$ and taking into consideration that $(X,\mathscr{F},\mu)$ is a probability space, we can rewrite these as

$$\mathbb{E}_\nu(G|\mathscr{G}_m)=G^g_m+G^b_m+G^h_m \text{ for each } m\in\mathbb{N}_0,$$
$$G^g_0=\mathbb{E}_\nu(G|\mathscr{G}_0), G^b_0=G^h_0\equiv 0,$$
$$\|G^g_n\|_{\mathrm{L}^\infty(X\times Y)}\le 2s^{\frac{p}{p+1}}, \|G^g_n\|_{\mathrm{L}^1(X\times Y)}\le 4\|\mathbb{E}_\nu(G|\mathscr{G}_n)\|_{\mathrm{L}^1(X\times Y)},$$
$$(\mu\times\nu)\big(\max_{0\le m\le n}|G^b_m|>0\big)\le 3s^{-\frac{p}{p+1}}\|\mathbb{E}_\nu(G|\mathscr{G}_n)\|_{\mathrm{L}^1(X\times Y)},$$
$$\sum_{m=0}^{n-1}\|G^h_{m+1}-G^h_m\|_{\mathrm{L}^1(X\times Y)}\le 4\|\mathbb{E}_\nu(G|\mathscr{G}_n)\|_{\mathrm{L}^1(X\times Y)}.$$

Let us observe the same paraproduct where, instead of martingale $(\mathbb{E}_\nu(G|\mathscr{G}_m))_{m\in\mathbb{N}_0}$ we have its good, bad and harmless part. From the assumption (3.14) we have

$$\begin{aligned}
&(\mu\times\nu)\Big(\Big|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(G^g_{i+1}-G^g_i)\Big|>\frac{s}{2}\Big)\\
&\le\Big(\frac{s}{2}\Big)^{-r}\Big\|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(G^g_{i+1}-G^g_i)\Big\|^r_{\mathrm{L}^r(X\times Y)}\\
&\le 2^r s^{-r}\|F\|^r_{\mathrm{L}^p(X\times Y)}\|G^g_n\|^r_{\mathrm{L}^q(X\times Y)}\le 2^r s^{-r}\|G^g_n\|^{\frac{r(q-1)}{q}}_{\mathrm{L}^\infty(X\times Y)}\|G^g_n\|^{\frac{r}{q}}_{\mathrm{L}^1(X\times Y)}\\
&\le 2^{\frac{pq}{p+q}}s^{-\frac{pq}{p+q}}\cdot 2^{\frac{p(q-1)}{p+q}}s^{\frac{p^2(q-1)}{(p+1)(p+q)}}\cdot 4^{\frac{p}{p+q}}\|\mathbb{E}_\nu(G|\mathscr{G}_n)\|^{\frac{p}{p+q}}_{\mathrm{L}^1(X\times Y)}\le 2^{\frac{p(2q+1)}{p+q}}s^{-\frac{p}{p+1}}\lesssim s^{-\frac{p}{p+1}}.
\end{aligned}$$

Here we also used the fact that $(G^g_m)_{m\in\mathbb{N}_0}$ is a martingale with respect to $(\mathscr{G}_m)_{m\in\mathbb{N}_0}$, so $\mathbb{E}_\nu(G^g_n|\mathscr{G}_i)=G^g_i$ for each $i\in\{0,\dots,n-1\}$. Also,

$$\begin{aligned}
(\mu\times\nu)\Big(\Big|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(G^b_{i+1}-G^b_i)\Big|>0\Big)&\le(\mu\times\nu)\big(\max_{0\le m\le n}|G^b_m|>0\big)\\
&\le 3s^{-\frac{p}{p+1}}\|\mathbb{E}_\nu(G|\mathscr{G}_n)\|_{\mathrm{L}^1(X\times Y)}\lesssim s^{-\frac{p}{p+1}}.
\end{aligned}$$

As for the harmless part,

$$
\begin{aligned}
&(\mu\times\nu)\Big(\Big|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(G^h_{i+1}-G^h_i)\Big|>\frac{s}{2}\Big)\\
&\leq\Big(\frac{s}{2}\Big)^{-\frac{p}{p+1}}\Big\|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(G^h_{i+1}-G^h_i)\Big\|_{\mathrm{L}^{\frac{p}{p+1}}(X\times Y)}^{\frac{p}{p+1}}\\
&\leq 2^{\frac{p}{p+1}}s^{-\frac{p}{p+1}}\Big\|\max_{0\leq m\leq n-1}|\mathbb{E}_\mu(F|\mathscr{F}_m)|\sum_{i=0}^{n-1}|G^h_{i+1}-G^h_i|\Big\|_{\mathrm{L}^{\frac{p}{p+1}}(X\times Y)}^{\frac{p}{p+1}}\\
&\leq 2^{\frac{p}{p+1}}s^{-\frac{p}{p+1}}\|\max_{0\leq m\leq n-1}|\mathbb{E}_\mu(F|\mathscr{F}_m)|\|_{\mathrm{L}^p(X\times Y)}^{\frac{p}{p+1}}\Big\|\sum_{i=0}^{n-1}|G^h_{i+1}-G^h_i|\Big\|_{\mathrm{L}^1(X\times Y)}^{\frac{p}{p+1}}\\
&\leq 2^{\frac{p}{p+1}}s^{-\frac{p}{p+1}}\|F\|_{\mathrm{L}^p(X\times Y)}^{\frac{p}{p+1}}\Big(\sum_{i=0}^{n-1}\|G^h_{i+1}-G^h_i\|_{\mathrm{L}^1(X\times Y)}\Big)^{\frac{p}{p+1}}\\
&\leq 2^{\frac{p}{p+1}+2}s^{-\frac{p}{p+1}}\|\mathbb{E}_\nu(G|\mathscr{G}_n)\|_{\mathrm{L}^1(X\times Y)}^{\frac{p}{p+1}}\lesssim s^{-\frac{p}{p+1}}.
\end{aligned}
$$

We used the fact that the martingale maximal operator $F\mapsto\max_{0\leq m\leq n-1}|\mathbb{E}_\mu(F|\mathscr{F}_m)|$ is bounded from $\mathrm{L}^p(X\times Y)$ to $\mathrm{L}^p(X\times Y)$ which follows from Theorem 1.5.7. Overall,

$$
\begin{aligned}
(\mu\times\nu)\Big(\Big|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(\mathbb{E}_\nu(G|\mathscr{G}_{i+1})-\mathbb{E}_\nu(G|\mathscr{G}_i))\Big|>s\Big)&\\
\leq(\mu\times\nu)\Big(\Big|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(G^g_{i+1}-G^g_i)\Big|>\frac{s}{2}\Big)&\\
+(\mu\times\nu)\Big(\Big|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(G^b_{i+1}-G^b_i)\Big|>0\Big)&\\
+(\mu\times\nu)\Big(\Big|\sum_{i=0}^{n-1}\mathbb{E}_\mu(F|\mathscr{F}_i)(G^h_{i+1}-G^h_i)\Big|>\frac{s}{2}\Big)&\lesssim s^{-\frac{p}{p+1}}.
\end{aligned}
$$

Since $s\in\langle 0,\infty\rangle$ was arbitrary, this gives (3.15). ■

It is important to emphasize that, while we are applying the Gundy's decomposition for each $x\in X$ and the cardinal number of X can be arbitrary, thanks to the fact that σ-algebras $\mathscr{G}_1,\ldots,\mathscr{G}_n$ are finitely generated, the set of values $\{\mathbb{E}_\nu(G|\mathscr{G}_i): i\in\{1,\ldots,n\}\}$ is finite for each $n\in\mathbb{N}$ so we are effectively decomposing only finitely many times.

3.2.6. Estimating forms to obtain the first pair of exponents

The inequality (3.13) is equivalent to

$$\Big| \iint_{\Omega_1\times\Omega_2} \sum_{i=0}^{n-1}\sum_{k=1}^{m_i} \mathbb{E}_1(F|\mathscr{F}_i)\mathbb{E}_2(G\mathbb{h}_{G_k^{(i)}})\mathbb{h}_{G_k^{(i)}} H d(\mathbb{P}_1\times\mathbb{P}_2)\Big| \lesssim \|F\|_{\mathrm{L}^p(\Omega_1\times\Omega_2)}\|G\|_{\mathrm{L}^q(\Omega_1\times\Omega_2)}\|H\|_{\mathrm{L}^{r'}(\Omega_1\times\Omega_2)}$$

for any choice of $H\in\mathrm{L}^{r'}(\Omega_1\times\Omega_2)$, r' being the conjugate exponent of r.

Let $\{F_1^{(i)},\dots,F_{n_i}^{(i)}\}$ be a collection of mutually disjoint atoms of $\mathscr{F}_i$, ordered similarly as $\{G_1^{(i)},\dots,G_{m_i}^{(i)}\}$ have been chosen, and let $\mathbb{h}_{F_k^{(i)}}$ be defined analogously as in (3.12) for each $k\in\{1,\dots,n_i\}$ and $i\in\mathbb{N}$.

The following idea is inspired by [33]. In order to bound the form

$$\begin{aligned}\Lambda(F,G,H) &:= \iint_{\Omega_1\times\Omega_2} \sum_{i=0}^{n-1}\sum_{k=1}^{m_i} \mathbb{E}_1(F|\mathscr{F}_i)\mathbb{E}_2(G\mathbb{h}_{G_k^{(i)}})\mathbb{h}_{G_k^{(i)}} H d(\mathbb{P}_1\times\mathbb{P}_2)\\ &= \int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i} F(\omega_1',\omega_2)\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)G(\omega_1,\omega_2')\mathbb{h}_{G_k^{(i)}}(\omega_2')\mathbb{h}_{G_k^{(i)}}(\omega_2)\\ &\qquad H(\omega_1,\omega_2)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1'), \end{aligned} \tag{3.16}$$

we are going to observe few additional forms:

$$\Theta_1(F_1,F_2,F_3,F_4) := \int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i} F_1(\omega_1',\omega_2)F_2(\omega_1,\omega_2')F_3(\omega_1,\omega_2)$$
$$F_4(\omega_1',\omega_2')\mathbb{h}_{G_k^{(i)}}(\omega_2')\mathbb{h}_{G_k^{(i)}}(\omega_2)\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1'),$$
$$\Theta_2(F_1,F_2,F_3,F_4) := \int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i}\sum_{j\in\{1,2\}} F_1(\omega_1',\omega_2)F_2(\omega_1,\omega_2')F_3(\omega_1,\omega_2)$$
$$F_4(\omega_1',\omega_2')\mathbb{h}^0_{G_{k,j}^{(i+1)}}(\omega_2')\mathbb{h}^0_{G_{k,j}^{(i+1)}}(\omega_2)\mathbb{h}_{F_l^{(i)}}(\omega_1')\mathbb{h}_{F_l^{(i)}}(\omega_1)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1'),$$
$$\Xi^{(i)}(F_1,F_2,F_3,F_4) := \int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i} F_1(\omega_1',\omega_2)F_2(\omega_1,\omega_2')F_3(\omega_1,\omega_2)$$
$$F_4(\omega_1',\omega_2')\mathbb{h}^0_{G_{k,j}^{(i+1)}}(\omega_2')\mathbb{h}^0_{G_{k,j}^{(i+1)}}(\omega_2)\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')$$

for each $i\in\{0,\dots,n\}$. First we are going to show that

$$\Theta_1(F_1,F_2,F_3,F_4)+\Theta_2(F_1,F_2,F_3,F_4) = \Xi^{(n)}(F_1,F_2,F_3,F_4)-\Xi^{(0)}(F_1,F_2,F_3,F_4). \tag{3.17}$$

Let us fix $\omega_1, \omega_1' \in \Omega_1, \omega_2, \omega_2' \in \Omega_2$, $i \in \{0, \dots, n-1\}$, $k \in \{1, \dots, m_i\}$ and $l \in \{1, \dots, n_i\}$. Notice that

$$\begin{aligned}
&\mathbb{h}^0_{G_k^{(i)}}(\omega_2')\mathbb{h}^0_{G_k^{(i)}}(\omega_2) + \mathbb{h}_{G_k^{(i)}}(\omega_2')\mathbb{h}_{G_k^{(i)}}(\omega_2) \\
&\quad= \frac{1}{\mathbb{P}_2(G_k^{(i)})}\big(\mathbb{1}_{G_{k,1}^{(i+1)}}(\omega_2') + \mathbb{1}_{G_{k,2}^{(i+1)}}(\omega_2')\big)\big(\mathbb{1}_{G_{k,1}^{(i+1)}}(\omega_2) + \mathbb{1}_{G_{k,2}^{(i+1)}}(\omega_2)\big)\big) \\
&\qquad+ \Bigg(\sqrt{\frac{\mathbb{P}_2(G_{k,2}^{(i+1)})}{\mathbb{P}_2(G_{k,1}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}}\mathbb{1}_{G_{k,1}^{(i+1)}}(\omega_2') - \sqrt{\frac{\mathbb{P}_2(G_{k,1}^{(i+1)})}{\mathbb{P}_2(G_{k,2}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}}\mathbb{1}_{G_{k,2}^{(i+1)}}(\omega_2')\Bigg) \\
&\qquad\cdot \Bigg(\sqrt{\frac{\mathbb{P}_2(G_{k,2}^{(i+1)})}{\mathbb{P}_2(G_{k,1}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}}\mathbb{1}_{G_{k,1}^{(i+1)}}(\omega_2) - \sqrt{\frac{\mathbb{P}_2(G_{k,1}^{(i+1)})}{\mathbb{P}_2(G_{k,2}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}}\mathbb{1}_{G_{k,2}^{(i+1)}}(\omega_2)\Bigg) \\
&\quad= \Bigg(\frac{1}{\mathbb{P}_2(G_k^{(i)})} + \frac{\mathbb{P}_2(G_{k,2}^{(i+1)})}{\mathbb{P}_2(G_{k,1}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}\Bigg)\mathbb{1}_{G_{k,1}^{(i+1)}}(\omega_2')\mathbb{1}_{G_{k,1}^{(i+1)}}(\omega_2) \\
&\qquad+ \Bigg(\frac{1}{\mathbb{P}_2(G_k^{(i)})} + \frac{\mathbb{P}_2(G_{k,1}^{(i+1)})}{\mathbb{P}_2(G_{k,2}^{(i+1)})\mathbb{P}_2(G_k^{(i)})}\Bigg)\mathbb{1}_{G_{k,2}^{(i+1)}}(\omega_2')\mathbb{1}_{G_{k,2}^{(i+1)}}(\omega_2) \\
&\quad= \frac{1}{\mathbb{P}_2(G_{k,1}^{(i+1)})}\mathbb{1}_{G_{k,1}^{(i+1)}}(\omega')\mathbb{1}_{G_{k,1}^{(i+1)}}(\omega) + \frac{1}{\mathbb{P}_2(G_{k,2}^{(i+1)})}\mathbb{1}_{G_{k,2}^{(i+1)}}(\omega_2')\mathbb{1}_{G_{k,2}^{(i+1)}}(\omega_2) \\
&\quad= \mathbb{h}^0_{G_{k,1}^{(i+1)}}(\omega_2')\mathbb{h}^0_{G_{k,1}^{(i+1)}}(\omega_2) + \mathbb{h}^0_{G_{k,2}^{(i+1)}}(\omega_2')\mathbb{h}^0_{G_{k,2}^{(i+1)}}(\omega_2).
\end{aligned}$$

Completely analogously,

$$\begin{aligned}
&\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1) + \mathbb{h}_{F_l^{(i)}}(\omega_1')\mathbb{h}_{F_l^{(i)}}(\omega_1) \\
&\qquad\qquad= \mathbb{h}^0_{F_{l,1}^{(i+1)}}(\omega_1')\mathbb{h}^0_{F_{l,1}^{(i+1)}}(\omega_1) + \mathbb{h}^0_{F_{l,2}^{(i+1)}}(\omega_1')\mathbb{h}^0_{F_{l,2}^{(i+1)}}(\omega_1).
\end{aligned}$$

From these two identities we can use a trivial formula:

$$\begin{aligned}
&\sum_{j_1 \in \{1,2\}} \mathbb{h}^0_{G_{k,j_1}^{(i+1)}}(\omega_2')\mathbb{h}^0_{G_{k,j_1}^{(i+1)}}(\omega_2)\Bigg(\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1) + \mathbb{h}_{F_l^{(i)}}(\omega_1')\mathbb{h}_{F_l^{(i)}}(\omega_1) \\
&\quad - \sum_{j_2 \in \{1,2\}} \mathbb{h}^0_{F_{l,j_2}^{(i+1)}}(\omega_1')\mathbb{h}^0_{F_{l,j_2}^{(i+1)}}(\omega_1)\Bigg) + \mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)\Bigg(\mathbb{h}^0_{G_k^{(i)}}(\omega_2')\mathbb{h}^0_{G_k^{(i)}}(\omega_2) \\
&\quad + \mathbb{h}_{G_k^{(i)}}(\omega_2')\mathbb{h}_{G_k^{(i)}}(\omega_2) - \sum_{j_1 \in \{1,2\}} \mathbb{h}^0_{G_{k,j_1}^{(i+1)}}(\omega_2')\mathbb{h}^0_{G_{k,j_1}^{(i+1)}}(\omega_2)\Bigg) = 0.
\end{aligned}$$

We can rewrite this expression as

$$\begin{aligned}
&\mathbb{h}_{G_k^{(i)}}(\omega_2')\mathbb{h}_{G_k^{(i)}}(\omega_2)\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1) + \sum_{j_1\in\{1,2\}} \mathbb{h}^0_{G_{k,j_1}^{(i+1)}}(\omega_2')\mathbb{h}^0_{G_{k,j_1}^{(i+1)}}(\omega_2)\mathbb{h}_{F_l^{(i)}}(\omega_1')\mathbb{h}_{F_l^{(i)}}(\omega_1)\\
&\quad = \sum_{j_1\in\{1,2\}}\sum_{j_2\in\{1,2\}} \mathbb{h}^0_{G_{k,j_1}^{(i+1)}}(\omega_2')\mathbb{h}^0_{G_{k,j_1}^{(i+1)}}(\omega_2)\mathbb{h}^0_{F_{l,j_1}^{(i+1)}}(\omega_1')\mathbb{h}^0_{F_{l,j_1}^{(i+1)}}(\omega_1)\\
&\qquad - \mathbb{h}^0_{G_k^{(i)}}(\omega_2')\mathbb{h}^0_{G_k^{(i)}}(\omega_2)\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1).
\end{aligned}$$

By multiplying this equation with $F_1(\omega_1',\omega_2)F_2(\omega_1,\omega_2')F_3(\omega_1,\omega_2)F_4(\omega_1',\omega_2')$, summing over $l\in\{1,\dots,n_i\}$, $k\in\{1,\dots,m_i\}$ and $i\in\{0,\dots,n-1\}$ and integrating over $\omega_2,\omega_2'\in\Omega_2$ and $\omega_1,\omega_1'\in\Omega_1$ we obtain

$$\Theta_1(F_1,F_2,F_3,F_4)+\Theta_2(F_1,F_2,F_3,F_4) = \sum_{i=0}^{n-1}(\Xi^{(i+1)}(F_1,F_2,F_3,F_4)-\Xi^{(i)}(F_1,F_2,F_3,F_4)),$$

which corresponds to (3.17). Specially, since $F_1,F_2,F_3,F_4\geq 0$, each form $\Xi^{(i)}$ takes non-negative values, so specially

$$\Theta_1(F_1,F_2,F_3,F_4)+\Theta_2(F_1,F_2,F_3,F_4)\leq \Xi^{(n)}(F_1,F_2,F_3,F_4). \tag{3.18}$$

The double application of Cauchy-Schwarz inequality gives us

$$\begin{aligned}
|\Theta_1(F_1,F_2,F_3,F_4)| &\leq \sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\Big|\int_{\Omega_1}\int_{\Omega_1}\Big(\int_{\Omega_2}F_1(\omega_1',\omega_2)F_3(\omega_1,\omega_2)\mathbb{h}_{G_k^{(i)}}(\omega_2)d\mathbb{P}_2(\omega_2)\Big)\\
&\qquad\cdot\Big(\int_{\Omega_2}F_2(\omega_1,\omega_2')F_4(\omega_1',\omega_2')\mathbb{h}_{G_k^{(i)}}(\omega_2')d\mathbb{P}_2(\omega_2')\Big)\\
&\qquad\cdot\sum_{l=1}^{n_i}\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big|\\
&\leq \sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\Big[\int_{\Omega_1}\int_{\Omega_1}\Big(\int_{\Omega_2}F_1(\omega_1',\omega_2)F_3(\omega_1,\omega_2)\mathbb{h}_{G_k^{(i)}}(\omega_2)d\mathbb{P}_2(\omega_2)\Big)^2\\
&\qquad\cdot\sum_{l=1}^{n_i}\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big]^{\frac12}\\
&\quad\cdot\Big[\int_{\Omega_1}\int_{\Omega_1}\Big(\int_{\Omega_2}F_2(\omega_1,\omega_2')F_4(\omega_1',\omega_2')\mathbb{h}_{G_k^{(i)}}(\omega_2')d\mathbb{P}_2(\omega_2')\Big)^2\\
&\qquad\cdot\sum_{l=1}^{n_i}\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big]^{\frac12}\\
&\leq\Big[\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\int_{\Omega_1}\int_{\Omega_1}\Big(\int_{\Omega_2}F_1(\omega_1',\omega_2)F_3(\omega_1,\omega_2)\mathbb{h}_{G_k^{(i)}}(\omega_2)d\mathbb{P}_2(\omega_2)\Big)^2
\end{aligned}$$

$$
\cdot \sum_{l=1}^{n_i} \mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big]^{\frac{1}{2}}
$$

$$
\cdot\Big[\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\int_{\Omega_1}\int_{\Omega_1}\Big(\int_{\Omega_2}F_2(\omega_1,\omega_2')F_4(\omega_1',\omega_2')\mathbb{h}_{G_k^{(i)}}(\omega_2')d\mathbb{P}_2(\omega_2')\Big)^2
$$

$$
\cdot \sum_{l=1}^{n_i} \mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big]^{\frac{1}{2}}
$$

$$
=\Big[\int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i}F_1(\omega_1',\omega_2)F_3(\omega_1,\omega_2)F_1(\omega_1',\omega_2')F_3(\omega_1,\omega_2')
$$

$$
\cdot\mathbb{h}_{G_k^{(i)}}(\omega_2)\mathbb{h}_{G_k^{(i)}}(\omega_2')\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big]^{\frac{1}{2}}
$$

$$
\Big[\int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i}F_2(\omega_1',\omega_2)F_4(\omega_1,\omega_2)F_2(\omega_1',\omega_2')F_4(\omega_1,\omega_2')
$$

$$
\cdot\mathbb{h}_{G_k^{(i)}}(\omega_2)\mathbb{h}_{G_k^{(i)}}(\omega_2')\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big]^{\frac{1}{2}}
$$

$$
=\Theta_1(F_1,F_3,F_3,F_1)^{\frac{1}{2}}\Theta_1(F_2,F_4,F_4,F_2)^{\frac{1}{2}}.
$$

Analogously,

$$
|\Theta_2(F_1,F_2,F_3,F_4)|
$$

$$
\leq\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i}\sum_{j\in\{1,2\}}\Big|\int_{\Omega_2}\int_{\Omega_2}\Big(\int_{\Omega_1}F_2(\omega_1,\omega_2')F_3(\omega_1,\omega_2)\mathbb{h}_{F_l^{(i)}}(\omega_1)d\mathbb{P}_1(\omega_1)\Big)
$$

$$
\cdot\Big(\int_{\Omega_1}F_1(\omega_1',\omega_2)F_4(\omega_1',\omega_2')\mathbb{h}_{F_l^{(i)}}(\omega_1')d\mathbb{P}_1(\omega_1')\Big)
$$

$$
\cdot\mathbb{h}^0_{G_{k,j}^{(i+1)}}(\omega_2)\mathbb{h}^0_{G_{k,j}^{(i+1)}}(\omega_2')d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')\Big|
$$

$$
\leq\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i}\sum_{j\in\{1,2\}}\Big[\int_{\Omega_2}\int_{\Omega_2}\Big(\int_{\Omega_1}F_2(\omega_1,\omega_2')F_3(\omega_1,\omega_2)\mathbb{h}_{F_l^{(i)}}(\omega_1)d\mathbb{P}_1(\omega_1)\Big)^2
$$

$$
\cdot\mathbb{h}^0_{G_{k,j}^{(i+1)}}(\omega_2)\mathbb{h}^0_{G_{k,j}^{(i+1)}}(\omega_2')d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')\Big]^{\frac{1}{2}}
$$

$$
\cdot\Big[\int_{\Omega_2}\int_{\Omega_2}\Big(\int_{\Omega_1}F_1(\omega_1',\omega_2)F_4(\omega_1',\omega_2')\mathbb{h}_{F_l^{(i)}}(\omega_1')d\mathbb{P}_1(\omega_1')\Big)^2
$$

$$
\cdot\mathbb{h}^0_{G_{k,j}^{(i+1)}}(\omega_2)\mathbb{h}^0_{G_{k,j}^{(i+1)}}(\omega_2')d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')\Big]^{\frac{1}{2}}
$$

$$
\leq\Big[\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i}\sum_{j\in\{1,2\}}\int_{\Omega_2}\int_{\Omega_2}\Big(\int_{\Omega_1}F_2(\omega_1,\omega_2')F_3(\omega_1,\omega_2)\mathbb{h}_{F_l^{(i)}}(\omega_1)d\mathbb{P}_1(\omega_1)\Big)^2
$$

$$\cdot \mathbb{h}^0_{G^{(i+1)}_{k,j}}(\omega_2)\mathbb{h}^0_{G^{(i+1)}_{k,j}}(\omega_2')d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')\Big]^{\frac{1}{2}}$$

$$\cdot\Big[\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i}\sum_{j\in\{1,2\}}\int_{\Omega_2}\int_{\Omega_2}\Big(\int_{\Omega_1}F_1(\omega_1',\omega_2)F_4(\omega_1',\omega_2')\mathbb{h}_{F_l^{(i)}}(\omega_1')d\mathbb{P}_1(\omega_1')\Big)^2$$

$$\cdot \mathbb{h}^0_{G^{(i+1)}_{k,j}}(\omega_2)\mathbb{h}^0_{G^{(i+1)}_{k,j}}(\omega_2')d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')\Big]^{\frac{1}{2}}$$

$$=\Big[\int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i}\sum_{j\in\{1,2\}}F_2(\omega_1,\omega_2')F_3(\omega_1,\omega_2)F_2(\omega_1',\omega_2')F_3(\omega_1',\omega_2)$$

$$\cdot \mathbb{h}^0_{G^{(i+1)}_{k,j}}(\omega_2')\mathbb{h}^0_{G^{(i+1)}_{k,j}}(\omega_2)\mathbb{h}_{F_l^{(i)}}(\omega_1')\mathbb{h}_{F_l^{(i)}}(\omega_1)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big]^{\frac{1}{2}}$$

$$\cdot\Big[\int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i}\sum_{j\in\{1,2\}}F_1(\omega_1',\omega_2)F_4(\omega_1',\omega_2')F_1(\omega_1,\omega_2)F_4(\omega_1,\omega_2')$$

$$\cdot \mathbb{h}^0_{G^{(i+1)}_{k,j}}(\omega_2')\mathbb{h}^0_{G^{(i+1)}_{k,j}}(\omega_2)\mathbb{h}_{F_l^{(i)}}(\omega_1')\mathbb{h}_{F_l^{(i)}}(\omega_1)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big]^{\frac{1}{2}}$$

$$=\Theta_2(F_3,F_2,F_3,F_2)^{\frac{1}{2}}\Theta_2(F_1,F_4,F_1,F_4)^{\frac{1}{2}}.$$

Notice that, by Hölder's and Jensen's inequality,

$$\Xi^{(i)}(F,H,H,F)=\int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}F(\omega_1',\omega_2)F(\omega_1',\omega_2')H(\omega_1,\omega_2')H(\omega_1,\omega_2)$$

$$\cdot\sum_{k=1}^{m_i}\mathbb{h}^0_{G_k^{(i)}}(\omega_2')\mathbb{h}^0_{G_k^{(i)}}(\omega_2)\sum_{l=1}^{n_i}\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')$$

$$\le\Big(\int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}F(\omega_1',\omega_2)^2F(\omega_1',\omega_2')^2$$

$$\cdot\sum_{k=1}^{m_i}\mathbb{h}^0_{G_k^{(i)}}(\omega_2')\mathbb{h}^0_{G_k^{(i)}}(\omega_2)\sum_{l=1}^{n_i}\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big)^{\frac{1}{2}}$$

$$\cdot\Big(\int_{\Omega_1}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2}H(\omega_1,\omega_2')^2H(\omega_1,\omega_2)^2$$

$$\cdot\sum_{k=1}^{m_i}\mathbb{h}^0_{G_k^{(i)}}(\omega_2')\mathbb{h}^0_{G_k^{(i)}}(\omega_2)\sum_{l=1}^{n_i}\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1')\Big)^{\frac{1}{2}}$$

$$=\Big(\sum_{k=1}^{m_i}\int_{\Omega_1}\Big(\int_{\Omega_2}F(\omega_1',\omega_2)^2\mathbb{h}^0_{G_k^{(i)}}(\omega_2)d\mathbb{P}_2(\omega_2)\Big)$$

$$\cdot\Big(\int_{\Omega_2}F(\omega_1',\omega_2')^2\mathbb{h}^0_{G_k^{(i)}}(\omega_2')d\mathbb{P}_2(\omega_2')\Big)d\mathbb{P}_1(\omega_1')\Big)^{\frac{1}{2}}$$

$$\cdot\Big(\sum_{k=1}^{m_n}\int_{\Omega_1}\Big(\int_{\Omega_2}H(\omega_1,\omega_2)^2\mathbb{h}^0_{G_k^{(i)}}(\omega_2)d\mathbb{P}_2(\omega_2)\Big)$$

$$
\begin{aligned}
&\qquad\cdot\Big(\int_{\Omega_2} H(\omega_1,\omega_2')^2 \mathbb{h}^0_{G_k^{(i)}}(\omega_2')d\mathbb{P}_2(\omega_2')\Big)d\mathbb{P}_1(\omega_1)\Big)^{\frac{1}{2}}\\
=&\Big(\int_{\Omega_1}\Big(\int_{\Omega_2} F(\omega_1',\omega_2)^2 d\mathbb{P}_2(\omega_2)\Big)^2 d\mathbb{P}_1(\omega_1')\Big)^{\frac{1}{2}}\\
&\qquad\cdot\Big(\int_{\Omega_1}\Big(\int_{\Omega_2} H(\omega_1,\omega_2)^2 d\mathbb{P}_2(\omega_2)\Big)^2 d\mathbb{P}_1(\omega_1)\Big)^{\frac{1}{2}} \le \|F\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)}\|H\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\Xi^{(i)}(G,\mathbb{1}_{\Omega_1\times\Omega},\mathbb{1}_{\Omega_1\times\Omega_2},G)\\
&\quad=\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_2} G(\omega_1',\omega_2)G(\omega_1',\omega_2')\sum_{k=1}^{m_i}\mathbb{h}^0_{G_k^{(i)}}(\omega_2')\mathbb{h}^0_{G_k^{(i)}}(\omega_2)d\mathbb{P}_2(\omega_2)d\mathbb{P}_2(\omega_2')d\mathbb{P}_1(\omega_1')\\
&\quad=\int_{\Omega_1}\Big(\int_{\Omega_2} G(\omega_1',\omega_2)d\mathbb{P}_2(\omega_2)\Big)^2 d\mathbb{P}_1(\omega_1')\le \|G\|^2_{\mathrm{L}^2(\Omega_1\times\Omega_2)}.
\end{aligned}
$$

Using (3.18) we get

$$
\begin{aligned}
\Theta_1(F,H,H,F) &\le \|F\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)}\|H\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)} - \Theta_2(F,H,H,F),\\
\Theta_1(G,\mathbb{1}_{\Omega_1\times\Omega_2},\mathbb{1}_{\Omega_1\times\Omega_2},G) &\le \|G\|^2_{\mathrm{L}^2(\Omega_1\times\Omega_2)} - \Theta_2(G,\mathbb{1}_{\Omega_1\times\Omega_2},\mathbb{1}_{\Omega_1\times\Omega_2},G)\\
&= \|G\|^2_{\mathrm{L}^2(\Omega_1\times\Omega_2)},\\
\Theta_2(F,F,F,F) &\le \|F\|^4_{\mathrm{L}^4(\Omega_1\times\Omega_2)} - \Theta_1(F,F,F,F) \le \|F\|^4_{\mathrm{L}^4(\Omega_1\times\Omega_2)},\\
\Theta_2(H,H,H,H) &\le \|H\|^4_{\mathrm{L}^4(\Omega_1\times\Omega_2)} - \Theta_1(H,H,H,H) \le \|H\|^4_{\mathrm{L}^4(\Omega_1\times\Omega_2)}.
\end{aligned}
$$

The second formula follows from the cancellation of the form Θ_2 and the Haar function over the standard dyadic interval appearing in the expression. The third, and similarly the fourth one, follow from

$$
\begin{aligned}
\Theta_1(F,F,F,F) = \int_{\Omega_1}\int_{\Omega_1}\sum_{i=0}^{n-1}\sum_{k=1}^{m_i}\sum_{l=1}^{n_i}\Big(\int_{\Omega_2} F(\omega_1',\omega_2)F(\omega_1,\omega_2)\mathbb{h}_{G_k^{(i)}}(\omega_2)d\mathbb{P}_2(\omega_2)\Big)^2&\\
\mathbb{h}^0_{F_l^{(i)}}(\omega_1')\mathbb{h}^0_{F_l^{(i)}}(\omega_1)d\mathbb{P}_1(\omega_1)d\mathbb{P}_1(\omega_1') \ge 0.&
\end{aligned}
$$

Now,

$$
\begin{aligned}
&|\Lambda(F,G,H)| = |\Theta_1(F,G,H,\mathbb{1}_{\Omega_1\times\Omega_2})| \le \Theta_1(F,H,H,F)^{\frac{1}{2}}\Theta_1(G,\mathbb{1}_{\Omega_1\times\Omega_2},\mathbb{1}_{\Omega_1\times\Omega_2},G)^{\frac{1}{2}}\\
&= \big(\|F\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)}\|H\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)} - \Theta_2(F,H,H,F)\big)^{\frac{1}{2}}\|G\|_{\mathrm{L}^2(\Omega_1\times\Omega_2)}\\
&\le \big(\|F\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)}\|H\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)} + \Theta_2(H,H,H,H)^{\frac{1}{2}}\Theta_2(F,F,F,F)^{\frac{1}{2}}\big)^{\frac{1}{2}}\|G\|_{\mathrm{L}^2(\Omega_1\times\Omega_2)}\\
&\le \big(\|F\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)}\|H\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)} + \|H\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)}\|F\|^2_{\mathrm{L}^4(\Omega_1\times\Omega_2)}\big)^{\frac{1}{2}}\|G\|_{\mathrm{L}^2(\Omega_1\times\Omega_2)}\\
&= \sqrt{2}\|F\|_{\mathrm{L}^4(\Omega_1\times\Omega_2)}\|G\|_{\mathrm{L}^2(\Omega_1\times\Omega_2)}\|H\|_{\mathrm{L}^4(\Omega_1\times\Omega_2)}
\end{aligned}
$$

This shows that we have obtained (3.13) for the triple of exponents $(p,q,r') = (4,2,4)$.

3.2.7. Obtaining the whole range of exponents

Similarly as we did in (3.3), we can show that

$$\sum_{i=0}^{n-1} \mathbb{E}_1(F|\mathscr{F}_i)(\mathbb{E}_2(G|\mathscr{G}_{i+1}) - \mathbb{E}_2(G|\mathscr{G}_i)) + \sum_{i=0}^{n-1} \mathbb{E}_2(G|\mathscr{G}_{i+1})(\mathbb{E}_1(F|\mathscr{F}_{i+1}) - \mathbb{E}_1(F|\mathscr{F}_i))$$
$$= \mathbb{E}_1(F|\mathscr{F}_n)\mathbb{E}_2(G|\mathscr{G}_n) - \mathbb{E}_1(F|\mathscr{F}_0)\mathbb{E}_2(G|\mathscr{G}_0).$$

If we replaced the sum in the inequality (3.11) with $\sum_{i=0}^{n-1} \mathbb{E}_2(G|\mathscr{G}_{i+1})(\mathbb{E}_2(F|\mathscr{F}_{i+1}) - \mathbb{E}_2(F|\mathscr{F}_i))$, the proof would still follow and we would be able to deduct estimate for $p=4$ and $q=2$. By triangle inequality and by Theorems 1.1.3 and 1.2.6,

$$\begin{aligned}
&\Big\| \sum_{i=0}^{n-1} \mathbb{E}_1(F|\mathscr{F}_i)(\mathbb{E}_2(G|\mathscr{G}_{i+1}) - \mathbb{E}_2(G|\mathscr{G}_i)) \Big\|_{\mathrm{L}^r(\Omega_1\times\Omega_2)} \\
&\quad \le \Big\| \sum_{i=0}^{n-1} \mathbb{E}_2(G|\mathscr{G}_{i+1})(\mathbb{E}_2(F|\mathscr{F}_{i+1}) - \mathbb{E}_2(F|\mathscr{F}_i)) \Big\|_{\mathrm{L}^r(\Omega_1\times\Omega_2)} \\
&\qquad + \|\mathbb{E}_1(F|\mathscr{F}_n)\mathbb{E}_2(G|\mathscr{G}_n)\|_{\mathrm{L}^r(\Omega_1\times\Omega_2)} + \|\mathbb{E}_1(F|\mathscr{F}_0)\mathbb{E}_2(G|\mathscr{G}_0)\|_{\mathrm{L}^r(\Omega_1\times\Omega_2)} \\
&\quad \lesssim \|F\|_{\mathrm{L}^q(\Omega_1\times\Omega_2)} \|G\|_{\mathrm{L}^p(\Omega_1\times\Omega_2)}.
\end{aligned}$$

If we compare this inequality with (3.11), we can see that we managed to swap places of functions F and G and therefore obtain additional estimate for reversed pair of exponents. We can consider following the proof from Subsection 3.2.6 onward in order to conclude that we can interchange exponents for which the boundedness is obtained. This gives us the bound for $(p,q,r') = (2,4,4)$ as well. By Marcinkiewicz's interpolation theorem, the same boundedness is obtained for each tuple $(p_t,q_t,r_t') = \big((\frac{t}{4}+\frac{1-t}{2})^{-1},(\frac{t}{2}+\frac{1-t}{4})^{-1},(\frac{t}{4}+\frac{1-t}{4})^{-1}\big) = \big(\frac{4}{2-t},\frac{4}{t+1},4\big)$ where $t \in [0,1]$. We can additionally expand the range of exponents with Lemma 3.2.1 in order to obtain range $(p,q) \in \cup_{t\in[0,1]}\big\{\frac{4}{2-t}\big\} \times \big\langle \frac{4}{t+2},\frac{4}{t+1}\big]$.

Moreover, by this same trick of interchanging F and G we can expand the range onto $q_t = \frac{4}{t+1}$ and all $p \in \langle \frac{4}{t+2}, \frac{4}{2-t}\rangle$. To summarize, (3.4) is valid for all exponents

$$(p,q) \in \left[\frac{4}{3},4\right]^2 \cap \left\{(x,y) : \frac{3}{4} \le \frac{1}{x} + \frac{1}{y} \le 1\right\}.$$

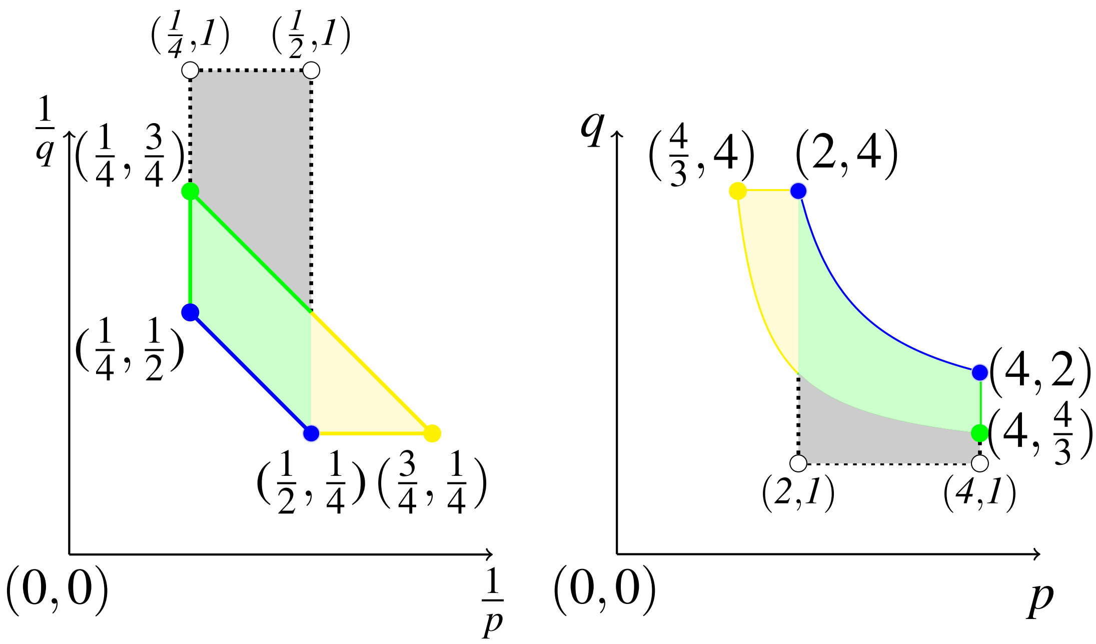

Figure 3.1: The range of exponents for which the boundedness is shown.

The whole procedure of obtaining the exponents can be presented by the Figure 3.1. The first result was obtained for the blue point $\left(\frac{1}{p},\frac{1}{q}\right) = \left(\frac{1}{4},\frac{1}{2}\right)$. With that and with the swap of F and G we got the similar conclusion for the point $\left(\frac{1}{2},\frac{1}{4}\right)$ and then, by interpolating, we obtained the blue line connecting two dots that present reciprocal values. By obtaining the weak boundedness for the line from $\left(\frac{1}{4},1\right)$ to $\left(\frac{1}{2},1\right)$ (presented with dotted line connecting white dots on the Figure), but restricting ourselves to the region where $r \geq 1$ we rejected the grey area and proved boundedness for the green region which, by the same swap as before, expanded to the yellow region. This shows that we have strong boundedness for the complete range on the diagram, including border lines and circles.

CONCLUSION

In this thesis we obtained the first characterizations of L^p boundedness of dyadic entangled multilinear singular integral forms associated with hypergraphs. We managed to prove the L^p boundedness of such forms where the kernel is composed of products of Haar functions and in a certain range of exponents. In case of general dyadic Calderón-Zygmund kernels, the L^p boundedness in the same range was shown to be equivalent to each of the following: the weak boundedness property and a T(1)-type condition; the L^p boundedness for only one choice of exponents in the stated range; domination by a sparse form; the weighted L^p boundedness for certain tuples of Muckenhoupt weights.

We also suggested a new approach to answering Kakutani's question by introducing the notion of the ergodic-martingale paraproduct and investigating its properties. By observing its L^p estimates, making several reductions and applying methods from harmonic analysis we showed that such paraproducts converge in L^p spaces for a specific triple of exponents. After that we applied interpolation methods in order to obtain a bigger range of exponents satisfying the Hölder scaling property.

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Curriculum Vitae

Mario Stipčić was born on the 17^{th} of April 1991 in Split, Croatia. He finished primary school in Trogir and secondary school in Split. In 2010 he started studying at Department of Mathematics, Faculty of Science, University of Zagreb where he obtained bachelor's degree in Mathematics in 2013. In 2015, at the same department and faculty, he earned master's degree with the highest distinction (*summa cum laude*) in Theoretical mathematics, with master thesis *Wavelet characterizations of Sobolev spaces* under the supervision of Associate Professor Vjekoslav Kovač. He was awarded by Department of Mathematics as the most successful student in both bachelor's and master's studies. He was also awarded rector's award for the best student work of scientific content, titled *Applications of random dyadic sistems to decompositions of functions and operators* in coautorship with Ivana Valentić and under supervision of Associate Professor Vjekoslav Kovač.

After graduation he started working as a teaching assistant at Department of Mathematics, Faculty of Science, University of Zagreb, where he was a member of Seminar for probability theory and also a member of research project *Multilinear and Nonlinear Harmonic Analysis and Applications*. During his doctoral studies he published two articles, delivered five talks at conferences and participated at several scientific events. In 2020 he also participated in Simposium of students of doctoral studies at Faculty of Science, University of Zagreb, at which he won the award for the best presentation.

Research publications

(1) V. Kovač, M. Stipčić, *Convergence of the ergodic-martingale paraproduct*. Statistics & Probability Letters, 164:108826, 2020.

(2) M. Stipčić, *T(1) theorem for dyadic singular integral forms associated with hypergraphs*. Journal of Mathematical Analysis and Applications, 481:123496, 2019.