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DOCTORAL THESIS

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Geometrijska struktura teorija polja s poopćenim baždarnim simetrijama

DOKTORSKI RAD

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SUPERVISOR

Larisa Jonke obtained her PhD in theoretical physics at the University of Zagreb. She was a postdoctoral researcher at the Ludwig-Maximilian University in Munich, Julius Wess chair, as an Alexander von Humboldt fellow. As a visiting scientist she worked at University Bonn, Leibniz University Hannover and LMU Munich. Currently, she is a senior scientific associate and head of the Quantum Gravity and Mathematical Physics Group at the Division of Theoretical Physics, Ruđer Bošković Institute.

Her research interest focuses on theoretical and mathematical physics, more specifically: generalised geometry; sigma models; matrix models; field theories on non-commutative space; random matrices; integrable models and low-dimensional field theories. In the course of her research she has co-authored 50 research papers with over 750 citations. She was the PI of two projects funded by the Croatian Science Foundation (CSF) and a member of the Management Committee of the COST action "Quantum Structure of Spacetime" and a member of the Steering Committee of the ESF research network "Quantum Geometry and Quantum Gravity". She taught two undergraduate courses at the University of Zagreb and supervised several students and postdoctoral researchers. She also served as an external expert for CSF, the Irish Research Council (for Science, Engineering and Technology) (IRCSET/ IRC) and the Italian Ministry for Education, University and Research (MUIR).

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ABSTRACT

Generalised gauge field theories are interacting field theories with a gauge symmetry generalised beyond the standard Lie algebra case. Such theories have become increasingly important in modern physics, especially high energy physics as the search for a consistent description of quantum gravity makes these generalised symmetries unavoidable. Two theories we shall analyse in particular are the Courant sigma model and double field theory. They are relevant in the understanding of dualities and symmetries with higher form gauge fields. The aim of this thesis is to explore deeper the relation between the two and to embed them into the appropriate framework for studying higher gauge symmetries – L_∞-algebras. This will be broken down into three parts. The first dedicated to delving deeper into the BRST symmetry of them and exploring how the known classical projection procedure relating them appears on the BRST level. It is at this point that we explicitly see the necessity of the strong constraint for double field theory to be covariant. Having understood the BRST structure and knowing that the Courant algebroid, the geometric structure behind the sigma model, is an L_∞-algebra we are motivated to see how the full field theory fits into this framework. This is the second part. In it we construct the L_∞-algebra for the Courant sigma model in such a way that all physical data can be expressed using the objects and operations defined by the algebra itself. Then the intimate relationship between L_∞ and the Batalin-Vilkovisky formalism is utilised to obtain its BV/BRST formulation. Finally, in the last part, we go back to double field theory and embed its geometric structure, the DFT algebroid, into L_∞ much the same way as the Courant algebroid is. However, because of the section condition of double field theory this cannot be done as is, given that the cohomological vector in this formulation squares to zero only upon applying the section condition. In order to get around this we turn to the well known but extremely rarely used extension of L_∞-algebras – curved L_∞-algebras. It is precisely this curving that encapsulates the strong constraint violating terms and restores the existence of the cohomological vector. We finish off with a sigma model built upon this new curved algebra that brings about the section condition as an on-shell requirement.

Keywords: L_∞-algebra, Batalin-Vilkovisky, Double Field Theory, Gauge symmetry, Courant algebroid.

PROŠIRENI SAŽETAK

Teorije polja s poopćenim baždarnim simetrijama integralni su dio moderne fizike, posebice fizike visokih energija s obzirom da potraga za kvantnim opisom gravitacije čini takve poopćene simetrije nezaobilaznima. Mi ćemo se fokusirati prvenstveno na dvije takve teorije: Courantov sigma model i dvostruka teorija polja (DFT). Obje teorije motivirane su teorijom zatvorenih struna i pojavnosti tzv. T-dualnosti u njoj. Cilj ovog rada jest da istraži dublje vezu među ovim dvjema teorijama te da ih smjesti u formalizam prilagođen višim baždarnim simetrijama – formalizam L_∞-algebri. Ovo će biti napravljeno u tri dijela detaljnije objašnjena u nastavku.

BRST simetrija Courantovog sigma modela i dvostruke teorije polja.

Roytenberg je u [18] pokazao da je BV akcija Courantovog sigma modela, ovdje nad dvostrukom baznom mnogostrukošću:

$$\mathbf{S}_{\mathrm{C}}[\mathbf{X},\mathbf{A},\mathbf{F}] = \int_{T[1]\Sigma_3} \mu \left(\mathbf{F}_A \,\mathbf{d}\mathbf{X}^A + \tfrac{1}{2}\hat{\eta}_{\hat{I}\hat{J}}\mathbf{A}^{\hat{I}}\,\mathbf{d}\mathbf{A}^{\hat{J}} - \rho^A{}_{\hat{J}}(\mathbf{X})\mathbf{A}^{\hat{J}}\mathbf{F}_A + \tfrac{1}{6}T_{\hat{I}\hat{J}\hat{K}}(\mathbf{X})\mathbf{A}^{\hat{I}}\mathbf{A}^{\hat{J}}\mathbf{A}^{\hat{K}} \right),$$

gdje su:

$$\begin{aligned}
\mathbf{X}^A &= X^A + F^{\dagger A} + t^{\dagger A} + v^{\dagger A},\\
\mathbf{A}^{\hat{I}} &= \epsilon^{\hat{I}} + \mathbb{A}^{\hat{I}} + \hat{\eta}^{\hat{I}\hat{J}}\mathbb{A}^\dagger_{\hat{J}} + \hat{\eta}^{\hat{I}\hat{J}}\epsilon^\dagger_{\hat{J}},\\
\mathbf{F}_A &= v_A + t_A + F_A + X^\dagger_A,
\end{aligned}$$

BV superpolja, μ mjera na $T[1]\Sigma_3$, $\hat{\eta}$ invarijantna metrika grupe $O(2d,2d)$ te ρ i T funkcije od X. Indeksi označeni slovima s početka abecede označavaju tangentne vektore i 1-forme na odredišnoj mnogostrukosti $A,\ldots = 1,\ldots,2d$, a iz sredine abecede $\hat{I},\ldots = 1,\ldots,4d$ vektore vektorskog svežnja nad odredišnom mnogostrukošću sa strukturom Courantovog algebroida. Ova akcija preko BV zagrade također definira poopćenu BRST transformaciju komponentnih polja. U izvoru [45] je bilo pokazano da postoji projekcija s udvostručenog Courantovog sigma modela na DFT:

$$\mathsf{p}_+ : \mathbf{A} \longmapsto \mathbf{A}_+ \equiv A,$$

gdje su:

$$\mathbf{A}_{\pm}^{I} = \tfrac{1}{2}\left(\mathbf{A}^{I} \pm \eta^{IJ}\,\tilde{\mathbf{A}}_{J}\right),$$

komponente dobivene miješanjem vektorskih i kovektorskih komponenti općeg prereza standardnog Courantovog algebroida novom strukturom η, $O(d,d)$ invarijantnom metrikom. Ovakva projekcija daje za BV akciju DFT-a:

$$\begin{aligned}\mathbf{S}_{\mathrm{DFT}}[\mathbf{X},\mathbf{A}_{+},\mathbf{F}] = \int_{T[1]\Sigma_3} \mu\Big(&\mathbf{F}_A\,\mathrm{d}\mathbf{X}^A + \eta_{IJ}\mathbf{A}_{+}^{I}\mathrm{d}\mathbf{A}_{+}^{J} - (\rho_{+})^{A}{}_{I}(\mathbf{X})\mathbf{A}_{+}^{I}\mathbf{F}_A + \\ &+ \tfrac{1}{3}\hat{T}_{IJK}(\mathbf{X})\mathbf{A}_{+}^{I}\mathbf{A}_{+}^{J}\mathbf{A}_{+}^{K}\Big).\end{aligned}$$

Međutim, BRST transformacije su također projicirane što znači da iako sve $-$ komponente iščezavaju to ne znači da njihove BRST varijacije također moraju iščeznuti. Naprotiv, transformacija i polja i duha superpolja $\mathbf{A}_{-}$ ne iščezava što nameće pitanje konzistentnosti projekcije. Naime, moramo osigurati kako BRST transformacija neće vratiti ono što je projekcija uništila tj. moramo zahtijevati $Q\mathbf{A}_{-} = 0$. Ovaj dodatni zahtjev implicira fiksiranje duhova t i v:

$$\begin{aligned} v_A &= \tfrac{1}{2}\Theta_{AJK}(X)\epsilon_{+}^{J}\epsilon_{+}^{K},\\ t_A &= \Theta_{AJK}(X)\mathbb{A}_{+}^{J}\epsilon_{+}^{K} + \tfrac{1}{2}\partial_B\Theta_{ALM}F^{\dagger B}\epsilon_{+}^{L}\epsilon_{+}^{M}.\end{aligned}$$

novom funkcijom Θ. No ovime ponovo imamo problem ekvivalentan prvotnom osim što duhovi t i v nisu fiksirani trivijalno kao komponentna polja $\mathbf{A}_{-}$. Dakle moramo provjeriti samosuglasnost ovakvog BRST transformacije ovako fiksiranih duhova s BRST transformacijama naslijeđenim iz Courantovog sigma modela. Rezultat jest uvjet na funkciju Θ iskazanu kroz iščezavanje dviju struktura koje zovemo R i S dane relacijama (2.2.39) i (2.2.38). Vratimo li se na baždarnu razinu i izračunamo varijaciju jednadžbi gibanja s obzirom na projicirane varijacije DFT-a, vidjet ćemo kako su one kovarijantne samo do na tzv. jaki uvjet:

$$\partial_A\partial^A(\cdots) = 0,$$

gdje je u zagradi bilo kakav produkt polja, odnosno na razini teorije svjetskog volumena:

$$\partial^A(\cdots)F_A = 0.$$

Courantov sigma model i L_∞-algebre.

Sada želimo konstruirati Courantov sigma model koristeći L_∞-algebre. Teoriju polja L_∞-algebri možemo razdijeliti na tri L_∞ razine. Prva algebra je simetrijska i sadrži samo algebarske elemente koji opisuju strukturu vrsti polja kakva će se pojaviti, tu osnovnu simetrijsku algebru zovemo L. Zatim algebarske elemente želimo pretvoriti u polja koja poprimaju vrijednosti u simetrijskoj algebri. To napravimo na način da konstruiramo novu

L_∞-algebru, L', tenzorskim produktom simetrijske algebre L i de Rhamovog kompleksa diferencijalnih formi $\Omega^\bullet(M)$. No možemo ići i dalje jer nam L_∞ opis omogućuje da konzistentno napravimo i proširenje na BV opis teorije. Za potrebe toga prelazimo na treću razinu L_∞-algebri, $\hat{\mathsf{L}}$, koja je tenzorski produkt L' i komutativne algebre $C^\infty(\mathsf{L}'[1])$ koja uvodi stupnjeve duha. Stoga sve što nam treba za potpun opis teorije je početna simetrijska L_∞-algebra definirana gradiranim vektorskim prostorom $\mathsf{L} = \mathsf{L}_1 \oplus \mathsf{L}_0 \oplus \mathsf{L}_{-1}$ i preslikavanjima μ:

$$
\begin{aligned}
\mathsf{L}_1 \ni \quad & \mu_n(l_{(1)1}, \ldots, l_{(1)n-1}, l_{(0)}) = l^{a_1}_{(1)1} \cdots l^{a_{n-1}}_{(1)n-1} \partial_{a_1} \cdots \partial_{a_{n-1}} \rho^a{}_I l^I_{(0)}, \\
\mathsf{L}_0 \ni \quad & \mu_n(l_{(1)1}, \ldots, l_{(1)n-1}, l_{(-1)}) = -l^{a_1}_{(1)1} \cdots l^{a_{n-1}}_{(1)n-1} \partial_{a_1} \cdots \partial_{a_{n-1}} \rho^a{}_J l_{(-1)a} \eta^{IJ}, \\
\mathsf{L}_{-1} \ni \quad & \mu_m(l_{(1)1}, \ldots, l_{(1)m-2}, l_{(-1)}, l_{(0)}) = -l^{a_1}_{(1)1} \cdots l^{a_{m-2}}_{(1)m-2} \partial_{a_1} \cdots \partial_{a_{m-2}} \partial_a \rho^b{}_I l_{(-1)b} l^I_{(0)}, \\
\mathsf{L}_0 \ni \quad & \mu_m(l_{(1)1}, \ldots, l_{(1)m-2}, l_{(0)1}, l_{(0)2}) = l^{a_1}_{(1)1} \cdots l^{a_{m-2}}_{(1)m-2} \partial_{a_1} \cdots \partial_{a_{m-2}} T_{JKL} l^K_{(0)1} l^L_{(0)2} \eta^{IJ}, \\
\mathsf{L}_{-1} \ni \quad & \mu_r(l_{(1)1}, \ldots, l_{(1)r-3}, l_{(0)1}, l_{(0)2}, l_{(0)3}) = l^{a_1}_{(1)1} \cdots l^{a_{r-3}}_{(1)r-3} \partial_{a_1} \cdots \partial_{a_{r-3}} \partial_a T_{IJK} l^I_{(0)1} l^J_{(0)2} l^K_{(0)3}.
\end{aligned}
$$

Ovakvim izborom dobijemo upravo akciju (4.B.1) i BRST transformacije (4.B.2)–(4.B.13) kakve slijede iz AKSZ pristupa. No kako znamo da je Courantov algebroid [52] također L_∞-algebra pitamo se kakva je veza između algebre tog geometrijskog objekta ciljane mnogostrukosti i algebre same pripadne teorije polja. Ovo je ostvareno morfizmom L_∞-algebri, dakle kolekcijom preslikavanja ϕ takvih da čuvaju L_∞ strukturu. Ovaj morfizam je konstruiran i njegove komponente iznose:

$$
\begin{aligned}
\phi_1(h) &= X^* h, \\
\phi_1(e) &= X^* e\Big|_p, \\
\phi_1(f) &= -X^* \tilde{\mathrm{d}} f\Big|_p, \\
\phi_i(h_1, \ldots, h_{i-1}, e)^I &= X^*(h_1^{a_1} \cdots h_{i-1}^{a_{i-1}} \tilde{\partial}_{a_1} \cdots \tilde{\partial}_{a_{i-1}} e^I)\Big|_p, \\
\phi_i(h_1, \ldots, h_{i-2}, e_1, e_2)_a &= X^*(h_1^{a_1} \cdots h_{i-2}^{a_{i-2}} \tilde{\partial}_{a_1} \cdots \tilde{\partial}_{a_{i-2}} (\eta_{IJ} e^I_{[1} \tilde{\partial}_a e^J_{2]}))\Big|_p, \\
\phi_i(h_1, \ldots, h_{i-1}, f)_a &= -X^*(h_1^{a_1} \cdots h_{i-1}^{a_{i-1}} \tilde{\partial}_{a_1} \cdots \tilde{\partial}_{a_{i-1}} \tilde{\partial}_a f)\Big|_p,
\end{aligned}
$$

za $i \geqslant 2$. Ovdje je p točka bazne mnogostrukosti Courantovog algebroida oko koje razvijamo i koja je takva da je njena koordinatizacija dana s $x = 0$.

DFT algebroid i zakrivljene L_∞-algebre.

Zanimljivo je istražiti kako geometrijska struktura, analogna Courantovom algebroidu, dobivena prethodno opisanom projekcijom odnosno DFT algebroid, ulazi u formalizam L_∞-algebri. Naime, fundamentalna razlika pri prijelazu iz Courantovog u DFT algebroid jest jaki uvjet definiran gore, zbog njega se naivni pokušaj konstrukcije L_∞-algebre lomi u trenutku zahtjeva relacija homotopije. No bez relacija homotopije to nije konzistentna algebra pa stoga autori [59] to nazivaju L_∞-algebrom do na jaki uvjet. Rješenje problema jest uvođenjem poznatog, ali vrlo rijetko korištenog proširenja zvanog zakrivljene L_∞-algebre. Ključna razlika jest u postojanju novog konstantnog preslikavanja μ_0 koje modificira sve relacije homotopije dodatnim članom:

$$\cdots + (-1)^n \mu_{i+1}(\mu_0, l_1, \ldots, l_i) = 0.$$

Upravo je ovaj član zaslužan za mogućnost opisa struktura koje ne zadovoljavaju jaki uvjet. Pogledamo li relacije za Courantov algebroid vidimo zašto:

$$\begin{aligned}
(\rho \circ \mathcal{D}) f &= 0 \\
\rho[e_1, e_2]_C - [\rho(e_1), \rho(e_2)] &= 0 \\
\mathrm{Jac}(e_1, e_2, e_3) - \mathcal{D}\mathcal{N}_c(e_1, e_2, e_3) &= 0
\end{aligned}$$

sve desne strane ovih relacija iščezavaju i sve slijede iz relacija homotopije pripadne L_∞-algebre. No u slučaju DFT algebroida te iste desne strane relacija, sada sa strukturama koje odgovaraju DFT algebroidu, više nisu nula već imaju netrivijalan doprinos. Upravo je ta netrivijalna desna strana proizvod postojanja dodatnog člana u relacijama homotopije zakrivljenih L_∞-algebri. Stoga definiramo strukturu L_∞ formulacije DFT algebroida (u minimalnom slučaju), opet definirajući prostor kao $\mathsf{L}_{-1} \oplus \mathsf{L}_0 \oplus \mathsf{L}_2$ te preslikavanja:

$$\begin{aligned}
\mu_1(f) &= \mathcal{D} f, \\
\mu_2(e_1, e_2) &= [\![e_1, e_2]\!], \\
\mu_2(e, f) &= \langle e, \mathcal{D} f \rangle, \\
\mu_3(e_1, e_2, e_3) &= \mathcal{N}(e_1, e_2, e_3), \\
\mu_3(\mu_0, e, f) &= [\![e, \mathcal{D} f]\!] - \mathcal{D}\langle e, \mathcal{D} f \rangle, \\
\mu_3(\mu_0, f_1, f_2) &= 2\langle \mathcal{D} f_1, \mathcal{D} f_2 \rangle, \\
\mu_4(\mu_0, e_1, e_2, e_3) &= \mathcal{D}\mathcal{N}(e_1, e_2, e_3) - \mathrm{Jac}(e_1, e_2, e_3), \\
\mu_5(\mu_0, e_1, e_2, e_3, e_4) &= \tfrac{1}{2}\langle \mathcal{D}\mathcal{N}(e_1, e_2, e_3), e_4 \rangle - \tfrac{1}{2}\langle \mathrm{Jac}(e_1, e_2, e_3), e_4 \rangle + \\
&\quad + \text{antisymm.}(1, 2, 3, 4),
\end{aligned}$$

gdje antisymm. označava sve potrebne permutacije kako bi se postigla potpuna antisimetrija u 1, 2, 3 i 4. Sada preostaje samo pitanje interpretacije jakog uvjeta. Naime, znamo da primjenom jakog uvjeta DFT algebroid pada nazad na Courantov algebroid (sada nad neudvostručenom mnogostrukošću). Znamo za oboje njima pripadnu L_∞ formulaciju stoga kao i u slučaju Courantovog sigma modela tražimo vezu među njima kroz L_∞-morfizam. Ovaj morfizam smo konstruirali te je dan sljedećim komponentnim preslikavanjima:

$$\begin{aligned}
\phi_1(f) &= \tfrac{1}{2} f\Big|_M , \\
\phi_1(e) &= e\Big|_M , \\
\phi_2(\mu_0, f) &= \tfrac{1}{2}(\tilde{D}f)\Big|_M , \\
\phi_3(\mu_0, f, e) &= \tfrac{1}{4}\langle e, \tilde{D}f\rangle\Big|_M , \\
\phi_3(\mu_0, e_1, e_2) &= [e_1, e_2]_C\Big|_M - [\![e_1, e_2]\!]\Big|_M , \\
\phi_4(\mu_0, e_1, e_2, e_3) &= \Big(\tfrac{1}{2}\mathcal{N}(e_1, e_2, e_3) - \mathcal{N}_c(e_1, e_2, e_3)\Big)\Big|_M .
\end{aligned}$$

Odnosno dijagramatski:

$$\begin{array}{ccccccc}
\text{DFT :} & \mathsf{L}_{-1} = C^\infty(\mathcal{M}) & \oplus & \mathsf{L}_0 = \Gamma(L) & \oplus & \mathsf{L}_2 \\
\phi \downarrow & \phi_1 \downarrow & & \phi_1 \downarrow & & \phi_1 \downarrow \\
\text{CA :} & \mathsf{L}'_{-1} = C^\infty(M) & \oplus & \mathsf{L}'_0 = \Gamma(E) & \oplus & \varnothing \; .
\end{array}$$

Na kraju smo iskoristili ovu *zakrivljenu* algebru kako bismo konstruirali sigma model koji za odredišnu mnogostrukost ima upravo DFT algebroid. Pri ovoj konstrukciji konstruiran je kohomološki vektor pripadan dvostrukoj teoriji polja čije je postojanje implicirala konzistentnost L_∞-algebre DFT algebroida:

$$\begin{aligned}
Q = \eta^{AB}\frac{\partial}{\partial \mathsf{\eta}^{AB}} &+ \Big(\rho^A{}_I(X)A^I - \mathsf{\eta}^{AB}F_B - \tfrac{1}{2}\rho_{B[I}(X)\partial_{\underline{D}}\rho^B{}_{J]}(X)\mathsf{\eta}^{AD}A^I A^J\Big)\frac{\partial}{\partial X^A} + \\
&+ \Big(\hat{\eta}^{IM}\rho^A{}_I(X)F_A - \tfrac{1}{2}\hat{\eta}^{IM}T_{IJK}(X)A^J A^K + \tfrac{1}{2}\mathsf{\eta}^{AB}\hat{\eta}^{IM}\rho_{C[I}(X)\partial_{\underline{B}}\rho^C{}_{J]}(X)A^J F_A + \\
&\quad + \tfrac{1}{3!}\mathsf{\eta}^{AB}\bar{\mathcal{Z}}_{ABLIJK}(X)\hat{\eta}^{LM}A^I A^J A^K\Big)\frac{\partial}{\partial A^M} + \\
&+ \Big(- \partial_E\rho^B{}_J(X)A^J F_B + \tfrac{1}{2}\mathsf{\eta}^{AD}\partial_E\Big(\rho_{C[K}(X)\partial_{\underline{D}}\rho^C{}_{L]}(X)\Big)A^K A^L F_A - \\
&\quad - \tfrac{1}{3!}\partial_E T_{IJK}(X)A^I A^J A^K + \tfrac{1}{4!}\mathsf{\eta}^{AB}\partial_E\bar{\mathcal{Z}}_{ABIJKL}(X)A^I A^J A^K A^L\Big)\frac{\partial}{\partial F_E},
\end{aligned}$$

Kako bismo sigma model mogli zapisati preko akcije koja varijacijskim principom vraća jednadžbe gibanja moramo uvesti novi prostor L_{-2} takav da postoji unutarnji produkt između L_{-2} i L_2. Ovo proširenje je napravljeno na minimalni način zahtjevom da svi viši produkti koji ukljucuju $G \in \mathsf{L}_{-2}$ iščezavaju. Maurer-Cartanova akcija dobivena jest:

$$\begin{aligned}
S[X, A, F, G] = \int_{\Sigma_3} G_{AB}\eta^{AB} &+ F_A \wedge \mathrm{d}X^A + \tfrac{1}{2}\hat{\eta}_{IJ}A^I \wedge \mathrm{d}A^J - \rho^A{}_I(X)A^I \wedge F_A + \\
&+ \tfrac{1}{6}T_{IJK}(X)A^I \wedge A^J \wedge A^K .
\end{aligned}$$

s reducibilnom baždarnom simetrijom reda 2 uzrokovanom postojanjem novog polja, 3-forme G.

Ključne riječi: L_∞-algebra, Batalin-Vilkovisky, dvostruka teorija polja, baždarna simetrija, Courantov algebroid.

CONTENTS

CHAPTER 1

INTRODUCTION TO GAUGE FIELD THEORY

Field theory occurs in most aspects of theoretical physics in some form or another. Nowhere as much as it does in high energy physics, where it is the basis for the description of almost all phenomena. The widely accepted current foundation of high energy physics is the Standard Model, a gauge field theory based on the $U(1) \times SU(2) \times SU(3)$ gauge group. Although extremely successful, it has certain drawbacks, the biggest of which is the glaring absence of gravity, in that it models three of the four fundamental forces known today. This large hole in its description of the quantum level of physics today is the main motivation for many alternate theories that try to integrate gravity into a wider framework for understanding the quantum physics of nature. The most widely recognised of such frameworks is string theory with its fundamental shift of elementary objects being not 0-dimensional objects, points, but 1-dimensional (or even higher) objects, strings. However groundbreaking this paradigm shift is, one still cannot avoid the underlying structure of gauge field theory. Now even richer in structure and symmetry but nonetheless still a gauge field theory. Therefore the study of gauge symmetries and the field theories built upon them remains one of the main interests of high energy theoretical physics, and mathematical physics.

So, what makes this symmetry so important to field theory and its attempt to describe nature? If one is to think about the fundamentals of physics at the most basic level there are essentially two concepts present: the fundamental objects and the fundamental interaction of these objects. Gauge field theory is constructed to explain the latter. Take for example the simplest of gauge theories, electromagnetism. Classically, we have charged particles interacting via an electromagnetic 4-potential that mediates this force. This 4-potential is not in one-to-one correspondence with the physics the particles experience since there is another potential field that gives the same interaction. This symmetry is called a gauge symmetry or gauge redundancy. This is why when we calculate interactions classically, we assume an additional constraint on the potential, a gauge or gauge fixing.

In electrodynamics the most common such constraints are the Coulomb, Weyl or Lorenz gauges. However, not all gauges are a complete fixing of the redundancy (such as the Weyl gauge just mentioned), and leave a remnant symmetry structure. Such gauge fixings are called incomplete gauges. In chapter 2 we shall in the same manner do a partial gauge fixing in order to reduce the theories symmetry to be able to relate two field theories of different gauge redundancies. The same applies to the quantum regime, with one difference. If we take the most widely used quantisation procedure, the path integral method, in order to calculate the interaction of particles *all physical* gauge field (potential) configurations affect the end result. Hence it is not enough to simply gauge fix the potential, now one must identify *all* the field configurations that are gauge equivalent in order to count them only once. This is a nontrivial problem and one that necessitates the introduction of special tools such as the Becchi-Rouet-Stora [1–3] and Tyutin [4] (BRST) formalism. This formalism had groundbreaking implications in the mathematical understanding of gauge theories and "physical states", more specifically it introduced the study of cohomologies into the spotlight of gauge field theory. On the basis of this the more general and advanced Batalin-Vilkovisky (BV) formalism [5–7] was built. It is necessary in the case of more general gauge symmetries called reducible gauge symmetries where the gauge parameters that control the redundancy are not themselves independent. In this case by introducing just ghosts in the quantum picture one would, in fact, dispose of too many degrees of freedom and therefore need more ghosts that fix parts of the lower ghosts that are redundant. This is what the BV procedure introduces, a tower of ghosts, ghosts for ghosts and so on until all the degeneracies are properly taken care of.

As mentioned above, string theory also comes with a sea of gauge field theories with generalised symmetries in some sense, therefore one needs BV to handle such cases. One theory in particular is of special interest to us and will be the focus of most chapters of this thesis, either directly or indirectly. This is Double Field Theory (DFT) [8,9]. Its goal is to include a symmetry known as T-duality manifestly into field theory. T-duality is a symmetry inherent to fundamental objects of an extended nature since it is based on the possibility of the fundamental object to see the nontrivial topology of spacetime.

1.1 | BRST and BV formalism

Gauge field theories contain by construction a redundancy in their description of physics. This is an integral part of the theory but presents challenges when one wants to extract physical data about the system in question. In order to obtain "pure" results, unaffected by this redundancy, we consider gauge invariant objects as those representing physical values. Therefore, we can calculate an observable in any gauge and hence have the freedom to choose the most convenient gauge condition. This changes when one wishes to quantise a theory as then all possible field configurations must be taken into account and therein lies the problem, discerning physically different configurations from those related by a gauge redundancy of the formulation. It is precisely this problem that the BRST or, its generalisation, BV methods aim to solve and it is this formalism we present in the following. To be able to fully grasp the formalism a brief recap on graded geometry is also given.

1.1.1 | Gauge redundancy

In this section we shall explore the idea of gauge symmetry or *redundancy* and its role in field theory. Before moving on to a proper treatment of the subject it is useful to illustrate this symmetry by analogy with complex analysis. Take, for example, the integral:

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1} dx,$$

and its computation using the theorem of residues (this illustration is due to [10]). First, one must extend the space from $\mathbb{R}$ to $\mathbb{C} \cong \mathbb{R}^2$, and in this the one-form $dx/(x^2+1)$ has been generalised to a closed complex 1-form $\omega = dz/(z^2+1)$. Therefore our integral is now:

$$\int_{\mathbb{R}\times\{0\}} \omega.$$

As ω is defined at ∞ but not at points $\pm i$ it is really a 1-form on $P^1\mathbb{C}\backslash\{\pm i\}$,[1] and since the integration domain $\mathbb{R} \cup \{\infty\}$ is a cycle in $P^1\mathbb{C}\backslash\{\pm i\}$, the integral does not change for all cycles in the same homology class. So we may change the integration domain to a small circle around i that we call Γ, meaning we have ended up so far at :

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1} dx = \int_{\Gamma} \omega.$$

Since Γ is arbitrarily small we may expand ω in to its Laurent series around i:

$$\frac{dz}{z^2+1} = \left(- \sum_{n=-1}^{\infty} \left(\frac{i}{2}\right)^{n+2} w^n \right) dw$$

where we have introduced the substitution $w = z - i$. Finally, one can integrate this relation (around a circle centred on 0) via the residue theorem to obtain:

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1} dx = \pi.$$

So, how does this relate to our problem of gauge symmetry? Let's break it down into four steps.

1. Identify the problem: calculate the integral $\int_M \omega$ with M an differential manifold

[1] $P^1\mathbb{C}$ denotes the complex procjective line, in other words the Riemann sphere.

and ω a top form.

2. Double the space: embed M as a cycle into a double dimensional manifold N and generalise ω to a closed form Ω on N.

3. Change the cycle: choose another cycle in N that is in the same homology class as M, one that has a power series expansion of Ω in its neighbourhood in N.

4. Expand and integrate: expand the 1-form Ω and apply the integration to obtain a series of the original integral.

The simplified analogy goes as follows: our initial problem is the gauge redundancy of the path integral, therefore BV extends this space by introducing "antifields" to each corresponding physical field just like in step 2 of the above example. The requirement that Ω be closed ($d\Omega = 0$) is a gauge invariance condition just as we have for gauge fields. Then in step 3, by changing the integration cycle we are doing a change of gauge that still produces the same result due to the *gauge invariance condition* essentially just as if going to another Lagrangian submanifold. In the end one arrives at the perturbative expansion of your desired now gauged theory.

1.1.2 | Graded geometry

Before defining the BRST or Batalin-Vilkovisky formalism we need to give a brief introduction to graded geometry as this is the mathematical basis of the formalism (following mostly [11] and [12], with the help of [13]). The idea of grading can be understood as a generalisation of differential forms, the essence of which is captured in the commutativity of the wedge product:

$$\omega \wedge \chi = (-1)^{pq} \chi \wedge \omega,$$

where ω and χ are p and q-forms respectively. The aim is to introduce coordinates on a *super*manifold that anticommute and then extend that from a $\mathbb{Z}_2$ to an arbitrary $\mathbb{Z}$ grading.

Graded vector spaces. We begin by defining a graded vector space and its related objects. A *graded vector space* is the formal sum of a collection of vector spaces $\{V_i\}_{i\in\mathbb{Z}}$, V_i is called the degree i homogeneous subspace of V. A degree shifted (by n) graded vector space denoted $V[n]$ is again a graded vector space with the homogeneous subspaces given by $(V[n])_i = V_{i+n}$. One can also define the symmetric:

$$\bigodot^\bullet(V) = T^\bullet(V) \backslash I_\odot,$$

and antisymmetric (exterior) tensor algebras:

$$\bigwedge\nolimits^{\bullet}(V) = T^{\bullet}(V)\backslash I_{\wedge},$$

with $I_{\odot}$ the ideal generated by elements $v_1 \otimes v_2 - (-1)^{|v_1||v_2|} v_2 \otimes v_1$ and $I_{\wedge}$ by elements $v_1 \otimes v_2 + (-1)^{|v_1||v_2|} v_2 \otimes v_1$.

Graded manifolds. Moving on now to manifolds, a graded manifold $\mathcal{M}$ is a locally ringed space locally isomorphic to $(U, C^{\infty}(U) \otimes \bigodot(W^*))$ where U is an open subset of $\mathbb{R}^n$, W is a graded vector space and the $\mathbb{Z}$ degree is preserved. It is logical to extend now to graded vector bundles which are, roughly speaking, a formal sum of (ungraded) vector bundles $E = \bigoplus_{i \in \mathbb{Z}} E_i$ over $\mathcal{M}$ that is itself a graded manifold. A simple example is the graded manifold $T[1]M$ with functions on it being equivalent to forms in $\Omega(M)$. A graded vector field X on a graded manifold $\mathcal{M}$ is a graded derivation on the algebra of smooth functions on $\mathcal{M}$. Therefore X is a graded linear map:

$$X : C^{\infty}(\mathcal{M}) \to C^{\infty}(\mathcal{M})[k]$$

such that it satisfies the graded Leibniz rule:

$$X(fg) = X(f)g + (-1)^{k|f|} f X(g),$$

for all homogeneous $f, g \in C^{\infty}(\mathcal{M})$. A special class of graded vector fields are *cohomological*[2] vector fields, degree +1 fields that commute with themselves. This implies the canonical existence of a graded commutator of vector fields:

$$[X, Y] = X \circ Y - (-1)^{|X||Y|} Y \circ X,$$

that is again a graded vector field, however now of degree $|[X, Y]| = |X| + |Y|$. This means that a cohomological vector Q:

$$Q : C^{\infty}(\mathcal{M}) \to C^{\infty}(\mathcal{M})[1],$$

is a differential on the algebra of smooth functions $C^{\infty}(\mathcal{M})$ because:

$$0 = [Q, Q] = 2Q \circ Q \quad \Rightarrow \quad Q^2 = 0.$$

Graded manifolds endowed with a cohomological vector field are called differential graded manifolds or *Q-manifolds* for short.

An interesting example is a real, finite dimensional Lie algebra $\mathfrak{g}$. The shifted space

[2] Sometimes called homological, although this is usually reserved for the degree -1 case.

$\mathfrak{g}[1]$ has a natural cohomological vector field Q which is essentially the Chevalley-Eilenberg differential on $\wedge\mathfrak{g}^* \cong C^\infty(\mathfrak{g}[1])$. This can be seen from the fact that $\mathfrak{g}$ being a Lie algebra has a bracket defined by the set of structure constants $C^k{}_{ij}$:

$$[e_i, e_j] = C^k{}_{ij} e_k,$$

where $\{e_i\}$ is a basis on $\mathfrak{g}$. We define Q to be:

$$Q = \tfrac{1}{2} C^k{}_{ij} x^i x^j \frac{\partial}{\partial x^k},$$

where $\{x^i\}$ are (graded) coordinates on $\mathfrak{g}[1]$ (dual to $\{e_i\}$). Now if one were to calculate $[Q, Q]$ they would obtain the Jacobiator corresponding to the bracket defined above (expressed through the structure constants), thus the requirement that Q be cohomological is equivalent to $[\,\cdot\,,\cdot\,]$ being in fact a Lie bracket i.e. satisfying the Jacobi identity. At this point it is interesting to explore what happens in this example if: one changes the base manifold from a point to a nontrivial manifold therefore transforming $\mathfrak{g}$ into a vector bundle E, or allows the space to have (in general infinitely many) different types of structure constants and be fully $\mathbb{Z}$ graded. The first case corresponds to Lie algebroids (vector bundles that have a bracket and anchor map to the tangent bundle that satisfy the Jacobi and Leibniz rules) if Q is cohomological on $E[1]$, see [14]. The second is more relevant to our discussion in chapter 3 and beyond, since this is in a one-to-one correspondence with L_∞-algebras as is further explained in section 3.1.2.

Graded symplectic geometry. So far we have constructed graded manifolds and the special cohomological vector, now we add another structure integral to our story: graded symplectic structures. A graded symplectic form on a graded manifold $\mathcal{M}$ is a homogeneous degree k two-form ω that is closed with respect to the de Rham differential and is *non-degenerate*. By non-degenerate we mean that ω taken as a map to the cotangent bundle induces an isomorphism of vector bundles:

$$\omega : T\mathcal{M} \to T^*[k]\mathcal{M}.$$

Thus we define a graded manifold $\mathcal{M}$ with the additional structure of a symplectic form ω to be a symplectic graded manifold or *P-manifold* $(\mathcal{M}, \omega)$. To clarify nomenclature we mention also symplectic and Hamiltonian vector fields that are defined as those vector fields X along which the Lie derivative of ω vanishes and those that contract the graded symplectic form into an exact 1-form:[3]

$$\mathcal{L}_X \omega = 0$$
$$\iota_X \omega = -dH$$

where H is called the Hamiltonian function. If a graded symplectic manifold is also equipped with a symplectic cohomological vector field Q this is called a differential graded symplectic manifold (dg symplectic manifold) or, more recently, a *QP-manifold*. As a symplectic form ω defines a graded Poisson bracket (also know as a Gerstenhaber bracket):

$$\{f, g\} \equiv X_f g,$$

where X_f is the Hamiltonian vector field corresponding to the function f. It can be shown that Q of a QP-manifold will always be Hamiltonian except in the specific case of $|\omega| \equiv k = -1$. In the case when $k \neq -1$ one can write the cohomological vector using the Poisson bracket with the corresponding Hamiltonian function as:

$$Q = \{S, \cdot\},$$

and therefore the nilpotency of Q becomes:

$$[Q, Q]f = \{\{S, S\}, f\},$$

implying $\{S, S\}$ must be a constant. By doing some degree counting one may notice that function S is of degree $k+1$ (since the bracket is of degree $-k$), this makes $|\{S, S\}| = k+2$ implying if $k \neq -2$ one necessarily has:

$$\{S, S\} = 0.$$

This famous expression is known as the classical master equation if one can identify S with the action functional. Examples of such QP-manifolds are Poisson manifolds ($k = 1$) and Courant algebroids ($k = 2$) which will be explored in more detail in section 2.1.1. QP-manifolds are integral in the AKSZ construction [15] of topological sigma models such as the Poisson sigma model [16, 17] or Courant sigma model [18–20].

1.1.3 | BV algebra and cohomology

A quick recap of classical field theory is given before moving on to BRST or BV in order to see where precisely the need for these more advanced methods arises. Sources

[3] We would like to caution the reader that conventions vary in the sign of the rhs for Hamiltonian vector fields between mathematics and physics oriented literature.

include [10], [11] and [21] (for a classic reference see also [22]).

Classical Field Theory. The Lagrangian approach to classical physics states a classical field theory is specified with three ingredients:

1. a spacetime manifold M;
2. a space of fields $\mathcal{F}$, generally the space of sections of some bundle over M with the structure of an infinite dimensional manifold;
3. an action functional $S : \mathcal{F} \to \mathbb{R}$.

An additional requirement on this action functional is that it is local in the sense that S can be written as:

$$S[\phi] = \int_M L(x, j^n_x \phi),$$

where $j^n_x \phi$ is the n-jet of field ϕ. The classical physics of such a system is completely captured by the critical locus of S: $\mathrm{Crit}(S) = \{\phi \in \mathcal{F} \mid dS[\phi] = 0\}$, or, stated in the vernacular of physics: the principle of least action. In the variational principle this leads to (in cases where the boundary does not contribute) the Euler-Lagrange equations the solutions of which constitute $\mathrm{Crit}(S)$.

In order to transform this theory from the classical to the quantum regime one introduces the concept of path integrals.[4] In the most basic sense the path integral formalism gives the expectation values of an observable in the following way. We are given the classical data of the theory, so a manifold M, space of fields $\mathcal{F}$ and action functional S. We want to calculate the expectation value of observable $\mathcal{O}$ that is a function $\mathcal{O} : \mathcal{F} \to \mathbb{R}$, the principle of path integrals says this is given by:

$$\langle \mathcal{O} \rangle = \frac{1}{Z_S} \int_{\mathcal{F}} \mathcal{O}(\phi) \exp \tfrac{i}{\hbar} S[\phi] \, \mathcal{D}\phi,$$

where $\exp \frac{i}{\hbar} S[\phi] \, \mathcal{D}\phi$ is a measure on $\mathcal{F}$ and Z_S is the partition function:

$$\int_{\mathcal{F}} \exp \tfrac{i}{\hbar} S[\phi] \, \mathcal{D}\phi.$$

Normalising by the partition function makes this a probability measure, however, as $\mathcal{F}$ is infinite dimensional rigorous mathematical treatment of this approach is still in progress in the community. Nonetheless if one ignores the problems of infinite-dimensionality (as we shall) this approach is very successful.

The procedure outlined above works if the critical locus is non-degenerate, which is not the case if the system has symmetries such as those stemming from a gauge redundancy.

[4] There are other possibilities to quantise a theory, however, the path integral method makes dealing with symmetries the simplest and relies on the Lagrangian formulation of classical physics that we focus on in this work.

This is tackled by introducing a gauge-fixing, however, this is not trivial to do. Essentially what one needs to do is restrict the integration to the subspace of only gauge inequivalent field configurations i.e. reduce the integration domain to $\mathcal{F}/G$ where G is the group of gauge transformations.

BRST. BRST is a method of quantising or gauge fixing gauge theories. It cannot handle all kinds of gauge symmetries and is superseded by the BV formalism. Nevertheless we begin with the idea of BRST as this is the basis on which BV is built upon.

The framework is as follows. We embed the manifold of classical fields $\mathcal{F}$ into the 0 degree body of a ($\mathbb{Z}$-) graded manifold $\mathcal{F}_{\text{BRST}}$. This grading is what is usually called the ghost number meaning physical fields have ghost number 0. Additionally $\mathcal{F}_{\text{BRST}}$ is endowed with a cohomological vector field and measure that is assumed to be Q-invariant:

$$\int_{\mathcal{F}_{\text{BRST}}} Qf\,\mu = 0,$$

for any function f on $\mathcal{F}_{\text{BRST}}$ (for more details on this see [23]). BRST is related to the classical case by two conditions: first is that the classical action is a BRST cocycle i.e. $QS = 0$, and the second is $\mathcal{F}_{\text{BRST}}$ is a resolution of $\mathcal{F}/G$ or in other words the zeroth cohomology of $\mathcal{F}_{\text{BRST}}$ is isomorphic to the space of functions on $\mathcal{F}/G$. The gauge fixing is done by a choice of gauge fixing fermion Ψ which is a degree -1 function on $\mathcal{F}_{\text{BRST}}$, then for the path integral we have:

$$\int_{\mathcal{F}_{\text{BRST}}} \exp \tfrac{i}{\hbar} S\,\mu = \int_{\mathcal{F}_{\text{BRST}}} \exp \tfrac{i}{\hbar}(S + Q\Psi)\,\mu,$$

that holds because of the invariance of the measure. By requiring the zeroth cohomology to match $\mathcal{F}/G$ we ensure each gauge orbit is counted only once in the integral.

BV. The Batalin-Vilkovisky procedure is a generalisation of the BRST formalism in that to each field an additional *antifield* is attributed. We shall now define this construction. A BV manifold is a P(-1)-manifold or a graded symplectic manifold of degree -1 with a measure compatible with this symplectic structure in such a way that locally it is just the coordinate Berezinian measure. Usually we take the BV manifold to be of form $T^*[-1]\mathcal{M}$. This measure defines a divergence as:

$$\int_{\mathcal{M}} Xf\,\mu = \int_{\mathcal{M}} \operatorname{div}_\mu X \cdot f\,\mu,$$

with Xf understood as the differential action of vector field X on functions f on $\mathcal{M}$. With this divergence one can now define the BV Laplacian as:[5]

$$\Delta_\mu f \equiv \tfrac{(-1)^{|f|}}{2}\mathrm{div}_\mu\{f,\,\cdot\,\}$$

Its important to note that the Gerstenhaber bracket and Laplacian are not independent even though they are usually included separately in the definition of a BV algebra as a Gerstenhaber algebra with a compatible Laplacian in the sense of:

$$\Delta\{f,g\} = \{\Delta f, g\} + (-1)^{|f|+1}\{f, \Delta g\}.$$

A Lagrangian submanifold $\mathcal{L}$ is in the graded case defined as in the ordinary case, as a submanifold on which the symplectic form vanishes and that has maximal dimension. When we take the BV manifold $T^*[-1]\mathcal{M}$ there is a special Lagrangian submanifold given by a gauge fixing fermion of degree -1, in local Darboux coordinates $\{x^i, x_i^\dagger\}$:[6]

$$\mathcal{L}_\Psi = \left\{(x^i, x_i^\dagger)\,\middle|\, x_i^\dagger = \frac{\partial\Psi}{\partial x^i}\right\}.$$

A Lagrangian will always have a measure induced from the measure on the BV manifold. The main result of this formalism are the two following statements (due to [6] and [24]). If $\Delta f = 0$ and $\mathcal{L}$ and $\mathcal{L}'$ are homologically equivalent Lagrangians then:

$$\int_{\mathcal{L}} f\,\mu_{\mathcal{L}} = \int_{\mathcal{L}'} f\,\mu_{\mathcal{L}'};$$

if f is Δ-exact then:

$$\int_{\mathcal{L}} f\,\mu_{\mathcal{L}} = 0.$$

This theorem assures us that integrating over differently gauge fixed actions does not change the result.

We now have all the ingredients necessary to construct a BV gauge field theory. We start with the classical information $(M, \mathcal{F}_{\mathrm{cl}}, S_{\mathrm{cl}})$, then through the above explained procedure obtain the BRST fields $\mathcal{F}_{\mathrm{BRST}}$ and identify this with the $\mathcal{M}$ manifold on which we built up our BV manifold:

$$\mathcal{F}_{\mathrm{BV}} = T^*[1]\mathcal{F}_{\mathrm{BRST}}.$$

This induces the aforementioned antifields, in essence the fibre coordinates on $\mathcal{F}_{\mathrm{BV}}$. The remaining unknown is the BV action functional. For the scope of this thesis we shall only

[5]The measure μ will only be written in the subscript of Δ in this definition to emphasize its origin but will later be omitted for brevity.

[6]A dagger in superscript will always physically mean the corresponding antifield.

consider the zeroth order in the formal power series of S as a function on $\mathcal{F}_{\mathrm{BV}}$:

$$S = S_0 + S_1\hbar + S_2\hbar^2 + \cdots ,$$

since we are not explicitly interested in quantum calculations that follow once one knows S_0. Thus from now on we shall by S_{BV} denote S_0 for which the following must hold in order to be consistent with the classical action:

$$S_{\mathrm{BV}}|_{\mathcal{F}_{\mathrm{BRST}}} = S_{\mathrm{cl}}.$$

Remembering BV is an extension of BRST means that we inherit the cohomological BRST operator Q that can now be expressed using the Gerstenhaber bracket as:

$$Q = \{S_{\mathrm{BV}}, \cdot\},$$

that makes the BRST invariance of the action:

$$\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0, \tag{1.1.1}$$

the classical master equation again (equivalent to the requirement that Q be cohomological). Therefore the main challenge becomes finding a BV action S_{BV} that satisfies the classical master equation. Notice that we have obtained the master equation just as in the previous section for QP-manifolds, however, it is important to emphasize that in the previous case it arose as a geometrical consequence of construction, whereas now it has appeared as a separate requirement since this is not a QP-manifold. In fact it could not be one since it was shown that the QP compatibility allows the cohomological vector to be both symplectic and Hamiltonian in the cases when the degree of the graded symplectic structure is different from -1, the precise case we have in BV.

1.2 | Double field theory

Double field theory, an attempt to make T-duality a manifest symmetry of field theory [9, 25–27], is a special field theory of focus in this thesis and onto which all the above machinery will be applied. Thus, in this section we shall briefly go through the motivation for double field theory, namely T-duality, and its relation to the split orthogonal group that constitutes the foundation of the formulation of DFT. Then a short description of DFT is given introducing objects such as the generalised metric before going into a change of formalism by swapping the information contained within the generalised metric with that of a generalised vielbein. The section ends with the basics of the flux formulation needed in later chapters for the correspondence to the algebraic structure called a DFT algebroid.

1.2.1 | Foundation

T-duality. In the simplest sense T-duality is the physical equivalence of sigma models over a background with a circle dimension of radius R and $1/R$. For example, this can most easily be seen in bosonic string theory (covered in most string theory books e.g. [28] but here following the approach of [29] and [30]). There one can show that 26 spacetime dimensions are needed for the theory to be consistent, implying 22 of them need to be "hidden" somehow. This is done via the process of compactification, in essence, by making these dimensions small compact subspaces such that they cannot be observed directly (at least at energy scales presently available). The simplest such subspace is a circle of radius R and this will make our example. Take the target space to be $M = \mathbb{R}^{1,24} \times S^1$ and worldsheet Σ parametrised by coordinates (τ, σ). The 25th component (corresponding to the circle direction) of the target coordinates of Σ understood as maps $X : \Sigma \to M$ is then required to satisfy:

$$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi R m,$$

where $m \in \mathbb{Z}$ is the winding number. By calcualting the Fourier modes of the string one can obtain the Virasoro generators and arrive at the mass spectrum of the string:

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 + \cdots ,$$

the dots indicating terms irrelevant to the compactification and following observation, and $n \in \mathbb{Z}$ the momentum excitation. This expression is completely agnostic to the substitution:

$$n \leftrightarrow m \quad \text{and} \quad R \leftrightarrow \frac{\alpha'}{R},$$

with $\alpha' \in \mathbb{R}$ a dimension -2 constant parameter inverse to the string tension (up to 2π). This is T-duality. One can replace the circle S^1 with a different one of inverse radius and the physics (other than the switching of winding and momentum modes) would not change. This means that if we have coordinate X^{25} on the first and $\tilde{X}_{25}$ on the second, the duality becomes the equivalence:

$$X^{25} \leftrightarrow \tilde{X}_{25}.$$

Double field theory is a realisation of a field theory over a doubled space (spanned by coordinates x and $\tilde{x}$) such that this doubled nature has T-duality manifestly between the two sets of coordinates.

$O(d,d)$-symmetry. Our aim now is to see how a T-duality as described above generalises to $O(d,d)$ symmetry. Continuing with string theory, and in addition to [29] also following [9] and [31], closed strings must satisfy the level matching condition, a constraint stemming from the reparametrisation invariance of the worldsheet in the σ direction. In the more general case of the compactified space being an n-torus i.e. $M = \mathbb{R}^{1,26-n} \times T^n$ the level matching condition becomes:

$$N - \tilde{N} = p_i w^i,$$

with N and $\tilde{N}$ the number operators, and p and w the momentum and winding operators. If $N = \tilde{N} = 1$, for example, this condition implies the *weak constraint*:

$$\partial_i \tilde{\partial}^i (\,\cdot\,) = 0,$$

where the dot indicates any one field. This comes from the fact that $p_i = -i\partial_i$ and $w^i = -i\tilde{\partial}^i$ by analogy. One must take care not to confuse the weak constraint that is a physical requirement with the *strong* constraint of DFT that is just an artefact of the formulation, as we will see later. However, the form can be deceiving as it has the same expression:

$$\partial_i \tilde{\partial}^i (\cdots) = 0,$$

the difference being in the argument that is now any product of fields as opposed to the weak constraint that had only one field acted upon. Coming back to the level matching condition, one may combine these winding and momentum operators into a $2d$ vector (in bosonic strings d being 26):

$$v = \begin{pmatrix} w^i \\ p_i \end{pmatrix}$$

making the condition:

$$N - \tilde{N} = \tfrac{1}{2} v^T \eta v \quad \text{where} \quad \eta = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}.$$

Written in this form the relevant term in the string Hamiltonian becomes:

$$H \supset \tfrac{1}{2} v^T \mathcal{H}(G,B) v,$$

where $\mathcal{H}(G,B)$ is the generalised metric depending on the background metric G and 2-form Kalb-Ramond field B:

$$\mathcal{H}(G,B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}.$$

Its inverse is given by:

$$\mathcal{H}^{-1} = \eta\mathcal{H}\eta.$$

Both the level matching condition and the Hamiltonian must be invariant under a T-duality transformation which we take to be an invertible $d \times d$ matrix O. Therefore we say:

$$v = O^T v',$$

implying $v^T \eta v = v'^T O \eta O^T v$, meaning for the level matching condition to be invariant O must satisfy:

$$O\eta O^T = \eta.$$

Remember that η is an off-block diagonal unit matrix so can be transformed into the block diagonal form $\mathrm{diag}(1_d, -1_d)$. Hence O defines an element of the split orthogonal group $O(d,d)$. This transformation reflects on the generalised metric as well, this can be seen by the second claim that the Hamiltonian must be invariant:

$$v^T \mathcal{H}(G,B) v = v'^T O \mathcal{H}(G,B) O^T v',$$

defining the transformed metric as:

$$\mathcal{H}(G',B') = O\mathcal{H}(G,B)O^T.$$

There are two points to emphasise, first, since the momentum and winding numbers are integers the group is over the set $\mathbb{Z}$, and second, the physical symmetry itself is actually $O(n,n)$ as there are only n compactified dimensions that can dualise. As was shown the symmetry is formally extended to $O(d,d)$, however, another extension is made with regards to the first observation, namely $O(d,d)$ is extended from the integers to all of $\mathbb{R}$. This is the group double field theory will be based upon, occasionally denoted $O(d,d,\mathbb{R})$ for clarity.

1.2.2 | Frame formulation

In order to make the description of DFT and $O(d,d)$ symmetry more in line it is useful to introduce a more covariant notation. First one needs a basis of $O(d,d)$ such that the invariant metric η takes the off-block diagonal form above. The coordinates on which this metric can be used to raise and lower indices are given by:

$$X^A = \begin{pmatrix} \tilde{x}_a \\ x^a \end{pmatrix}; \qquad \partial_A = \begin{pmatrix} \tilde{\partial}^a \\ \partial_a \end{pmatrix}.$$

Notice the index $A = 1, \ldots, 2d$ is doubled putting on an equal footing both the original coordinates and their duals. These coordinates now transform with respect to an $O(d,d)$ transformation O in the following way:

$$X'^A = O^A{}_B X^B.$$

The field content i.e. the generalised dilaton (containing the dilaton and determinant of the target metric) and metric (containing the target space metric and B-field) are $O(d,d)$ scalars and symmetric tensors of order 2:

$$\begin{aligned} d'(X') &= d(X) \\ \mathcal{H}_{AB} &= O^C{}_A O^D{}_B \mathcal{H}'_{CD}. \end{aligned}$$

$\mathcal{H}$ is symmetric from the fact that η defines its inverse by raising the indices:

$$\mathcal{H}^{AB} = \eta^{AC}\eta^{BD}\mathcal{H}_{CD} \quad \text{and} \quad \mathcal{H}^{AB}\mathcal{H}_{BC} = \delta^A_C.$$

DFT defined in terms of $\mathcal{H}$ and d has a gauge symmetry generated by the generalised Lie derivative:

$$\mathsf{L}_\xi A_A = \xi^B \partial_B A_A + A_B \partial_A \xi^B - A_B \partial^B \xi_A,$$

with A and ξ generic $O(d,d)$ vectors and ξ understood as the gauge parameter. The commutator of two gauge transformations induces the gauge algebra closure by the C-bracket of DFT up to terms controlled by the strong constraint:

$$[\mathsf{L}_{\xi_1}, \mathsf{L}_{\xi_2}] = \mathsf{L}_{-[\![\xi_1,\xi_2]\!]}\Big|_{\text{strong constraint}}, \tag{1.2.1}$$

as will be seen in chapter 2. Taking cue from general relativity one can replace the generalised metric by introducing frame fields (Refs. [27, 32–34]). Hence, the fundamental fields become the generalised dilaton d and the generalised vielbein $\mathcal{E}_I{}^A$ with $I = 1, \ldots, 2d$ being the flat indices. A convenient choice for the flattened metric is that it coincides with the $O(d,d)$ invariant metric η making the bein precisely an element of the $O(d,d)$ group:

$$\hat{\eta}_{IJ} = \mathcal{E}_I{}^A \mathcal{E}_J{}^B \eta_{AB} = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}. \tag{1.2.2}$$

It is important to note here that even though η and $\hat{\eta}$ have formally the same form its indices give away their different nature, later this will become even more important to distinguish as one will end up defining the curvature of the underlying L_∞-algebra with the other just a structure constant. The choice of vielbein induces a flattened generalised metric:

$$S_{IJ} = \begin{pmatrix} s^{ij} & 0 \\ 0 & s_{ij} \end{pmatrix},$$

where $i, j = 1, \dots, d$ are the indices split by $\hat{\eta}$ and s are two flat Lorentz $O(1, d-1)$ metrics that satisfy:

$$\mathcal{H}_{AB} = \mathcal{E}^I{}_A \mathcal{E}^J{}_B S_{IJ}.$$

Having defined the vielbein we are now able to construct two objects or "generalised fluxes" with which we can construct an $O(d,d)$ scalar action:

$$\begin{aligned} \mathcal{F}_{IJK} &= 3\mathcal{E}_{[I}{}^A \partial_{\underline{A}} \mathcal{E}_J{}^B \mathcal{E}_{K]B}, \\ \mathcal{F}_I &= \mathcal{E}^{JA}\partial_A \mathcal{E}_J{}^B \mathcal{E}_{IB} + 2\mathcal{E}_I{}^A \partial_A d, \end{aligned}$$

that satisfy the Bianchi identities:[7]

$$\begin{aligned} \mathcal{E}_{[I}{}^A \partial_{\underline{A}} \mathcal{F}_{JKL]} - \tfrac{3}{4}\mathcal{F}_{[IJ}{}^M \mathcal{F}_{KL]M} &\equiv \hat{\mathcal{Z}}_{IJKL} \\ \mathcal{E}^{KA}\partial_A \mathcal{F}_{KIJ} + 2\mathcal{E}_{[I}{}^A \partial_{\underline{A}} \mathcal{F}_{J]} - \mathcal{F}^K \mathcal{F}_{KIJ} &\equiv \hat{\mathcal{Z}}_{IJ} \\ \mathcal{E}^{IA}\partial_A \mathcal{F}_I - \tfrac{1}{2}\mathcal{F}^I \mathcal{F}_I + \tfrac{1}{12}\mathcal{F}^{IJK}\mathcal{F}_{IJK} &\equiv \hat{\mathcal{Z}} \end{aligned} \tag{1.2.3}$$

The action defining DFT is now:

$$S = \int dX \exp(-2d)\ \mathcal{R}(\mathcal{E}, d), \tag{1.2.4}$$

with $\mathcal{R}$:

$$\begin{aligned} \mathcal{R} = &(S^{IJ} - \hat{\eta}^{IJ})(2\mathcal{E}_I{}^A \partial_A \mathcal{F}_J - \mathcal{F}_I \mathcal{F}_J) \\ &+ \mathcal{F}_{IJK}\mathcal{F}_{LMN}\left(-\tfrac{1}{6}\hat{\eta}^{IL}\hat{\eta}^{JM}\hat{\eta}^{KN} + \tfrac{1}{4}S^{IL}\hat{\eta}^{JM}\hat{\eta}^{KN} - \tfrac{1}{12}S^{IL}S^{JM}\hat{\eta}^{KN}\right), \end{aligned} \tag{1.2.5}$$

playing the "role" of both the spacetime Ricci scalar, dilaton kinetic term (rescaled to incorporate the metric determinant) and B-field strength H term of supergravity as a low energy effective action:

$$S_{\text{eff}} = \int dx \sqrt{-G} e^{-2\phi}\left(R + 4(\partial\phi)^2 - \tfrac{1}{12}H^2\right),$$

to the worldsheet string theory action:

$$S_{\text{WS}} = \tfrac{1}{4\pi\alpha'} \int d\sigma \sqrt{g}\Big(\left(G_{\mu\nu}g^{\alpha\beta} + iB_{\mu\nu}\epsilon^{\alpha\beta}\right)\partial_\alpha X^\mu \partial_\beta X^\nu + \alpha'\phi R^{(2)}\Big).$$

[7] Underlined indices are skipped in the antisymmetrisation.

where R and $R^{(2)}$ are the spacetime and worldsheet Ricci scalars, and we have performed a Wick rotation to Euclidian signature on the worldsheet. Relations (1.2.4) and (1.2.5) do not assume the strong constraint as will not be assumed throughout this thesis unless specifically stated. Additionally, in the remainder we shall drop the dilaton field for simplicity, see [35] for a geometric description of the dilaton field.

CHAPTER 2
BRST SYMMETRY OF DOUBLED MEMBRANE SIGMA MODELS

Double field theory (DFT) [9, 26, 27, 36], seen as an attempt to realise the T-duality of closed string theory in section 1.2 manifestly, is at the level of low-energy supergravity based on a generalised geometry of a tangent bundle extended by 1-forms [37,38]. This generalised tangent bundle is then equipped with a bracket, a symmetric bilinear form and a map to a tangent bundle defining the structure of Courant algebroid [39–41]. The symmetric bilinear form defines an $O(d, d)$ structure relevant for T-duality on a d-dimensional target space, while the symmetries of the generalised tangent bundle unify diffeomorphisms and 2-form gauge transformations of the Kalb-Ramond field.[1] Moreover, the properties of the Courant bracket are used to systematically determine background fluxes of string theory and their Bianchi identities [42, 43].

Furthermore, in Ref. [18] Roytenberg used graded geometry to show that given the data of a Courant algebroid one can uniquely construct the Batalin-Vilkovisky (BV) master action, S_{BV} of section 1.1.3, for a membrane sigma model which is a first-order functional for generalised Wess-Zumino terms in three dimensions. (See also Refs. [19, 20, 44] for earlier work in the same direction.) Therefore, the aim of this chapter is to connect the BV knowledge of the Courant sigma model with the known [45] classical projection that produces a sigma model over a doubled target that is identified with the DFT sigma model, and check these two are compatible.

The chapter is based mostly on [46] and is split in two parts: the first that recaps the Courant algebroid and sigma model theory, and the second that introduces the projection and applies it to the first part.

[1]These are not, however, all of the elements of $O(d, d)$. In addition to diffeomorphisms and B-field transformations one has factorised duality transformations and β-transformations, though these will be ignored.

2.1 | Gauge and BRST symmetries of the Courant sigma model

The Courant sigma model belongs to a general class of topological sigma models of AKSZ type [15] satisfying the classical master equation. In this particular case one can show that the conditions for gauge (or more generally BRST) invariance of the Courant sigma model and the on-shell closure of the algebra of gauge transformations follow from the classical master equation and correspond to the axioms of a Courant algebroid defined in section 2.1.1. Membrane sigma models were subsequently used for a systematic description of closed strings in non-geometric flux backgrounds [42, 45, 47–49].

The gauge transformations of the DFT membrane sigma model were obtained in [45] by projecting the standard gauge transformations of a Courant sigma model over a doubled target base manifold. However, the latter is the antifield zero sector of the classical BV action constructed using the AKSZ procedure. The master action is defined over a graded manifold in terms of superfields (fields of the same total degree) whose components include the classical fields, ghosts, ghosts for ghosts and antifields. The classical gauge transformations lift to the BRST transformations of the superfields as shown in sections 2.1.2 and 2.1.3.

2.1.1 | Courant algebroid

To begin we outline the definition of a standard Courant algebroid (due to [39, 40]) following [50]. A standard Courant algebroid is defined over $E = TM \oplus T^*M$, where M is a d-dimensional manifold. The resulting generalised vector stems from the generalised bundle E: $A = A_V + A_F \in \Gamma(E)$ where we have separated the vector part $A_V \in \Gamma(TM)$ and one-form part $A_F \in \Gamma(T^*M)$.

Let $E \to M$ be a vector bundle. Define an antisymmetric bracket of sections of the bundle, $[\cdot,\cdot]_C : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$, a non-degenerate symmetric bilinear form, $\langle\cdot,\cdot\rangle : \Gamma(E)\otimes\Gamma(E) \to C^\infty(M)$, and, finally, an anchor map, $\rho : E \to TM$. This quadruple $(E, [\cdot,\cdot]_C, \langle\cdot,\cdot\rangle, \rho)$ defines a Courant algebroid [40] up to some compatibility conditions discussed below.

A Courant algebroid allows for two types of operations, the antisymmetric Courant bracket or the Dorfman derivative [51]. The Courant bracket can be obtained as an antisymmetrisation of the Dorfman derivative. In the case of a standard Courant algebroid (the focus of this section) the bracket can be explicitly stated in terms of the vector and form parts of the sections:

$$[A, B]_C = [A_V, B_V] + \mathcal{L}_{A_V} B_F - \mathcal{L}_{B_V} A_F - \tfrac{1}{2}\,\mathrm{d}(\iota_{A_V} B_F - \iota_{B_V} A_F),$$

and can be further twisted by $T(A_V, B_V)$ where T is a closed three-form. The Dorfman derivative is this case is given by:

$$A \circ B = L_A B = [A_V, B_V] + \mathcal{L}_{A_V} B_F - \iota_{B_V} \mathrm{d} A_F.$$

It is obvious the Courant bracket is a generalisation of the Lie bracket and the Dorfman derivative of the Lie derivative. As for the pairing:

$$\langle A, B \rangle = \tfrac{1}{2}(\iota_{A_V} B_F + \iota_{B_V} A_F),$$

one can immediately see the $O(d,d)$ symmetry in the structure since:

$$\langle A, B \rangle = \begin{pmatrix} A_V & A_F \end{pmatrix} \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \begin{pmatrix} B_V \\ B_F \end{pmatrix},$$

where the matrix is the $O(d,d)$ metric that we denote $\hat{\eta}_{IJ}$ and the indices I, J go from $1, \ldots, 2d$.[2] The three ingredients added to the bundle (the bracket, pairing and anchor) must satisfy five compatibility conditions. The first is the Jacobi identity for the Courant bracket,

$$[[A, B]_C, C]_C + \text{cyclic} = \tfrac{1}{3}\,\mathcal{D}\langle [A, B]_C, C \rangle + \text{cyclic}, \tag{2.1.1}$$

where the differential operator $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by

$$\langle \mathcal{D} f, A \rangle = \tfrac{1}{2}\,\rho(A) f, \tag{2.1.2}$$

for any $A, B, C \in \Gamma(E)$ and $f \in C^\infty(M)$. Second is the Leibniz rule for the bracket:

$$[A, f\, B]_C = f\,[A, B]_C + \big(\rho(A) f\big)\, B - \langle A, B \rangle\, \mathcal{D} f, \tag{2.1.3}$$

third the homomorphism property of the anchor with respect to the bracket:

$$\rho[A, B]_C = [\rho(A), \rho(B)], \tag{2.1.4}$$

fourth is the fact that $\mathcal{D} f$ is in the kernel of the anchor:

$$\rho \circ \mathcal{D} = 0 \quad \Longleftrightarrow \quad \langle \mathcal{D} f, \mathcal{D} g \rangle = 0. \tag{2.1.5}$$

This condition becomes key when going to the DFT case as it will contain most explicitly the strong constraint of DFT. The fifth and final property is the compatibility between

[2] The hat on $\hat{\eta}$ will always denote that the $O(D,D)$ metric acts on vectors of the algebroid bundle (in this case E). This is in opposition to η that will appear later and will also be an $O(D,D)$ metric but one that acts on tangent vectors of the base manifold. One must be careful as D can mean both d or $2d$, the difference should be clear from context and the range of the corresponding indices.

the Courant bracket and the symmetric pairing,

$$\rho(C)\langle A,B\rangle = \langle [C,A]_C + \mathcal{D}\langle C,A\rangle, B\rangle + \langle A, [C,B]_C + \mathcal{D}\langle C,B\rangle\rangle. \tag{2.1.6}$$

It is important to note that although given five, only three are in fact independent requirements and one only really needs the Jacobi identity, compatibility and any one of the rest. Roytenberg and Weinstein [52] showed three properties of a CA that will become useful later.

Property 2.1.7. *In a Courant algebroid E with anchor ρ, differential $\mathcal{D}$ and Nijenhuis operator $\mathcal{N}_c(e_1,e_2,e_3) = \frac{1}{3}\langle [e_1,e_2],e_3\rangle$ + cyclic, the following three identities hold for $e_i \in \Gamma(E)$:*

1. $[e,\mathcal{D}f] = \mathcal{D}\langle e,\mathcal{D}f\rangle$,
2. $\mathcal{N}_c(e_1,e_2,\mathcal{D}f) = \frac{1}{4}\rho[e_1,e_2]f$,
3. $\langle \mathrm{Jac}(e_1,e_2,e_3),e_4\rangle + \textit{antisymm.}(1,2,3,4) =$
$= -2\langle [e_1,e_2],[e_3,e_4]\rangle + \textit{antisymm.}(1,2,3,4)$,
where antisymm.$(1,2,3,4)$ *indicates all terms needed for the antisymmetrisation of e_1, e_2, e_3 and e_4.*

Let's introduce a local basis for the sections of E, e^I where $I = 1,\dots,2d$. The structures defined thus far (bracket, pairing, anchor and derivation) produce the following coefficients in this basis:

$$\begin{aligned}
\langle e^I, e^J\rangle &= \frac{1}{2}\hat{\eta}^{IJ},\\
[e^I,e^J] &= \hat{\eta}^{IK}\hat{\eta}^{JL}T_{KLM}e^M,\\
\rho(e^I)f &= \hat{\eta}^{IJ}\rho^a{}_J\partial_a f,\\
\mathcal{D}f = \mathcal{D}_I f\, e^I &= \rho^a{}_I\partial_a f\, e^I.,
\end{aligned}$$

with $a = 1,\dots,d$ the tangent bundle index. In this local basis the five abstract properties become the following constraints on the functions $\rho^a{}_J$ and T_{IJK}

$$\hat{\eta}^{IJ}\,\rho^a{}_I\,\rho^b{}_J = 0, \tag{2.1.8}$$
$$\rho^a{}_I\,\partial_a\rho^b{}_J - \rho^a{}_J\,\partial_a\rho^b{}_I - \hat{\eta}^{KL}\,\rho^b{}_K\,T_{LIJ} = 0, \tag{2.1.9}$$
$$4\,\rho^a{}_{[L}\,\partial_a T_{IJK]} + 3\,\hat{\eta}^{MN}\,T_{M[IJ}\,T_{KL]N} = 0. \tag{2.1.10}$$

It is instructive to mention the correspondence between a Courant algebroid and QP2-manifolds [53] (for a pedagogical exposition see [54]). Recall from sec. 1.1.2 that a QP2-manifold (also called a differential graded symplectic manifold of degree 2) is defined by the triplet $(\mathcal{N},Q,\omega)$ with the base manifold usually taken to be of the form

$\mathcal{N} = T^*[2]T[1]M$. The Q is a cohomological vector field on $\mathcal{N}$ (a vector of grading one that squares to zero) and the P corresponds to the graded symplectic structure ω of degree 2 that induces a graded Poisson bracket on $C^\infty(\mathcal{N})$. The triple is a QP2 manifold if the two structures are compatible in the sense that:

$$\mathcal{L}_Q \omega = 0.$$

The manifold $\mathcal{N}$ is charted by three types of coordinates: degree 2 coordinate F_a coming from the cotangent bundle fibre shifted by two, two degree 1 coordinates (one from the shifted tangent and one from the shifted cotangent bundle) that can be combined into one double coordinate A^I and a degree 0 coordinate of the base manifold x^a. The Q-structure defines a degree three Hamiltonian function Θ via:

$$Q = \{\Theta, \,\cdot\,\},$$

where $\{\,\cdot, \cdot\,\}$ is the Poisson bracket induced by the symplectic structure:

$$\omega = \mathrm{d}X^a \,\mathrm{d}F_a + \tfrac{1}{2}\hat{\eta}_{IJ}\,\mathrm{d}A^I \mathrm{d}A^J. \tag{2.1.11}$$

The requirement $Q^2 = 0$, therefore, yields the classical master equation:

$$\{\Theta, \Theta\} = 0,$$

which in turn yields (2.1.8) - (2.1.10) in local coordinates when

$$\Theta = \rho^a{}_I(x)\, F_a\, A^I + \tfrac{1}{3!}\, T_{IJK}(x)\, A^I\, A^J\, A^K. \tag{2.1.12}$$

This choice of Hamiltonian function gives the following cohomological vector field [54, 55]:

$$\begin{aligned} Q = -\rho^a{}_I(x) A^I \frac{\partial}{\partial x^a} + \hat{\eta}^{IJ} \rho^a{}_J(x) F_a \frac{\partial}{\partial A^I} + \partial_a \rho^b{}_I(x) F_b A^I \frac{\partial}{\partial F_a} + \\ + \tfrac{1}{2}\hat{\eta}^{IJ} T_{JKL}(x) A^K A^L \frac{\partial}{\partial A^I} + \tfrac{1}{3!}\partial_a T_{IJK}(x) A^I A^J A^K \frac{\partial}{\partial F_a}. \end{aligned} \tag{2.1.13}$$

2.1.2 | Courant sigma model as a reducible gauge theory

First we discuss the gauge symmetries of the Courant sigma model for a membrane worldvolume Σ_3, defined over a doubled target space $\mathcal{M}$. We immediately start with a doubled target space $\mathcal{M}$ since later in the chapter our procedure will require this of us, however, everything stated within this and the next section holds for any base space. The action functional for the classical model is:

$$S_{\mathrm{C}}[X,\mathbb{A},F]=\int_{\Sigma_3}\Big(F_A\wedge \mathrm{d}X^A+\tfrac{1}{2}\hat{\eta}_{\hat{I}\hat{J}}\mathbb{A}^{\hat{I}}\wedge \mathrm{d}\mathbb{A}^{\hat{J}}-\rho^A{}_{\hat{J}}(X)\mathbb{A}^{\hat{J}}\wedge F_A+\tfrac{1}{6}T_{\hat{I}\hat{J}\hat{K}}(X)\mathbb{A}^{\hat{I}}\wedge\mathbb{A}^{\hat{J}}\wedge\mathbb{A}^{\hat{K}}\Big)\,, \tag{2.1.14}$$

where $A=1,\dots,2d$ is a target space index, $\hat{I}=1,\dots,4d$ is the bundle index and we have considered scalar fields as components of maps $X=(X^A):\Sigma_3\to\mathcal{M}$, 1-forms $\mathbb{A}\in\Omega^1(\Sigma_3,X^*\mathbb{E})$, and an auxiliary 2-form $F\in\Omega^2(\Sigma_3,X^*T^*\mathcal{M})$, and locally we consider the generalised tangent bundle $\mathbb{E}=\mathrm{T}\mathcal{M}\oplus\mathrm{T}^*\mathcal{M}$. The fields $(X^A)=(X^a,\tilde{X}_a)$ are identified with the pullbacks of the coordinate functions, $X^a=X^*(x^a)$ and $\tilde{X}_a=X^*(\tilde{x}_a)$. The symmetric bilinear form of the Courant algebroid over $\mathbb{E}$ corresponds to the $O(2d,2d)$-invariant metric

$$\hat{\eta}=(\hat{\eta}_{\hat{I}\hat{J}})=\begin{pmatrix}0 & 1_{2d}\\ 1_{2d} & 0\end{pmatrix},$$

not to be confused with the $O(d,d)$ metric η that acts on sections of $T\mathcal{M}$ that will appear later. $\rho^A{}_{\hat{J}}$ are related to the components of the anchor map $\rho:\mathbb{E}\to T\mathcal{M}$ and $T_{\hat{I}\hat{J}\hat{K}}$ are related to a general twist of the Courant algebroid, generating a generalised Wess-Zumino term. For a local basis $(e_{\hat{I}})$ of $\mathbb{E}$, they are related to $X^*\left(\langle e_{\hat{I}},[e_{\hat{J}},e_{\hat{K}}]\rangle\right)$, where $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ are the non-degenerate symmetric bilinear form and the bracket of the Courant algebroid over $\mathbb{E}$ respectively.

The action (2.1.14) is invariant under the following infinitesimal gauge transformations [19]

$$\delta_{(\epsilon,t)}X^A=\rho^A_{\hat{J}}\,\epsilon^{\hat{J}}, \tag{2.1.15}$$
$$\delta_{(\epsilon,t)}\mathbb{A}^{\hat{I}}=\mathrm{d}\epsilon^{\hat{I}}+\hat{\eta}^{\hat{I}\hat{N}}T_{\hat{N}\hat{J}\hat{K}}\mathbb{A}^{\hat{J}}\epsilon^{\hat{K}}-\hat{\eta}^{\hat{I}\hat{J}}\rho^A{}_{\hat{J}}\;t_A, \tag{2.1.16}$$
$$\delta_{(\epsilon,t)}F_A=-\mathrm{d}t_A-\partial_A\rho^B{}_{\hat{J}}\,\mathbb{A}^{\hat{J}}\wedge t_B-\epsilon^{\hat{J}}\partial_A\rho^B{}_{\hat{J}}\,F_B+\tfrac{1}{2}\epsilon^{\hat{J}}\partial_AT_{\hat{I}\hat{L}\hat{J}}\,\mathbb{A}^{\hat{I}}\wedge\mathbb{A}^{\hat{L}}, \tag{2.1.17}$$

where $\epsilon^{\hat{I}}$ is a scalar gauge parameter, dependent on the worldvolume coordinates, and t_A is an additional one-form gauge parameter. These transformations define a first-stage reducible gauge symmetry, typical for gauge theories that include differential forms with degree larger than one [22, 56]. For completeness, and although this is simpler to do directly in the BV formalism, it is instructive to check the gauge invariance of the field equations of the model and the closure of the algebra of gauge transformations. Varying (2.1.14) with respect to $F_A,\mathbb{A}^{\hat{I}}$ and X^A respectively, we find the field equations:[3]

$$\mathcal{D}X^A\equiv\tfrac{\delta S}{\delta F_A}=\mathrm{d}X^A-\rho^A{}_{\hat{J}}\,\mathbb{A}^{\hat{J}}=0, \tag{2.1.18}$$
$$\mathcal{D}\mathbb{A}^{\hat{I}}\equiv\hat{\eta}^{\hat{I}\hat{J}}\tfrac{\delta S}{\delta A^{\hat{J}}}=\mathrm{d}\mathbb{A}^{\hat{I}}-\hat{\eta}^{\hat{I}\hat{K}}\rho^A{}_{\hat{K}}F_A+\tfrac{1}{2}\hat{\eta}^{\hat{I}\hat{K}}T_{\hat{K}\hat{J}\hat{L}}\mathbb{A}^{\hat{J}}\wedge\mathbb{A}^{\hat{L}}=0, \tag{2.1.19}$$

[3]Both the Courant (and later DFT) algebroid differential and equations of motion are denoted with a calligraphic $\mathcal{D}$, however, due to their completely different nature it should be obvious from context what is implied.

$$\mathcal{D}F_A \equiv \tfrac{\delta S}{\delta X^A} = \mathrm{d}F_A + \partial_A\rho^B{}_{\hat{K}}\,\mathbb{A}^{\hat{K}} \wedge F_B - \tfrac{1}{6}\partial_A T_{\hat{J}\hat{K}\hat{L}}\,\mathbb{A}^{\hat{J}} \wedge \mathbb{A}^{\hat{K}} \wedge \mathbb{A}^{\hat{L}} = 0. \tag{2.1.20}$$

Let us now examine how the field equation (2.1.18) transforms. We find:

$$\delta_{(\epsilon,t)}\mathcal{D}X^A = \epsilon^{\hat{J}}\partial_B\rho^A{}_{\hat{J}}\,\mathcal{D}X^B - \hat{\eta}^{\hat{J}\hat{K}}\rho^A{}_{\hat{J}}\rho^B{}_{\hat{K}}\,t_B + \epsilon^{\hat{J}}\mathbb{A}^{\hat{K}}(2\rho^B{}_{[\hat{K}}\partial_{\underline{B}}\rho^A{}_{\hat{J}]} - \rho^A{}_{\hat{N}}\hat{\eta}^{\hat{N}\hat{M}}T_{\hat{M}\hat{K}\hat{J}}),$$

where underlined indices are not antisymmetrised. This directly implies that:

$$\hat{\eta}^{\hat{J}\hat{K}}\rho^A{}_{\hat{J}}\rho^B{}_{\hat{K}} = 0, \tag{2.1.21}$$

$$2\rho^B{}_{[\hat{K}}\partial_{\underline{B}}\rho^A{}_{\hat{J}]} - \rho^A{}_{\hat{N}}\hat{\eta}^{\hat{N}\hat{M}}T_{\hat{M}\hat{K}\hat{J}} = 0, \tag{2.1.22}$$

whereupon the field equation transforms covariantly. Next we examine the transformation of the equation (2.1.19) and obtain:

$$\begin{aligned}\delta_{(\epsilon,t)}\mathcal{D}\mathbb{A}^{\hat{I}} = {} & -\hat{\eta}^{\hat{I}\hat{N}}(\partial_A T_{\hat{N}\hat{J}\hat{K}}\epsilon^{\hat{K}}\mathbb{A}^{\hat{J}} - \partial_A\rho^I{}_{\hat{N}}\,t_I) \wedge \mathcal{D}X^A + \hat{\eta}^{\hat{I}\hat{N}}T_{\hat{N}\hat{J}\hat{K}}\epsilon^{\hat{K}}\mathcal{D}\mathbb{A}^{\hat{J}} + \\ & + \tfrac{1}{2}\hat{\eta}^{\hat{I}\hat{K}}\Big(3\rho^A{}_{[\hat{N}}\partial_A T_{\hat{J}\hat{L}]\hat{K}} - \rho^A{}_{\hat{K}}\partial_A T_{\hat{N}\hat{J}\hat{L}} - 3T_{\hat{K}\hat{R}[\hat{N}}\hat{\eta}^{\hat{R}\hat{P}}T_{\hat{J}\hat{L}]\hat{P}}\Big)\epsilon^{\hat{N}}\mathbb{A}^{\hat{J}} \wedge \mathbb{A}^{\hat{L}},\end{aligned}$$

where we used the condition in (2.1.22). We observe that the field equation transforms covariantly provided one more condition holds, namely:

$$3\rho^A{}_{[\hat{N}}\partial_A T_{\hat{J}\hat{L}]\hat{K}} - \rho^A{}_{\hat{K}}\partial_A T_{\hat{N}\hat{J}\hat{L}} - 3T_{\hat{K}\hat{R}[\hat{N}}\hat{\eta}^{\hat{R}\hat{P}}T_{\hat{J}\hat{L}]\hat{P}} = 0. \tag{2.1.23}$$

It is then easily confirmed that transforming the field equation (2.1.20) does not produce any further conditions. Moreover, the three conditions (2.1.21), (2.1.22) and (2.1.23) are precisely the local coordinate expressions for the three independent axioms of a Courant algebroid.

Closure of the algebra of gauge transformations gives:

$$\begin{aligned}[\delta_{(\epsilon_1,t_1)},\delta_{(\epsilon_2,t_2)}]X^A &= \rho^A{}_{\hat{J}}\;\epsilon_{12}^{\hat{J}}, \\ \epsilon_{12}^{\hat{I}} &\equiv \hat{\eta}^{\hat{I}\hat{J}}T_{\hat{J}\hat{K}\hat{L}}\epsilon_1^{\hat{K}}\epsilon_2^{\hat{L}},\end{aligned}$$

where we used the condition in (2.1.22) to define ϵ_{12}. Furthermore we have:

$$\begin{aligned}[\delta_{(\epsilon_1,t_1)},\delta_{(\epsilon_2,t_2)}]\mathbb{A}^{\hat{I}} &= \delta_{(\epsilon_{12},t_{12})}\mathbb{A}^{\hat{I}} - \hat{\eta}^{\hat{I}\hat{J}}\partial_A T_{\hat{J}\hat{K}\hat{L}}\epsilon_1^{\hat{K}}\epsilon_2^{\hat{L}}\mathcal{D}X^A, \\ t_{12A} &\equiv \partial_A T_{\hat{K}\hat{L}\hat{J}}\,\epsilon_1^{\hat{K}}\epsilon_2^{\hat{L}}\mathbb{A}^{\hat{J}} + 2\partial_A\rho^B{}_{\hat{K}}\,\epsilon_{[1}^{\hat{K}}\;t_{2]B},\end{aligned}$$

where we used the conditions in (2.1.22) and (2.1.23). The closure on the field F_I does not introduce any further conditions. Therefore we conclude that the Courant sigma model is gauge invariant *on-shell*, provided that Eqs. (2.1.21, 2.1.22, 2.1.23) hold. (Sometimes this is referred to as a reducible gauge theory with an open gauge algebra.)

2.1.3 | The BV action and BRST transformations

On-shell closure of the algebra of gauge transformations implies that the natural description of the gauge symmetries for the Courant sigma model is the BV/BRST formalism (for physics-oriented reviews, see [22, 56]). In particular, one can construct the classical master action [18]:

$$\mathbf{S}_{\rm C}[\mathbf{X},\mathbf{A},\mathbf{F}] = \int_{T[1]\Sigma_3}\mu\left(\mathbf{F}_A\,\mathbf{d}\mathbf{X}^A + \tfrac{1}{2}\hat{\eta}_{\hat{I}\hat{J}}\,\mathbf{A}^{\hat{I}}\,\mathbf{d}\mathbf{A}^{\hat{J}} - \rho^A{}_{\hat{J}}(\mathbf{X})\mathbf{A}^{\hat{J}}\,\mathbf{F}_A + \tfrac{1}{6}T_{\hat{I}\hat{J}\hat{K}}(\mathbf{X})\mathbf{A}^{\hat{I}}\mathbf{A}^{\hat{J}}\mathbf{A}^{\hat{K}}\right), \tag{2.1.24}$$

where $\mu \equiv d^3\sigma d^3\theta$ is the Berezinian measure on the graded manifold $T[1]\Sigma_3$ spanned by coordinates (σ^μ,θ^μ) of degrees $(0,1)$ respectively, $\mathbf{d} = \theta^\mu\partial_\mu$ is the superworldvolume differential and superfields (in the sense of [57]) include the classical fields $(X,\mathbb{A},F)$, ghosts (ϵ,t,v) of ghost numbers $(1,1,2)$ and antifields:

$$\mathbf{X}^A = X^A + F^{\dagger A} + t^{\dagger A} + v^{\dagger A}, \tag{2.1.25}$$
$$\mathbf{A}^{\hat{I}} = \epsilon^{\hat{I}} + \mathbb{A}^{\hat{I}} + \hat{\eta}^{\hat{I}\hat{J}}\mathbb{A}^\dagger_{\hat{J}} + \hat{\eta}^{\hat{I}\hat{J}}\epsilon^\dagger_{\hat{J}}, \tag{2.1.26}$$
$$\mathbf{F}_A = v_A + t_A + F_A + X^\dagger_A. \tag{2.1.27}$$

Here $\mathbf{X}^A, \mathbf{A}^{\hat{I}}, \mathbf{F}_A$ are superfields with total degree $0,1,2$ respectively, where the total degree of a field ϕ is the sum of its ghost number $\mathrm{gh}(\phi)$ and its form degree $\deg(\phi)$. Antifields are denoted by a dagger $\dagger$ and we have $\mathrm{gh}(\phi)+\mathrm{gh}(\phi^\dagger) = -1$ and $\deg(\phi)+\deg(\phi^\dagger) = 3$.

The conditions given in Eqs. (2.1.21), (2.1.22) and (2.1.23) are obtained directly from the classical master equation $\{\mathbf{S}_{\rm C},\mathbf{S}_{\rm C}\} = 0$, where the bracket arises from the target manifold symplectic structure of type (2.1.11). Setting all ghosts and antifields to zero in the master action (2.1.24) reproduces the Courant sigma model (2.1.14), while the BRST transformations of the classical fields give the gauge transformations as in (2.1.15)–(2.1.17). For completeness and as a comparison for chapter 4 we present here the BRST transformations of all the fields,

$$Q_{\rm BV}X^A = \rho^A{}_{\hat{I}}\epsilon^{\hat{I}}, \tag{2.1.28}$$
$$Q_{\rm BV}\mathbb{A}^{\hat{I}} = \mathrm{d}\epsilon^{\hat{I}} - \hat{\eta}^{\hat{I}\hat{J}}\rho^A{}_{\hat{J}}t_A + \hat{\eta}^{\hat{I}\hat{J}}T_{\hat{J}\hat{K}\hat{L}}\mathbb{A}^{\hat{K}}\epsilon^{\hat{L}} - \hat{\eta}^{\hat{I}\hat{J}}\partial_B\rho^A{}_{\hat{J}}F^{\dagger B}v_A + \tfrac{1}{2}\hat{\eta}^{\hat{I}\hat{J}}\partial_B T_{\hat{J}\hat{K}\hat{L}}F^{\dagger B}\epsilon^{\hat{K}}\epsilon^{\hat{L}}, \tag{2.1.29}$$

$$\begin{aligned}Q_{\rm BV}F_A = {}& -\mathrm{d}t_A - \partial_A\rho^B_{\hat{I}}\epsilon^{\hat{I}}F_B - \partial_A\rho^B{}_{\hat{I}}\mathbb{A}^{\hat{I}}t_B + \tfrac{1}{2}\partial_A T_{\hat{I}\hat{J}\hat{K}}\epsilon^{\hat{I}}\mathbb{A}^{\hat{J}}\mathbb{A}^{\hat{K}} + \\ & + \tfrac{1}{2}\partial_A T_{\hat{I}\hat{J}\hat{K}}\hat{\eta}^{\hat{K}\hat{L}}\epsilon^{\hat{I}}\epsilon^{\hat{J}}\mathbb{A}^\dagger_{\hat{L}} - \partial_A\rho^B{}_{\hat{I}}\hat{\eta}^{\hat{I}\hat{J}}\mathbb{A}^\dagger_{\hat{J}}v_B + \tfrac{1}{2}\partial_A\partial_B\partial_C\rho^L{}_{\hat{I}}F^{\dagger B}F^{\dagger C}\epsilon^{\hat{I}}v_L - \\ & - \partial_A\partial_B\rho^C{}_{\hat{I}}F^{\dagger B}\epsilon^{\hat{I}}t_C + \partial_A\partial_B\rho^C{}_{\hat{I}}F^{\dagger B}\mathbb{A}^{\hat{I}}v_C - \tfrac{1}{12}\partial_A\partial_B\partial_C T_{\hat{I}\hat{J}\hat{K}}F^{\dagger B}F^{\dagger C}\epsilon^{\hat{I}}\epsilon^{\hat{J}}\epsilon^{\hat{K}} + \\ & + \tfrac{1}{6}\partial_A\partial_B T_{\hat{I}\hat{J}\hat{K}}t^{\dagger B}\epsilon^{\hat{I}}\epsilon^{\hat{J}}\epsilon^{\hat{K}} - \tfrac{1}{2}\partial_A\partial_B T_{\hat{I}\hat{J}\hat{K}}F^{\dagger B}\mathbb{A}^{\hat{I}}\epsilon^{\hat{J}}\epsilon^{\hat{K}} - \partial_A\partial_B\rho^C{}_{\hat{I}}t^{\dagger B}\epsilon^{\hat{I}}v_C,\end{aligned} \tag{2.1.30}$$

$$Q_{\rm BV}\epsilon^{\hat{I}} = \hat{\eta}^{\hat{I}\hat{J}}\rho^A{}_{\hat{J}}\,v_A - \tfrac{1}{2}\hat{\eta}^{\hat{I}\hat{J}}T_{\hat{J}\hat{K}\hat{L}}\epsilon^{\hat{K}}\epsilon^{\hat{L}}, \tag{2.1.31}$$

$$Q_{\mathrm{BV}}t_A = \mathrm{d}v_A - \partial_A\rho^B{}_{\hat{I}}\epsilon^{\hat{I}}t_B + \partial_A\rho^B{}_{\hat{I}}\,\mathbb{A}^{\hat{I}}v_B - \tfrac{1}{2}\partial_A T_{\hat{I}\hat{J}\hat{K}}\epsilon^{\hat{I}}\epsilon^{\hat{J}}\,\mathbb{A}^{\hat{K}} + \partial_A\partial_B\rho^C{}_{\hat{I}}\,F^{\dagger B}\epsilon^{\hat{I}}v_C - \\ - \tfrac{1}{6}\partial_A\partial_B T_{\hat{I}\hat{J}\hat{K}}\,F^{\dagger B}\epsilon^{\hat{I}}\epsilon^{\hat{J}}\epsilon^{\hat{K}}, \tag{2.1.32}$$

$$Q_{\mathrm{BV}}v_A = -\partial_A\rho^B{}_{\hat{I}}\epsilon^{\hat{I}}v_B + \tfrac{1}{6}\partial_A T_{\hat{I}\hat{J}\hat{K}}\epsilon^{\hat{I}}\epsilon^{\hat{J}}\epsilon^{\hat{K}}. \tag{2.1.33}$$

Note that one needs to introduce a ghost for ghost v because we are dealing with a first-stage reducible gauge theory, or said differently, there are "gauge invariances" for gauge transformations as expected in gauge theories that include higher differential forms.

2.2 | Gauge symmetries of the DFT membrane sigma model

In reference [45], the starting point of this whole chapter, a DFT membrane sigma model was proposed beginning from a Courant sigma model defined over a doubled target spacetime and adopting a suitable projection. As was seen in section 2.1.1, for Courant algebroids the bundle over a base manifold is extended ("doubled"), while in DFT one doubles the coordinates, i.e. the base space. In order to be able to relate the two we had to start from a large Courant algebroid defined over a manifold spanned locally by the set of doubled coordinates $\{X^a, \tilde{X}_a\}$. This naturally introduces an $O(2d, 2d)$ structure indicating that a suitable projection to a subbundle with $O(d, d)$ structure is due. This projection, demonstrated in section 2.2.1, was identified and all Courant algebroid structures were projected accordingly to DFT structures; for instance, the characteristic C-bracket of DFT (1.2.1) is obtained in this way from the Courant bracket of the large Courant algebroid. The properties of this bracket were analysed and used to define a DFT algebroid that will be of great importance later. Moreover, the flux formulation of DFT was used to identify the components of the anchor map in section 2.2.2 and with these data a DFT membrane sigma model was defined in 2.2.3. This worldvolume theory is gauge invariant only under a certain condition which corresponds to the strong constraint of the target space DFT.

It was shown in Ref. [58] the classical master action of the large Courant sigma model can be projected to the corresponding DFT action for projected superfields. This action does not satisfy the BV master equation (1.1.1) and cannot be constructed using AKSZ theory. This is an expected result, since already at the classical level the DFT membrane sigma model is gauge invariant only up to the worldvolume analogue of the strong constraint, and therefore one cannot expect BRST invariance of the full action. Here we complete this analysis by explicitly constructing the BRST transformations for all projected superfield components in sec. 2.2.4 of the full DFT membrane sigma model.

2.2.1 | DFT algebroid

The question that naturally arises is if one can provide a geometric description of DFT symmetries based on the C-bracket, before reducing the theory by imposing the strong constraint. Our starting point for this section is the collection of properties corresponding to a standard Courant algebroid from sec. 2.1.1 and their modification in the case of double field theory.

Projection procedure. To make this connection of DFT to a Courant algebroid, one must first double the target space of the canonical Courant algebroid. Therefore, we begin with a doubled target space manifold $\mathcal{M}$ with local coordinates[4] $X = (X^A) = (X^a, \tilde{X}_a)$ and define a large Courant algebroid $(E, [\,\cdot\,,\,\cdot\,]_C, \langle\,\cdot\,,\,\cdot\,\rangle, \rho)$ over a vector bundle over this doubled target space $\mathcal{M}$:

$$E = \mathbb{T}\mathcal{M} \equiv T\mathcal{M} \oplus T^*\mathcal{M}.$$

Now, we introduce a splitting on the large section:

$$\mathbb{A}^I_{\pm} = \tfrac{1}{2}\Big(\mathbb{A}^I \pm \hat{\eta}^{IJ}\,\widetilde{\mathbb{A}}_J\Big), \tag{2.2.1}$$

and anchor:

$$(\rho_{\pm})^A{}_J = \rho^A{}_J \pm \hat{\eta}_{JK}\,\tilde{\rho}^{AK}. \tag{2.2.2}$$

The $O(d,d)$ metric $\hat{\eta}$ on E is used, relating their standard and dual parts. The generalised tangent bundle is thus split into two subbundles:

$$E = L_+ \oplus L_- \ ,$$

where $L_{\pm}$ is the bundle whose space of sections is spanned locally by $e^{\pm}_I$. A general section of E can thus be written in terms of L_+ and L_- parts:

$$\mathbb{A} = \mathbb{A}^I_+\, e^+_I + \mathbb{A}^I_-\, e^-_I\,. \tag{2.2.3}$$

In order to obtain a DFT structure one first observes the indices of the fields are $2d$-dimensional. This is resolved by a projection to the subbundle L_+ of E through the bundle map:[5]

$$\mathsf{p}_+ : E \longrightarrow L_+\ , \quad (\mathbb{A}_V, \mathbb{A}_F) \longmapsto \mathbb{A}_+ \equiv A. \tag{2.2.4}$$

Under the projection, the components $\mathbb{A}^I_-$ in the generalised vector (2.2.3) vanish, and we rename $\mathbb{A}^I_+ = A^I$. This is a new generalised vector that is in fact a DFT vector [59].

[4]Throughout $A, B, \ldots = 1, \ldots, 2d$ and $a, b, \ldots = 1, \ldots, d$ are indices that correspond to double and standard spacetime respectively, while Latin indices from the middle of the alphabet $I, J, \ldots = 1, \ldots, 2d$ and $i, j, \ldots = 1, \ldots, d$ are reserved for bundle indices.

[5]We shall denote this subbundle L in sections where the projection procedure is not under consideration.

Additionally we shall need the inclusion map:

$$i : L_+ \longrightarrow E$$

and a composition map of the inclusion and projection maps:

$$a = i \circ \mathsf{p}_+ : E \longrightarrow E.$$

The projection acting twice on the standard Courant bracket,

$$\mathsf{p}_+\Big([a(\mathbb{A}), a(\mathbb{B})]_C\Big) = [\![A, B]\!]_{L_+},$$

produces precisely the C-bracket of DFT vectors (see e.g [60]). The same on the generalised Lie derivative,

$$\mathsf{p}_+\Big(\mathbb{L}_{a(\mathbb{A})} a(\mathbb{B})\Big) = \mathsf{L}_A B,$$

yields the generalised Lie derivative in DFT. The closure of the gauge transformations (1.2.1) is controlled by the strong constraint,

$$\eta^{AB}\,\partial_A(\,\cdots)\,\partial_B(\,\cdots) = 0, \tag{2.2.5}$$

for all fields of DFT and we shall *not* assume it unless explicitly stated. In the case of a Courant algebroid, the question of the strong constraint becomes moot, as the Dorfman derivative automatically satisfies condition (1.2.1). Finally the bilinear pairing is defined simply from the pairing on the large Courant algebroid by inclusion:

$$\langle \mathsf{p}_+(\mathbb{A}), \mathsf{p}_+(\mathbb{B})\rangle_{L_+} \equiv \langle a(\mathbb{A}), a(\mathbb{B})\rangle_E. \tag{2.2.6}$$

The explicit relation between DFT algebroid structures and the flux formulation of double field theory using a local basis is reviewed in the next section.

Definition and properties. Now we focus our attention to the global properties of a DFT algebroid and, therefore, with respect to the construction above, we are ready to formally define a DFT algebroid.

Definition 2.2.7. *Let $\mathcal{M}$ be a $2d$-dimensional manifold. A double field theory algebroid is a quadruple $(L, [\![\,\cdot\,,\,\cdot\,]\!], \langle\,\cdot\,,\,\cdot\,\rangle, \rho)$, where L is a vector bundle of rank $2d$ over $\mathcal{M}$ equipped with a skew-symmetric bracket $[\![\,\cdot\,,\,\cdot\,]\!] : \Gamma(L)\otimes\Gamma(L) \to \Gamma(L)$, a non-degenerate symmetric form $\langle\,\cdot\,,\,\cdot\,\rangle : \Gamma(L)\otimes\Gamma(L) \to C^\infty(\mathcal{M})$, and a smooth bundle map $\rho : L \to T\mathcal{M}$, such that:*

1. $\langle \mathcal{D}f, \mathcal{D}g\rangle = \frac{1}{4}\,\langle \mathrm{d}f, \mathrm{d}g\rangle$;

2. $[\![e_1, f\, e_2]\!] = f\,[\![e_1, e_2]\!] + \big(\rho(e_1)f\big)\, e_2 - \langle e_1, e_2\rangle\, \mathcal{D}f$;

3. $\langle [\![e_3, e_1]\!] + \mathcal{D}\langle e_3, e_1\rangle, e_2\rangle + \langle e_1, [\![e_3, e_2]\!] + \mathcal{D}\langle e_3, e_2\rangle\rangle = \rho(e_3)\langle e_1, e_2\rangle$;

for all $e_i \in \Gamma(L)$ *and* $f, g \in C^\infty(\mathcal{M})$*, where* $\mathcal{D} : C^\infty(\mathcal{M}) \to \Gamma(L)$ *is the derivative defined through* $\langle \mathcal{D}f, e\rangle = \frac{1}{2}\,\rho(e)f$*.*

From axiom 1 it follows that a pairing on the bundle L induces a symmetric pairing on $T\mathcal{M}$ that is actually a para-Hermitian metric on the doubled configuration space (see Refs. [61–65] for the description of double field theory in terms of para-Hermitian manifolds, and more generally in terms of Born geometry):

$$\begin{aligned} &\eta : T\mathcal{M} \times T\mathcal{M} \to C^\infty(\mathcal{M}) \\ &\eta = \tfrac{1}{2}\eta_{AB}\mathrm{d}X^A \vee \mathrm{d}X^B, \end{aligned} \tag{2.2.8}$$

or, in other words, an $O(d,d)$ metric with components:

$$\eta_{AB} = \begin{pmatrix} 0 & \delta^a_b \\ \delta^b_a & 0 \end{pmatrix}. \tag{2.2.9}$$

Here we introduced the symmetric tensor product $u \vee v = u \otimes v + v \otimes u$, in analogy with the more standard wedge product. Since η_{AB} is invertible it also defines a symmetric 2-vector:

$$\begin{aligned} &\eta^{-1} : T^*\mathcal{M} \times T^*\mathcal{M} \to C^\infty(\mathcal{M}) \\ &\eta^{-1} = \tfrac{1}{2}\eta^{AB}\partial_A \vee \partial_B. \end{aligned} \tag{2.2.10}$$

The action on functions is defined via the natural contraction with 1-forms:

$$\eta^{-1}(df) = \iota_{\eta^{-1}}df = \eta^{AB}\partial_A f \partial_B.$$

Additionally, we define the action of the symmetric 2-vector η^{-1} on a section $v = v^A\partial_A$ of $\Gamma(T\mathcal{M})$ using the Schouten-Nijenhuis bracket for symmetric vectors [66] as follows:

$$\eta^{-1}(v) \equiv [\eta^{-1}, v]_{SN} = \eta^{AB}\partial_A v^C \partial_C \vee \partial_B.$$

We would now like to further explore the structural data of the DFT algebroid. First, one notices that $\mathrm{Im}(\mathcal{D})$ is not in the kernel of map ρ:

$$(\rho \circ \mathcal{D})f = \tfrac{1}{2}\eta^{-1}(df). \tag{2.2.11}$$

Next, one can show that map ρ is not a homomorphism:

$$\rho([\![e_1, e_2]\!])(f) - [\rho(e_1), \rho(e_2)](f) = -\,\mathsf{SC}_\rho(e_1, e_2)f,$$

$$\mathsf{SC}_\rho(e_1,e_2)f \equiv \tfrac{1}{2}\eta(\rho(e_1),\eta^{-1}(df)(\rho(e_2))) - (e_1 \leftrightarrow e_2), \tag{2.2.12}$$

where the second bracket on the lhs in the first line is the standard Lie bracket of vector fields, and we introduced a shorthand notation for the rhs. Furthermore, the bracket does not satisfy the Jacobi identity, moreso than the Courant algebroid as even if we were to include the $\mathcal{D}$ exact term into a *modified* Jacobi identity this would still be broken. In fact we have:

$$\mathrm{Jac}(e_1,e_2,e_3) \equiv [\![[\![e_1,e_2]\!],e_3]\!] + \text{cyclic} = \mathcal{D}\mathcal{N}(e_1,e_2,e_3) + \mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,e_3), \tag{2.2.13}$$

where $\mathcal{N}$ is the Nijenhuis operator defined by:

$$\mathcal{N}(e_1,e_2,e_3) \equiv \tfrac{1}{3}\left\langle [\![e_1,e_2]\!],e_3\right\rangle + \text{cyclic}, \tag{2.2.14}$$

and we introduced the shorthand notation:[6]

$$\begin{aligned}&\mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,e_3) \equiv \\ &\tfrac{1}{2}\rho^{-1}\left\{(-\eta^{-1}(\rho(e_2))(\eta(\rho(e_1)) + \eta^{-1}(\rho(e_1))(\eta(\rho(e_2)) + [\rho(e_1),\rho(e_2)])\rho(e_3)\right\} + \text{cyclic.}\end{aligned}$$

Here we defined the inverse of the anchor map $\rho^{-1} : T\mathcal{M} \to L$ as shown in the following commutative diagram:

$$\begin{tikzcd} L \arrow[r, "\hat{\eta}"] & L^* \\ T\mathcal{M} \arrow[u, "\rho^{-1}"] \arrow[r, "\eta"] & T^*\mathcal{M} \arrow[u, "\rho^*"] \end{tikzcd} \quad : \quad \hat{\eta}\circ\rho^{-1} = \rho^*\circ\eta,$$

where the map $\hat{\eta} : L \to L^*$ is induced by the DFT algebroid symmetric form.[7] The existence of the inverse map ρ^{-1} is due to axiom 1 of definition 2.2.7 and the relation of the anchor to the generalised bein in DFT, as reviewed in section 2.2.2.

Finally, in analogy to three further properties of a Courant algebroid (Property 2.1.7), the following properties of a DFT algebroid will prove useful later in chapter 5.

Property 2.2.15. *The following identities hold in a DFT algebroid:*

1. $2\langle e_1, \mathcal{D}\langle e_2, \mathcal{D}f\rangle\rangle - 2\langle e_1, [\![e_2, \mathcal{D}f]\!]\rangle = \rho(\mathcal{D}f)\langle e_1, e_2\rangle + \mathsf{SC}_\rho(e_1,e_2)f,$

2. $\mathcal{N}(e_1,e_2,\mathcal{D}f) - \tfrac{1}{4}\rho[\![e_1,e_2]\!]f = -\tfrac{1}{4}\mathsf{SC}_\rho(e_1,e_2)f,$

3. $\mathcal{N}([\![e_1,e_2]\!],e_3,e_4) + \langle\mathcal{D}\mathcal{N}(e_1,e_2,e_3),e_4\rangle + \mathit{antisymm.}(1,2,3,4) =$
$= \tfrac{1}{2}\langle\mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,e_3),e_4\rangle + \mathit{antisymm.}(1,2,3,4).$

[6] Compared with definitions in [45], here we included $\mathcal{Z}_{IJKL}$ in the definition of $\mathsf{SC}_{\mathrm{Jac}}$, see sec. 2.2.2 for more details.

[7] In a slight abuse of notation, we denote the bilinear form and the map it induces with the same letter, both for $\hat{\eta}$ and η.

Proof. To prove property 1 we apply axiom 3 of Def. 2.2.7 to obtain $\rho(\mathcal{D}f)\langle e_1, e_2\rangle + \rho(e_1)\langle e_2, \mathcal{D}f\rangle - \rho(e_2)\langle e_1, \mathcal{D}f\rangle = 2\langle e_2, \langle e_1, \mathcal{D}f\rangle\rangle + 2\langle e_1 [\![\mathcal{D}f, e_2]\!]\rangle + 2\langle \mathcal{D}f, [\![e_1, e_2]\!]\rangle$. Then by the definition of $\mathcal{D}$ the last two terms on the lhs become the commutator of vector fields $[\rho(e_1), \rho(e_2)]$ on f. Finally, using (2.2.12) twice we obtain the desired relation. For property 2 first plug in the definition of the differential and Nijenhuis (2.2.14) to obtain $-\frac{1}{6}\langle [\![e_1, e_2]\!]\rangle + \frac{1}{3}\langle [\![\mathcal{D}f, e_1]\!], e_2\rangle - \frac{1}{3}\langle [\![\mathcal{D}f, e_2]\!], e_1\rangle$, then use property 1 on the last two terms and definition of $\mathcal{D}$ again to merge them into the commutator of vector fields as in the proof of the previous property, and finally use the broken homomorphism identity (2.2.12) to yield the rhs of property 2. Lastly, the proof for property 3 is to be done in two parts, first by (2.2.14) the rhs becomes $\frac{1}{3}\langle \mathrm{Jac}(e_1, e_2, e_3), e_4\rangle + \frac{2}{3}\langle [\![e_1, e_2]\!], [\![e_3, e_4]\!]\rangle + \langle \mathcal{D}\mathcal{N}(e_1, e_2, e_3), e_4\rangle$+antisymm.$(1, 2, 3, 4)$. Then, by the Jacobiator identity (2.2.13) this becomes $\frac{4}{3}\langle \mathrm{Jac}(e_1, e_2, e_3), e_4\rangle + \frac{2}{3}\langle [\![e_1, e_2]\!], [\![e_3, e_4]\!]\rangle - \langle \mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, e_3), e_4\rangle$+antisymm.$(1, 2, 3, 4)$. The second part is to express the first two terms using $\mathsf{SC}_{\mathrm{Jac}}$, to show this we start from $\langle \mathrm{Jac}(e_1, e_2, e_3), e_4\rangle$, by (2.2.13) and the definitions of $\mathcal{D}$ and $\mathcal{N}$ this becomes $\langle \mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, e_3), e_4\rangle + \frac{1}{6}\rho(e_4)(\langle [\![e_1, e_2]\!], e_3\rangle + \text{cyclic})$. Plugging in axiom 3 into the second term, summing over antisymm.$(1, 2, 3, 4)$ and applying (2.2.13) to eliminate terms with $\mathcal{D}\mathcal{N}$ again, yields:

$$2\langle \mathrm{Jac}(e_1, e_2, e_3), e_4\rangle + \langle [\![e_1, e_2]\!], [\![e_3, e_4]\!]\rangle - \frac{9}{4}\langle \mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, e_3), e_4\rangle + \text{antisymm.}(1, 2, 3, 4) = 0.$$

Multiplying this relation by 2/3 and substituting back completes the proof.

Comparing these identities with Property 2.1.7 one will notice they coincide if $\rho \circ \mathcal{D} = 0$, $\mathsf{SC}_\rho(e_1, e_2) = 0$ and $\mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, e_3) = 0$, all three of which hold in a Courant algebroid. Property 3 is worded slightly differently but as can be seen from the proof the source of this property is the same. Therefore, the rhs of all three properties contain the strong constraint violating terms.

2.2.2 | Relation of the DFT algebroid with the flux formulation of DFT

Here we review the correspondence between the structural data of a DFT algebroid and double field theory [45, 50], using a local basis. Starting from definition 2.2.7 of a DFT algebroid, we relate the $2d$-dimensional base manifold $\mathcal{M}$ spanned by $\{X^A\}$, $A = 1, \dots, 2d$ with the doubled configuration space of double field theory spanned by $\{x^a, \tilde{x}_a\}$, $a = 1, \dots, d$. Axiom 1 of definition 2.2.7 implies:

$$\hat{\eta}^{IJ} \rho^A{}_I \rho^B{}_J = \eta^{AB}. \tag{2.2.16}$$

Using the Leibniz rule, i.e., axiom 2, one can show that the bracket on a general section $E^I(X)e_I \in \Gamma(L)$ is the C-bracket of double field theory:

$$[\![E_1, E_2]\!]^J = \rho^A{}_I \left(E_1^I\, \partial_A E_2^J - \tfrac{1}{2}\, \hat{\eta}^{IJ}\, E_1^K\, \partial_A E_{2K} - E_1 \leftrightarrow E_2\right) + \hat{\eta}^{JM} \hat{T}_{MIK}\, E_1^I\, E_2^K, \quad (2.2.17)$$

while axiom 3 evaluated in a local basis imposes the antisymmetry of $\hat{T}$ in all three indices.

Relation (2.2.16) enables us to identify the components of the anchor map ρ with the generalised bein $\mathcal{E}^A{}_I$ of the flux formulation of DFT (1.2.2):

$$\hat{\eta}^{IJ} \mathcal{E}^A{}_I \mathcal{E}^B{}_J = \eta^{AB}.$$

Moreover, properties of the bracket (2.2.12) and (2.2.13) written in a local basis produce:

$$2\rho^B{}_{[I} \partial_{\underline{B}} \rho^A{}_{J]} - \rho^A{}_M \hat{\eta}^{MN} \hat{T}_{NIJ} = \eta_{BC} \rho^C{}_{[I} \partial^A \rho^B{}_{J]}, \quad (2.2.18)$$

$$3\hat{\eta}^{MN} \hat{T}_{M[JK} \hat{T}_{IL]N} + 4\rho^A{}_{[L} \partial_{\underline{A}} \hat{T}_{JKI]} = \mathcal{Z}_{JKIL}, \quad (2.2.19)$$

where:

$$\mathcal{Z}_{IJKL} = 3\eta_{AD}\eta_{BE}\eta^{CF} \rho^D{}_{[I} \partial_{\underline{F}} \rho^A{}_J\, \rho^E{}_K \partial_{\underline{C}} \rho^B{}_{L]}. \quad (2.2.20)$$

By direct comparison with the expression for fluxes and their Bianchi identities in double field theory, given as (see (1.2.3)):

$$\mathcal{F}_{IJK} = 3\mathcal{E}_{[I}{}^A \partial_{\underline{A}} \mathcal{E}_J{}^B \mathcal{E}_{K]B},$$
$$3\hat{\eta}^{MN} \mathcal{F}_{M[JK} \mathcal{F}_{IL]N} + 4\mathcal{E}_{[L}{}^M \partial_{\underline{M}} \mathcal{F}_{JKI]} = 4\hat{\mathcal{Z}}_{JKIL},$$

we observe that the twist of the bracket $\hat{T}$ can be identified with the 3-form flux $\mathcal{F}$ of double field theory and $\mathcal{Z}_{JKIL} = 4\hat{\mathcal{Z}}_{JKIL}$. The origin of the totally antisymmetric tensor $\mathcal{Z}_{IJKL}$ has been explained in [45], where it has been shown that at the level of the corresponding 3d DFT sigma model one can realise this term as a Wess-Zumino term on an extension of the membrane worldvolume to four dimensions, as in [67]. However, this distinction is not crucial in the present context, and in the remainder $\mathcal{Z}_{IJKL}$ is packaged into SC_{Jac} together with the rest of the strong-constraint breaking terms appearing in the expression for the Jacobiator of the C-bracket (2.2.13).

2.2.3 | DFT membrane sigma model

As shown in Ref. [45] and above one can define a DFT algebroid structure and a corresponding membrane sigma model starting from a large Courant algebroid over a $2d$ dimensional space $\mathcal{M}$ by applying a suitable projection. In particular, we considered sections $\mathbb{A}$ of the large Courant algebroid $\mathbb{E}$, decomposed in a suitable basis (see (2.2.1) and (2.2.2)), and projected (2.2.4) to the subbundle L_+ spanned by the local sections (e_I^+). Projection of the symmetric bilinear form (2.2.6) of $\mathbb{E}$, leads to the $O(d,d)$ invariant DFT metric:[8]

$$\langle \mathbb{A}, \mathbb{B} \rangle_{\mathbb{E}} = \tfrac{1}{2}\hat{\eta}_{\hat{I}\hat{J}}\mathbb{A}^{\hat{I}}\mathbb{B}^{\hat{J}} = \hat{\eta}_{IJ}(\mathbb{A}_+^I\mathbb{B}_+^J - \mathbb{A}_-^I\mathbb{B}_-^J) \quad \mapsto \quad \hat{\eta}_{IJ}A^IB^J = \langle A, B\rangle_{L_+}.$$

This works for general Courant algebroids over $\mathcal{M}$ with anchor $\rho^A{}_{\hat{J}} = (\rho^A{}_J, \tilde{\rho}^{AJ})$, yielding the C-bracket (2.2.17):

$$[\![A,B]\!]^J = (\rho_+)^A{}_I\left(A^I\partial_L A B^J - \tfrac{1}{2}\hat{\eta}^{IJ}A^K\partial_A B_K - (A \leftrightarrow B)\right) + \hat{T}_{IK}{}^J A^I B^K,$$

in terms of the map $\rho_+ : L_+ \to T\mathcal{M}$ with components $(\rho_\pm)^A{}_J = \rho^A{}_J \pm \hat{\eta}_{JK}\tilde{\rho}^{AK}$, and $\hat{T}$ chosen as:

$$\hat{T}_{IJK} \equiv \tfrac{1}{2}T_{IJK} = \tfrac{1}{2}\left(A_{IJK} + 3B_{[IJ}{}^L\hat{\eta}_{K]L} + 3C_{[I}{}^{LM}\hat{\eta}_{J\underline{L}}\hat{\eta}_{K]M} + D^{LMN}\hat{\eta}_{I[\underline{L}}\hat{\eta}_{J[\underline{M}}\hat{\eta}_{K]N}\right), \tag{2.2.21}$$

where A, B, C and D are the components of $T_{\hat{I}\hat{J}\hat{K}}$:

$$T_{\hat{I}\hat{J}\hat{K}} \equiv \begin{pmatrix} A_{IJK} & B_{IJ}{}^K \\ C_I{}^{JK} & D^{IJK} \end{pmatrix} . \tag{2.2.22}$$

Using these projected data the following DFT membrane sigma model was proposed:

$$S_{\mathrm{DFT}}[X,A,F] = \int_{\Sigma_3}\Big(F_A \wedge \mathrm{d}X^A + \hat{\eta}_{IJ}A^I \wedge \mathrm{d}A^J - (\rho_+)^A{}_J A^J \wedge F_A + \tfrac{1}{3}\hat{T}_{IJK}A^I \wedge A^J \wedge A^K\Big) . \tag{2.2.23}$$

Next, in parallel to the flux formulation of DFT [27,33,34,68–70] a parametrisation of the ρ_+ components is taken such that they coincide with the generalised vielbein as stated in 2.2.2. Relation (1.2.2) then implies they satisfy:

$$\hat{\eta}^{JK}(\rho_+)^A{}_J(\rho_+)^B{}_K = \eta^{AB}, \tag{2.2.24}$$

which is to be compared with the condition in (2.1.21). Moreover, the following set of infinitesimal gauge transformations was proposed:

$$\begin{aligned} \delta_\epsilon X^A &= \rho^A{}_J\epsilon^J, \\ \delta_\epsilon A^I &= \mathrm{d}\epsilon^I + \hat{\eta}^{IN}\hat{T}_{NJK}A^J\epsilon^K, \\ \delta_\epsilon F_A &= -\epsilon^J\partial_A\rho^B{}_J F_B + \epsilon^J A^K \wedge A^L\partial_A\hat{T}_{KLJ}. \end{aligned}$$

[8] Denoting $\mathbb{A}_+ = A$ and $\mathbb{B}_+ = B$.

It was shown the action (2.2.23) is invariant under these transformations provided:

$$2\rho^{B}{}_{[L}\partial_{\underline{B}}\rho^{A}{}_{M]} - \rho_{B[L}\partial^{A}\rho^{B}{}_{M]} = \rho^{A}{}_{J}\hat{\eta}^{JK}\hat{T}_{KLM}, \tag{2.2.25}$$

$$3\rho^{A}{}_{[K}\partial_{\underline{A}}\hat{T}_{MM']N} - \rho^{A}{}_{N}\partial_{A}\hat{T}_{KMM'} - 3\hat{\eta}^{PJ}\hat{T}_{P[MM'}\hat{T}_{K]NJ} = 0. \tag{2.2.26}$$

However, these conditions are not sufficient; one needs to additionally impose the following constraint:

$$\rho_{AL}\,\partial^{B}\rho^{A}{}_{M}\,\epsilon^{M}\,F_{B} = 0. \tag{2.2.27}$$

This is the way that the strong constraint of the target space DFT appears in the world-volume theory. To see this we expand $\partial^A(\cdots)F_A = \partial_a(\cdots)\tilde{F}^a + \tilde{\partial}^a(\cdots)F_a$ and notice that if nothing depends on $\tilde{X}^a$ the second term vanishes automatically and the first term due to $\tilde{F}^a$ being conjugate to $\tilde{X}^a$. The opposite holds for the same reason. However since the expression is $O(d,d)$ invariant the statement holds for any solution to the strong constraint.

2.2.4 | Projecting superfields

The classical action (2.2.23) is lifted to the full action in terms of superfields [58]:[9]

$$\begin{aligned}\mathbf{S}_{\mathrm{DFT}}[\mathbf{X},\mathbf{A}_+,\mathbf{F}] = \int_{T[1]\Sigma_3}\mu\Big(&\mathbf{F}_A\,\mathrm{d}\mathbf{X}^A + \hat{\eta}_{IJ}\mathbf{A}_+^I\mathrm{d}\mathbf{A}_+^J - (\rho_+)^A{}_I(\mathbf{X})\mathbf{A}_+^I\mathbf{F}_A + \\ &+ \tfrac{1}{3}\hat{T}_{IJK}(\mathbf{X})\mathbf{A}_+^I\mathbf{A}_+^J\mathbf{A}_+^K\Big),\end{aligned} \tag{2.2.28}$$

where in comparison with (2.1.24) we used the structures $(\rho_+,\hat{T},\eta)$ of a DFT algebroid and projected the superfield $\mathbf{A}\to\mathbf{A}_+$:

$$\mathbf{A}_+^I = \epsilon_+^I + \mathbb{A}_+^I + \hat{\eta}^{IJ}\mathbb{A}^{\dagger}_{+J} + \hat{\eta}^{IJ}\epsilon^{\dagger}_{+J},$$

by setting the $\mathbb{A}_-$ and ϵ_- to zero, an operation whose consistency will be addressed below. Next, we project the BRST transformations of the superfields of the large Courant sigma model (2.1.24). By splitting and projecting the BRST transformation of the field $\mathbb{A}^{\hat{I}}$, one obtains:

$$\begin{aligned}Q_{\mathrm{BV}}\mathbb{A}_+^I = {}&\mathrm{d}\epsilon_+^I - \tfrac{1}{2}\hat{\eta}^{IJ}\rho^A_{+J}\,t_A + \hat{\eta}^{IL}\hat{T}_{LJK}\mathbb{A}_+^J\epsilon_+^K - \tfrac{1}{2}\hat{\eta}^{IJ}\partial_A\rho^B_{+J}F^{\dagger A}v_B + \\ &+ \tfrac{1}{2}\hat{\eta}^{IJ}\partial_A\hat{T}_{JLM}F^{\dagger A}\epsilon_+^L\epsilon_+^M,\end{aligned} \tag{2.2.29}$$

$$Q_{\mathrm{BV}}\mathbb{A}_-^I = \tfrac{1}{2}\hat{\eta}^{IJ}\rho^A_{-J}\,t_A + \tfrac{1}{2}\hat{\eta}^{IL}\theta_{JKL}\,\mathbb{A}_+^J\epsilon_+^K + \tfrac{1}{2}\hat{\eta}^{IJ}\partial_A\rho^B_{-J}F^{\dagger A}v_B + \tfrac{1}{4}\hat{\eta}^{IJ}\partial_A\theta_{LMJ}F^{\dagger A}\epsilon_+^L\epsilon_+^M, \tag{2.2.30}$$

[9] Writing $\pm$ subscripts explicitly again.

and for the ghost field $\epsilon^{\hat{I}}$:

$$Q_{\mathrm{BV}}\epsilon_+^I = \tfrac{1}{2}\hat{\eta}^{IJ}\rho^A_{+J}v_A - \tfrac{1}{2}\hat{\eta}^{IL}\hat{T}_{LJK}\epsilon_+^J\epsilon_+^K, \tag{2.2.31}$$
$$Q_{\mathrm{BV}}\epsilon_-^I = -\tfrac{1}{2}\hat{\eta}^{IJ}\rho^A_{-J}v_A - \tfrac{1}{4}\hat{\eta}^{IL}\theta_{JKL}\epsilon_+^J\epsilon_+^K, \tag{2.2.32}$$

up to terms containing $\mathbb{A}_-$ and ϵ_- on the right-hand sides of the above equations; such terms will eventually drop out by setting the corresponding fields to zero, but this has to be done in a consistent way. The quantity θ_{IJK} is defined as:

$$\begin{aligned}\theta_{IJK} = &-A_{IJK} + 3\hat{\eta}_{L[K}B_{IJ]}{}^L - 4\hat{\eta}_{L[I}B_{J]K}{}^L - 3\hat{\eta}_{L[I}\hat{\eta}_{\underline{M}J}C_{K]}{}^{LM} - 4\hat{\eta}_{KL}\hat{\eta}_{M[I}C_{J]}{}^{ML} + \\ &+ \hat{\eta}_{KL}\hat{\eta}_{IM}\hat{\eta}_{JN}D^{MNL},\end{aligned}$$

with A, B, C and D being the components of $T_{\hat{I}\hat{J}\hat{K}}$ in (2.2.22).

The requirement that the projection onto L_+ be well-defined with respect to the BRST symmetry means that the transformations of $\mathbb{A}_-$ (2.2.30) and ϵ_- (2.2.32) must vanish. Therefore, setting $Q_{\mathrm{BV}}\mathbb{A}_- = Q_{\mathrm{BV}}\epsilon_- = 0$ leads to the fixing of the ghost fields t_A and v_A:

$$v_A = -\tfrac{1}{2}\eta_{AB}\eta^{NM}\rho^B_{-M}\theta_{JKN}\epsilon_+^J\epsilon_+^K \equiv \tfrac{1}{2}\Theta_{AJK}(X)\epsilon_+^J\epsilon_+^K, \tag{2.2.33}$$
$$t_A = \Theta_{AJK}(X)\mathbb{A}_+^J\epsilon_+^K + \tfrac{1}{2}\partial_B\Theta_{ALM}F^{\dagger B}\epsilon_+^L\epsilon_+^M. \tag{2.2.34}$$

We used the fact that ρ^A_{-J} satisfy (2.2.24), since one can write:

$$0 = \eta^{\hat{I}\hat{J}}\rho^A{}_{\hat{I}}\rho^B{}_{\hat{J}} = \tfrac{1}{2}\eta^{IJ}\left(\rho^A_{+J}\rho^B_{+I} - \rho^A_{-J}\rho^B_{-I}\right).$$

Fixing of the ghosts t and v is a consequence of choosing the map ρ_+ in such a way that it satisfies (2.2.24). Recall that the anchor map of an exact Courant algebroid has a kernel (2.1.5); in the standard case of the projection to the tangent bundle it is all of the cotangent bundle. However, a DFT algebroid is different and this can be seen as follows. Choosing the above parametrisation for ρ_+, this map has no kernel and therefore we have to fix the symmetry associated to the gauge parameter t that came from the Courant algebroid where the map had a kernel instead. As we have the fixed ghosts t_A and v_A, their BRST transformations must be consistent with those coming from the master action (2.1.32) and (2.1.33). Applying the BRST operator on (2.2.34) one obtains:

$$\begin{aligned}Q_{\mathrm{BV}}t_A = &\,\partial_B\Theta_{AJK}Q_{\mathrm{BV}}X^B\mathbb{A}_+^J\epsilon_+^K + \Theta_{AJK}Q_{\mathrm{BV}}\mathbb{A}_+^J\epsilon_+^K + \Theta_{AJK}\mathbb{A}_+^JQ_{\mathrm{BV}}\epsilon_+^K + \\ &+ \tfrac{1}{2}\partial_B\partial_C\Theta_{ALM}Q_{\mathrm{BV}}X^BF^{\dagger C}\epsilon_+^L\epsilon_+^M + \tfrac{1}{2}\partial_B\Theta_{ALM}Q_{\mathrm{BV}}F^{\dagger B}\epsilon_+^L\epsilon_+^M - \\ &- \partial_B\Theta_{ALM}F^{\dagger B}Q_{\mathrm{BV}}\epsilon_+^L\epsilon_+^M \\ = &\,\Theta_{AJK}\mathrm{d}\epsilon_+^J\epsilon_+^K + \tfrac{1}{2}\partial_B\Theta_{ALM}\mathcal{D}X^B\epsilon_+^L\epsilon_+^M + \tfrac{1}{4}\Big(\hat{\eta}^{JL}\rho^C_{+L}\Theta_{AJI}\partial_B\Theta_{CKM} + \\ &+ \partial_B\Big(2\hat{\eta}^{JL}\Theta_{AJI}\hat{T}_{LKM} - 2\partial_C\Theta_{AMI}\rho^C_{+K} - \hat{\eta}^{JL}\rho^C_{+L}\Theta_{AJI}\Theta_{CKM}\Big)\Big)F^{\dagger B}\epsilon_+^K\epsilon_+^M\epsilon_+^I +\end{aligned}$$

$$+\Big(\partial_B\Theta_{AJL}\rho^B_{+K} - \tfrac{1}{2}\Theta_{APL}\hat{\eta}^{PR}\rho^B_{+R}\Theta_{BJK} + \Theta_{AML}\hat{\eta}^{MN}\hat{T}_{NJK} - \\ - \tfrac{1}{2}\Theta_{AMJ}\hat{\eta}^{MN}\hat{T}_{NKL} + \tfrac{1}{4}\Theta_{AJM}\eta^{MN}\rho^B_{+N}\Theta_{BKL}\Big)\mathbb{A}^J_+\epsilon^K_+\epsilon^L_+, \tag{2.2.35}$$

using the BRST transformation for $F^{\dagger A}$:

$$Q_{\rm BV}F^{\dagger A} = \mathcal{D}X^A - \partial_B\rho^A_{+K}F^{\dagger B}\epsilon^K_+.$$

However, the projection of (2.1.32) implies the following transformation:

$$\begin{aligned} Q_{\rm BV}t_A &= {\rm d}v_A - \epsilon^B_+\partial_A\rho^C_{+B}t_C + \mathbb{A}^B_+\partial_A\rho^C_{+B}v_C - \partial_A\hat{T}_{JKL}\epsilon^J_+\epsilon^K_+\mathbb{A}^L_+ - \partial_A\partial_B\rho^C_{+L}\epsilon^L_+F^{\dagger B}v_C - \\ &\quad - \tfrac{1}{3}\partial_A\partial_B\hat{T}_{KLM}F^{\dagger B}\epsilon^K_+\epsilon^L_+\epsilon^M_+ \\ &= \tfrac{1}{2}\partial_B\Theta_{AKL}\mathcal{D}X^B\epsilon^K_+\epsilon^L_+ + \Theta_{AJK}{\rm d}\epsilon^J_+\epsilon^K_+ + \\ &\quad + \Big(\tfrac{1}{2}\partial_B\Theta_{AKL}\rho^B_{+J} - \partial_A\rho^B_{+K}\Theta_{BJL} + \tfrac{1}{2}\partial_A\rho^B_{+J}\Theta_{BKL} - \partial_A\hat{T}_{JKL}\Big)\mathbb{A}^J_+\epsilon^K_+\epsilon^L_+. \end{aligned} \tag{2.2.36}$$

Eqs. (2.2.35) and (2.2.36) should coincide. Therefore, the consistency condition is:

$$3S_{AJKL}\mathbb{A}^J_+\epsilon^K_+\epsilon^L_+ + \partial_B S_{AIJK}F^{\dagger B}\epsilon^I_+\epsilon^J_+\epsilon^K_+ - \tfrac{1}{2}R^C{}_{AI}\partial_B\Theta_{CJK}F^{\dagger B}\epsilon^I_+\epsilon^J_+\epsilon^K_+ = 0, \tag{2.2.37}$$

where,

$$S_{AJKL} \equiv \partial_B\Theta_{A[JK}\rho^B_{+L]} - \Theta_{AM[J}\hat{\eta}^{MN}\hat{T}_{\underline{N}KL]} + \tfrac{1}{2}\hat{\eta}^{MN}\rho^B_{+N}\Theta_{AM[J}\Theta_{\underline{B}KL]} - \tfrac{2}{3}\partial_A\hat{T}_{JKL} + \\ + \partial_A\rho^B_{+[J}\Theta_{\underline{B}KL]}, \tag{2.2.38}$$

$$R^A{}_{BK} \equiv \hat{\eta}^{IJ}\rho^A_{+J}\Theta_{BIK} + 2\partial_B\rho^A_{+K}. \tag{2.2.39}$$

The same can be done for ghost v_A and we obtain:

$$S_{AJKL}\epsilon^J_+\epsilon^K_+\epsilon^L_+ = 0. \tag{2.2.40}$$

Fixing function $\Theta_{AJK}(X)$ by setting:

$$R^I{}_{BK} = 0, \tag{2.2.41}$$

in (2.2.39) can be shown to imply $S_{AJKL} = 0$ meaning conditions (2.2.37) and (2.2.40) are automatically satisfied.

2.2.5 | Projected gauge transformations

Once we consistently projected all components of the superfields we obtain the following set of gauge transformations:[10]

$$\begin{aligned}
\delta_\epsilon X^A &= \rho^A{}_J \epsilon^J, \\
\delta_\epsilon A^I &= \mathrm{d}\epsilon^I + \Phi^I{}_{JK} A^J \epsilon^K, \\
\delta_\epsilon F_A &= -\mathrm{d}(\Theta_{AJK} A^J \epsilon^K) - \epsilon^J \partial_A \rho^B{}_J F_B + \epsilon^J A^K \wedge A^L (\partial_A \hat{T}_{KLJ} - \partial_A \rho^B{}_K \Theta_{BLJ}),
\end{aligned}$$

where we defined:

$$\Phi^I{}_{JK} \equiv \hat{\eta}^{IN}(\hat{T}_{NJK} - \tfrac{1}{2}\rho^A{}_N \Theta_{AJK}). \tag{2.2.42}$$

Note that the gauge variation of F_A now includes trivial gauge transformations proportional to the equations of motion.

As we did for the Courant sigma model case, we examine the transformation of the field equations obtained by varying the action (2.2.23) with respect to F_A, A^I and X^A respectively:

$$\mathcal{D}X^A \equiv \mathrm{d}X^A - \rho^A{}_J\, A^J = 0, \tag{2.2.43}$$
$$\mathcal{D}A^I \equiv \mathrm{d}A^I - \tfrac{1}{2}\hat{\eta}^{IK}\rho^A{}_K F_A + \tfrac{1}{2}\hat{\eta}^{IK}\hat{T}_{KJL} A^J \wedge A^L = 0, \tag{2.2.44}$$
$$\mathcal{D}F_A \equiv \mathrm{d}F_A + \partial_A \rho^B{}_K\, A^K \wedge F_B - \tfrac{1}{3}\partial_A \hat{T}_{JKL}\, A^J \wedge A^K \wedge A^L = 0. \tag{2.2.45}$$

The gauge transformation of the first field equation gives:

$$\delta_\epsilon \mathcal{D}X^A = \epsilon^J \partial_B \rho^A{}_J \mathcal{D}X^B + \epsilon^J A^K (2\rho^B{}_{[K} \partial_{\underline{B}} \rho^A{}_{J]} - \rho^A{}_N \Phi^N{}_{KJ}). \tag{2.2.46}$$

Therefore, the first condition from the covariance of the field equation is:

$$2\rho^B{}_{[K} \partial_{\underline{B}} \rho^A{}_{J]} - \rho^A{}_N \Phi^N{}_{KJ} = 0. \tag{2.2.47}$$

If we compare this expression with the DFT fluxes obtained by twisting the C-bracket (2.2.25) and (2.2.42), we obtain:

$$\Theta_{AKJ}(X) = -2\eta_{BA}\rho_{C[K}\partial^B \rho^C{}_{J]}, \tag{2.2.48}$$

which is precisely the fixing (2.2.41). Next we check the transformation of the field equation of A^I and obtain:

$$\begin{aligned}
\delta_\epsilon \mathcal{D}A^I = {}& \hat{\eta}^{IN}(\partial_A \hat{T}_{NJK} - \tfrac{1}{2}\partial_A \rho^B{}_N \Theta_{BJK})\epsilon^K \mathcal{D}X^A \wedge A^J + \hat{\eta}^{IN}\hat{T}_{NJK}\epsilon^K \mathcal{D}A^J + \\
& + \tfrac{1}{2}\hat{\eta}^{IN}\epsilon^K A^M \wedge A^{M'} \left(3\rho^A{}_{[K}\partial_{\underline{A}}\hat{T}_{MM']N} - \rho^A{}_N \partial_A \hat{T}_{[KMM']} - 3\hat{\eta}^{PJ}\hat{T}_{P[MM'}\hat{T}_{K]NJ}\right) +
\end{aligned}$$

[10] From now on we denote $\mathbb{A}_+ = A$ and drop all other $\pm$ subscripts.

$$+\ \tfrac{1}{4}\hat{\eta}^{IN}\underline{\rho^A{}_J\hat{\eta}^{JL}\rho^B{}_L}\epsilon^K\left(\Theta_{AKN}F_B+\Theta_{BMN}\Theta_{AM'K}A^M\wedge A^{M'}\right). \tag{2.2.49}$$

Here the underlined contribution is highlighted for later reference, as it would vanish in the case of a Courant algebroid. We see that the gauge variation of the field equation of A is covariant provided that:

$$3\rho^A{}_{[K}\partial_{\underline{A}}\hat{T}_{MM']N}-\rho^A{}_N\partial_A\hat{T}_{KMM'}-3\hat{\eta}^{PJ}\hat{T}_{P[MM'}\hat{T}_{K]NJ}=0.$$

This is one of the local coordinate expressions for a DFT algebroid. However, due to (2.2.24), the last line in (2.2.49) does not vanish, thus there is an additional obstruction. Let us look at this obstruction in more detail:

$$\begin{aligned}&\hat{\eta}^{IN}\underline{\rho^A{}_J\hat{\eta}^{JL}\rho^B{}_L}\epsilon^K\left(\Theta_{AKN}F_B+\Theta_{BMN}\Theta_{AM'K}A^M\wedge A^{M'}\right)=\\&=\hat{\eta}^{IN}\epsilon^K\left(\eta^{AB}\Theta_{AKN}F_B+\Theta_{BMN}\eta^{AB}\Theta_{AM'K}A^M\wedge A^{M'}\right).\end{aligned} \tag{2.2.50}$$

The first term in the parentheses can be rewritten using (2.2.48) as:

$$\eta^{AB}\Theta_{AKN}F_B=-2\rho_{A[K}\partial^B\rho^A{}_{N]}F_B,$$

which vanishes due to the already imposed condition (2.2.27). The second term in the round brackets gives explicitly:

$$\Theta_{AMN}\eta^{BA}\Theta_{BM'K}=4\eta_{AB}\rho_{C[M}\partial^B\rho^C{}_{N]}\rho_{D[M'}\partial^A\rho^D{}_{K]},$$

again after using (2.2.48). This term has precisely the form of the DFT strong constraint.

What about closure of the algebra of gauge transformations? On X^A we have:

$$\begin{aligned}[\delta_{\epsilon_1},\delta_{\epsilon_2}]X^A&=\rho^A{}_J\epsilon_{12}^J,\\ \epsilon_{12}^I&\equiv\Phi^I{}_{KL}\epsilon_1^K\epsilon_2^L,\end{aligned}$$

where we used the function (2.2.42) and condition (2.2.47) to define ϵ_{12}. On A^I we have:

$$\begin{aligned}[\delta_{\epsilon_1},\delta_{\epsilon_2}]A^I=&\ \delta_{\epsilon_{12}}A^I-\partial_A\Phi^I{}_{JK}\epsilon_1^J\epsilon_2^K\mathcal{D}X^A+\\&+3\left(\Phi^I{}_{N[M}\Phi^N{}_{JK]}-\rho^A{}_{[M}\partial_{\underline{A}}\Phi^I{}_{JK]}\right)\epsilon_1^J\epsilon_2^KA^M,\end{aligned}$$

where we used (2.2.47) and (2.2.26). The last line vanishes identically using (2.2.47), thus we have the on-shell closure of the algebra of gauge transformations. However, we obtain consistent gauge transformations of the field equations only after applying the strong constraint, c.f. the underlined term in Eq. (2.2.49).

2.3 | Summary

We have shown how to construct the gauge symmetry of the DFT worldvolume action by projecting the superfield components and BRST transformations of a Courant sigma model master BV action defined over doubled space. We obtained that the algebra of gauge transformations closes on-shell. However, the field equations transform covariantly only upon the use of a constraint, which is the analogue of the DFT strong constraint. This is in accord with the statement that the target space DFT action is invariant under the generalised diffeomorphisms only after using the strong constraint. Our approach establishes this result at the level of the worldvolume theory. To further our understanding of the underlying symmetry and relation between Courant sigma models and DFT we need to introduce the concept of L_∞-algebras. The remainder of our work will be dedicated to the exploration and application of precisely this structure.

CHAPTER 3

INTRODUCTION TO L_∞-ALGEBRAS

Once we include higher-degree gauge fields in our physical models, it is, in general, necessary to extend the standard description of symmetries based on Lie algebras and Lie groups to higher structures. Higher gauge fields appear naturally in string theory [28], but one can embed this generalisation into the standard field-theoretical framework, e.g. Refs. [71]. A systematic approach to these higher structures can be formulated using L_∞-algebras [72, 73], and recently, it was proposed that L_∞-algebras could provide a classification of perturbative gauge theories in general [74]. In this chapter (following mostly [75] and [76] unless otherwise stated) we shall briefly review the basic definitions and properties of L_∞-algebras following the conventions of [76]. The relation to other conventions, used for example in Ref. [74], is given in Appendix 3.A.

3.1 | Formulations of L_∞-algebras

L_∞-algebras can be formulated in terms of graded algebras or coalgebras.[1] Each formulation has its benefits and drawbacks however in this thesis we shall mostly focus on the algebraic counterpart. In order to provide a complete understanding, a brief introduction to the coalgebra formulation is provided. Before moving on it is important to note that the dualisation between algebra and coalgebra does not go both ways in the case of infinite dimensional vector spaces. This will not be of concern to us as in all physical cases the vector spaces of interest are topological spaces where one can restrict to continuous duals for which the double dual is indeed the original space. (See discussion in e.g. [77].)

[1]The history of development of the concept together with relevant references is given in `https://ncatlab.org/nlab/show/L-infinity-algebra#History`

3.1.1 | Algebra formulation

L_∞-algebras. An L_∞-algebra or strong homotopy Lie algebra (L, μ_i) is a graded vector space L with a collection of higher products that are graded totally antisymmetric multilinear maps

$$\mu_i : \underbrace{\mathsf{L} \times \cdots \times \mathsf{L}}_{i\text{-times}} \to \mathsf{L}.$$

of degree $2 - i$ which satisfy the homotopy Jacobi identities:

$$\sum_{j+k=i} \sum_{\sigma} \chi(\sigma; l_1, \ldots, l_i)(-1)^k \mu_{k+1}(\mu_j(l_{\sigma(1)}, \ldots, l_{\sigma(j)}), l_{\sigma(j+1)}, \ldots, l_{\sigma(i)}) = 0; \tag{3.1.1}$$
$$\forall i \in \mathbb{N}_0 \quad \forall l_1, \ldots, l_i \in \mathsf{L}.$$

Here $\chi(\sigma; l_1, \ldots, l_i)$ is the antisymmetric Koszul sign that includes the sign from the parity of the permutation of $\{1, \ldots, i\}$, σ, ordered as: $\sigma(1) < \cdots < \sigma(j)$ and $\sigma(j+1) < \cdots < \sigma(i)$.[2] By graded totally antisymmetric map we mean

$$\mu_i(\ldots, l_n, l_m, \ldots) = (-1)^{|l_n||l_m|+1} \mu_i(\ldots, l_m, l_n, \ldots),$$

where $|l_n|$ is the degree of element $l_n \in \mathsf{L}$. By careful observation of expression (3.1.1) one notices index i is an element of the naturals *including* 0. This inclusion of zero may seem trivial but produces a non-trivial extension called *curved* L_∞-algebras. As this subtlety is integral for the discussion of chapter 5 we will treat it separately. We restrict to the case $i \neq 0$ for now. It is instructive to explicitly write the first three relations in (3.1.1):

$$i = 1 : \;\; \mu_1(\mu_1(l)) = 0.$$

This relation states the map μ_1 is a differential on L.

$$i = 2 : \;\; \mu_1(\mu_2(l_1, l_2)) = \mu_2(\mu_1(l_1), l_2) - (-1)^{|l_1||l_2|} \mu_2(\mu_1(l_2), l_1).$$

From the second relation it is obvious the map μ_1 is a derivation with respect to the graded 2-bracket μ_2 on L, while the third relation:

$$\begin{aligned} i = 3 : \;\; \mu_1(\mu_3(l_1, l_2, l_3)) = {} & \mu_2(\mu_2(l_1, l_2), l_3) - (-1)^{|l_2||l_3|} \mu_2(\mu_2(l_1, l_3), l_2) + \\ & + (-1)^{|l_1|(|l_2|+|l_3|)} \mu_2(\mu_2(l_2, l_3), l_1) - \mu_3(\mu_1(l_1), l_2, l_3) + \\ & + (-1)^{|l_1||l_2|} \mu_3(\mu_1(l_2), l_1, l_3) - (-1)^{|l_3|(|l_1|+|l_2|)} \mu_3(\mu_1(l_3), l_1, l_2), \end{aligned}$$

is the Jacobi identity for a 2-bracket μ_2 up to homotopy given by μ_3. Should the maps μ_i for $i \geqslant 3$ all be zero one would recover the standard differential graded Lie algebra. It is

[2] Permutations ordered in this way are conventionally called unshuffles.

important to notice that when $i \neq 0$, since the map μ_1 is a differential,[3] the elements of the graded vector space $\mathsf{L} = \bigoplus_k \mathsf{L}_k$ form a cochain complex:

$$\cdots \xrightarrow{\mu_1} \mathsf{L}_k \xrightarrow{\mu_1} \mathsf{L}_{k+1} \xrightarrow{\mu_1} \cdots$$

Therefore one can use the additional properties of a cochain complex with differential μ_1 if considering *flat* L_∞-algebras such as for the construction of homotopy transfer (e.g. [77] and [78]).

Maurer-Cartan elements. A special subset of elements called the Maurer-Cartan (MC) elements of L_∞-algebra L are elements of homogeneous subspace L_1 satisfying the generalised *Maurer-Cartan equation*:

$$\sum_i \tfrac{1}{i!}\,\mu_i(x,\ldots,x) = 0, \qquad x \in \mathsf{L}_1,$$

and denoted $\mathsf{MC}(\mathsf{L}) \equiv \{x \in \mathsf{L}_1 \mid \sum_i \frac{1}{i!}\,\mu_i(x,\ldots,x) = 0\}$. These will become important when we move on to field theory as MC elements are the building blocks of MC homotopy theory. MC elements are additionally of great importance when examining curved L_∞-algebras as there is a fundamental difference in that for *curved* algebras $\mathsf{MC}(\mathsf{L})$ can be an empty set. One can see this if we expand the MC equation explicitly:

$$\mu_0 + \mu_1(x) + \tfrac{1}{2!}\mu_2(x,x) + \cdots = 0,$$

from this it is obvious that $x = 0$ is no longer a solution in the case $\mu_0 \neq 0$. Therefore if L is flat we always have at least one element in $\mathsf{MC}(\mathsf{L})$ namely $0 \in \mathsf{MC}(\mathsf{L})$. So curved L_∞-algebras that have MC elements constitute a special subset of curved L_∞-algebras. This subset is important as such algebras can always be "flattened" i.e. one can always uniquely construct a new L_∞-algebra that has vanishing μ_0. The construction, generalising that of Getzler [79] for differential graded Lie algebras, is as follows. Identify the MC elements x of (L,μ) then define a new L_∞-algebra $(\mathsf{L},\tilde{\mu})$ with the same spaces and maps:

$$\tilde{\mu}_i = \begin{cases} 0 & i = 0, \\ \sum_{n\in\mathbb{N}} \frac{1}{(n-1)!}\mu_n(x,\ldots,x,\,\cdot\,) & i = 1, \\ \mu_i & \text{else}. \end{cases}$$

The key fact that $\tilde{\mu}_1$ squares to zero stems from the fact that (L,μ) is an L_∞-algebra i.e. that products μ satisfy the homotopy relations.

[3] Why this is not so when i is allowed to be 0 will be shown in chapter 5.

Tensor L_∞-algebras. An important class of L_∞-algebras that are relevant for our construction in the following are induced by the tensor product of an L_∞-algebra with a differential graded commutative (associative) algebra. Let (L, μ) be an L_∞-algebra and (A, d) be a differential graded commutative algebra with product m that is associative. Define $\tilde{\mathsf{L}}$ as:

$$\tilde{\mathsf{L}} \equiv \bigoplus_{k\in\mathbb{Z}} (\mathsf{A}\otimes\mathsf{L})_k \quad \text{where} \quad (\mathsf{A}\otimes\mathsf{L})_k \equiv \bigoplus_{i+j=k} \mathsf{A}_i \otimes \mathsf{L}_j,$$

and maps $\tilde{\mu}$ by:

$$\tilde{\mu}_i \equiv \begin{cases} d\otimes \mathrm{id} + \mathrm{id}\otimes\mu_1 & i=1 \\ m_i\otimes\mu_i & \text{else.}\end{cases}$$

One can show $(\tilde{\mathsf{L}}, \tilde{\mu})$ is an L_∞-algebra since $\tilde{\mu}$ satisfy the homotopy relations (3.1.1). With respect to $\tilde{\mathsf{L}}$ a homogeneous element $\tilde{l} \in \tilde{\mathsf{L}}$ has degree given as the sum of its L and A degrees: $|\tilde{l}|_{\tilde{\mathsf{L}}} = |\tilde{l}|_{\mathsf{A}} + |\tilde{l}|_{\mathsf{L}}$.

For example, we can take the de Rham complex on a manifold M, $(\Omega^\bullet(M), \mathrm{d})$ and tensor it with an L_∞-algebra to again obtain an L_∞-algebra (L', μ_i'):[4]

$$\mathsf{L}' \equiv \Omega^\bullet(M, \mathsf{L}) \equiv \bigoplus_{k\in\mathbb{Z}} \Omega^\bullet_k(M, \mathsf{L}), \quad \Omega^\bullet_k(M,\mathsf{L}) \equiv \bigoplus_{i+j=k} \Omega^i(M)\otimes\mathsf{L}_j,$$

with higher products:

$$\mu_1'(\alpha_1\otimes l_1) = \mathrm{d}\alpha_1\otimes l_1 + (-1)^{|\alpha_1|}\alpha_1\otimes\mu_1(l_1), \tag{3.1.2}$$

$$\mu_i'(\alpha_1\otimes l_1,\dots,\alpha_i\otimes l_i) = (-1)^{i\sum_{j=1}^{i}|\alpha_j| + \sum_{j=0}^{i-2}|\alpha_{i-j}|\sum_{k=1}^{i-j-1}|l_k|}(\alpha_1\wedge\dots\wedge\alpha_i)\otimes\mu_i(l_1,\dots,l_i),$$
$$\forall i \geqslant 2, \quad \alpha_1,\dots,\alpha_i\in\Omega^\bullet(M), \quad l_1,\dots,l_i\in\mathsf{L}. \tag{3.1.3}$$

The signs above can be understood as a consequence of the fact that the tensored algebra does not "see" the individual degrees of the underlying algebras, only the sum. Therefore one needs to compensate for the mixed terms in the exponent of -1 that make the illusion of A degrees graded commuting with L degrees.

Cyclic L_∞-algebras. There is an additional structure one can (but does not have to) add to an L_∞-algebra: an inner product. We say an L_∞-algebra (over $\mathbb{R}$) together with an inner product i.e. a graded symmetric non-degenerate bilinear pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}} : \mathsf{L}\times\mathsf{L}\to\mathbb{R}, \quad \langle l_n, l_m\rangle_{\mathsf{L}} = (-1)^{|l_n||l_m|}\langle l_m, l_n\rangle_{\mathsf{L}},$$

[4]In the L_∞ tensor product hierarchy defined in the next section and used throughout this thesis L' will always mean an L_∞-algebra obtained by tensoring with the de Rham complex.

that satisfies the cyclicity condition:

$$\langle l_1, \mu_i(l_2, \ldots, l_{i+1})\rangle_{\mathsf{L}} = (-1)^{i+i(|l_1|+|l_{i+1}|)+|l_{i+1}|\sum_{j=1}^{i}|l_j|}\langle l_{i+1}, \mu_i(l_1, \ldots, l_i)\rangle_{\mathsf{L}}; \qquad \forall i \in \mathbb{N}_0, \tag{3.1.4}$$

is a cyclic L_∞-algebra. These will be of importance to all field theories that have an action functional description.

L_∞**-morphisms.** A morphism between two L_∞-algebras $(\tilde{\mathsf{L}}, \tilde{\mu}_i)$ and (L, μ_i) due to [80] is a collection of homogeneous maps $\phi_i : \tilde{\mathsf{L}} \times \ldots \times \tilde{\mathsf{L}} \to \mathsf{L}$ of degree $1-i$ for $i \in \mathbb{N}_0$ which are multilinear and totally graded anti-symmetric and obey:

$$\begin{aligned}
&\sum_{j+k=i}\sum_{\sigma\in\mathrm{Sh}(j;i)}(-1)^k\chi(\sigma; l_1, \ldots, l_i)\phi_{k+1}(\tilde{\mu}_j(l_{\sigma(1)}, \ldots, l_{\sigma(j)}), l_{\sigma(j+1)}, \ldots, l_{\sigma(i)}) = \\
&= \sum_{k_1+\cdots+k_j=i}\tfrac{1}{j!}\sum_{\sigma\in\mathrm{Sh}(k_1,\ldots,k_{j-1};i)}\chi(\sigma; l_1, \ldots, l_i)\zeta(\sigma; l_1, \ldots, l_i)\times \\
&\quad\times \mu_j(\phi_{k_1}(l_{\sigma(1)}, \ldots, l_{\sigma(k_1)}), \ldots, \phi_{k_j}(l_{\sigma(k_1+\ldots+k_{j-1}+1)}, \ldots, l_{\sigma(i)})),
\end{aligned} \tag{3.1.5}$$

where $\chi(\sigma; l_1, \ldots, l_i)$ is the graded Koszul sign and $\zeta(\sigma; l_1, \ldots, l_i)$ for a $(k_1, \ldots, k_{j-1}; i)$-shuffle σ is given by

$$\zeta(\sigma; l_1, \ldots, l_i) = (-1)^{\sum_{1\leqslant m<n\leqslant j}k_m k_n + \sum_{m=1}^{j-1}k_m(j-m) + \sum_{m=2}^{j}(1-k_m)\sum_{k=1}^{k_1+\ldots+k_{m-1}}|l_{\sigma(k)}|}.$$

Expression(3.1.5) is easier understood in the context of L_∞-algebras as coalgebras shown next. An isomorphism of L_∞-algebras is an L_∞-morphism with invertible ϕ_1 maps. However, as this is in most cases overly restrictive, one defines a weaker L_∞ *quasi*-isomorphism that requires only the homologies be isomorphic. These are of interest as two quasi-isomorphic algebras correspond to phisically equivalent theories (for a pedagogical exposition see e.g. [77]).

3.1.2 | Coalgebra formulation

L_∞**-algebras.** A coalgebra A (over the field of real numbers) is a vector space dual to a unital associative algebra. (see e.g. [81]) It is endowed with a coproduct $\Delta : A \to A \otimes A$ and $e : A \to \mathbb{R}$ such that Δ is coassociative:

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$$

and e is a counit:

$$(\mathrm{id} \otimes e)\Delta = \mathrm{id} \quad \text{and} \quad (e \otimes \mathrm{id})\Delta = \mathrm{id}.$$

An L_∞-algebra is a coalgebra $(\bigodot^\bullet \mathsf{V}, \Delta)$ with a codifferential $D : \bigodot^\bullet \mathsf{V} \to \bigodot^\bullet \mathsf{V}$ of degree 1 that satisfies the coLeibniz rule and squares to zero:

$$\Delta D = (D \otimes \mathrm{id})\Delta + (\mathrm{id} \otimes D)\Delta \quad \text{and} \quad D^2 = 0.$$

where $\bigodot^\bullet \mathsf{V} \equiv \bigoplus_{k\geqslant 0} \bigodot^k \mathsf{V}$ indicates the symmetric tensor algebra. In order to show the connection to the algebraic formulation of the previous section we define the shift isomorphism $s : \mathsf{V} \to \mathsf{V}[1]$:

$$s^\bullet : \bigwedge^\bullet \mathsf{V} \to \bigodot^\bullet \mathsf{V}[1]. \tag{3.1.6}$$

The codifferential can be decomposed as $D = \sum_{i\in\mathbb{N}_0} D_i$ and induces the algebraic product μ_i via the shift isomorphism as:

$$\mu_i = (-1)^{\frac{1}{2}i(i-1)} s^{-1} \circ D_i \circ s^{\otimes i}, \tag{3.1.7}$$

where the sign stems from:

$$\mathrm{id}^{\otimes i} = (-1)^{\frac{1}{2}i(i-1)} s^{\otimes i} \circ (s^{-1})^{\otimes i},$$

since swapping shift maps produces a sign.

L_∞-morphisms. A morphism Φ of degree zero between two coalgebras (A, Δ) and (A', Δ') is a morphism of coalgebras $\Phi : A \to A'$ if it is compatible with the coproducts in the sense:

$$\Delta'\Phi = (\Phi \otimes \Phi)\Delta.$$

As an L_∞-algebra is a codifferential coalgebra one can construct an L_∞-morphism as a morphism of coalgebras with the additional requirement that morphism Φ be compatible with the codifferentials in the following way:

$$\Phi D = D'\Phi.$$

As for the higher products, we can construct the components of morphism Φ in the algebra formulation using the shift isomorphism s by:

$$\phi_i = (-1)^{\frac{1}{2}i(i-1)} s^{-1} \circ \Phi_i^1 \circ s^{\otimes i},$$

where Φ_i^1 is the component of Φ that maps to the V' subspace of $\bigodot^\bullet \mathsf{V}'$ and takes arguments from $\bigodot^i \mathsf{V}$.

3.2 | Maurer-Cartan homotopy theory

Now that we know what L_∞-algebras are we need to explore how they interplay with physics, more specifically, field theory. This is known as Maurer-Cartan homotopy theory. It is based on the interpretation that the solutions of the equations of motion are L_∞ Maurer-Cartan elements of the underlying L_∞-algebra. This can then be expanded to the BRST and BV extensions of the gauge field theory. In the following we shall explore both the classical and BRST/BV levels of an L_∞ gauge field theory.

3.2.1 | Classical theory

Within the framework of L_∞-algebras one can define a generalisation of the Maurer-Cartan (MC) equation as follows. Take (L', μ'_i) and an element $a \in \mathsf{L}'_1$ which we call a gauge potential. One defines the corresponding curvature as:

$$f \equiv \mu'_1(a) + \frac{1}{2}\mu'_2(a,a) + \cdots = \sum_{i\geqslant 1} \frac{1}{i!}\mu'_i(a,\ldots,a). \tag{3.2.1}$$

The generalised or homotopy Maurer-Cartan equation is then $f = 0$.

Gauge transformations of gauge potentials a and their curvatures f are given by:

$$\delta_{c_0} a = \sum_{i\geqslant 0} \frac{1}{i!}\mu'_{i+1}(a,\ldots,a,c_0), \tag{3.2.2}$$

$$\delta_{c_0} f = \sum_{i\geqslant 0} \frac{1}{i!}\mu'_{i+2}(a,\ldots,a,f,c_0), \tag{3.2.3}$$

where $c_0 \in \mathsf{L}_0$ is the *level 0* gauge parameter. If a theory contains higher gauge symmetries we will have higher (*level k*) gauge parameters $c_{-k} \in \mathsf{L}_{-k}$, $k > 0$, with infinitesimal gauge transformations given by:

$$\delta_{c_{-k-1}} c_{-k} = \sum_{i\geqslant 0} \frac{1}{i!}\mu'_{i+1}(a,\ldots,a,c_{-k-1}). \tag{3.2.4}$$

One can show that the algebra of gauge transformations closes up to terms proportional to the curvature f. In this respect, the MC equation can be interpreted either as an equation of motion or as a constraint on the kinematical data of the theory. If we choose to think of the MC equation as dynamical, the next question is if there exists an action from which this MC equation follows by variational principle. The answer is yes if it is possible to define a bilinear pairing compatible with the L_∞-algebra structure i.e. if the algebra can be made cyclic.

If we have a tensored structure like (L', μ'_i) then (L', μ'_i) is cyclic provided (L, μ) is cyclic and M is an oriented, compact cycle. The induced inner product is then

$$\langle \alpha_1 \otimes l_1, \alpha_2 \otimes l_2 \rangle_{\mathsf{L}'} = (-1)^{|\alpha_2||l_1|} \int_M \alpha_1 \wedge \alpha_2 \; \langle l_1, l_2 \rangle_{\mathsf{L}}. \tag{3.2.5}$$

An inner product defined in this way allows us to write the action whose stationary point is the MC equation:

$$S_{\mathrm{MC}}[a] \equiv \sum_{i \geqslant 1} \frac{1}{(i+1)!} \langle a, \mu'_i(a, \dots, a) \rangle_{\mathsf{L}'}. \tag{3.2.6}$$

It follows from the homotopy Jacobi identities (3.1.1) that this action is gauge invariant with respect to the variation of field a as given in (3.2.2).

Bianchi identities can also be defined in terms of L_∞-algebras as:

$$\sum_{i \geqslant 0} \frac{1}{i!} \mu'_{i+1}(a, \dots, a, f) = 0, \tag{3.2.7}$$

satisfied simply by the homotopy relations. However, in the case when one has a cyclic algebra i.e. an action functional then it can be related to Noether's second theorem (see e.g. [82]) by the fact that δ_{c_0} is a variational symmetry of S_{MC}. This is easily seen from:

$$0 = \delta_{c_0} S_{\mathrm{MC}} = \sum_{i \geqslant 0} \frac{1}{i!} \langle \delta_{c_0} a, \mu'_{i+1}(a, \dots, a) \rangle = \langle \delta_{c_0} a, f \rangle,$$

and by comparing with the Noether theorem:

$$(\delta_{c_0} a)_i f^i = \mathrm{div},$$

producing the divergence div. On the other hand:

$$\langle \delta_{c_0} a, f \rangle = \sum_{i \geqslant 0} \frac{1}{i!} \langle \mu'_{i+1}(a, \dots, a, c_0), f \rangle = \sum_{i \geqslant 0} \frac{1}{i!} \langle \mu'_{i+1}(a, \dots, a, f), c_0 \rangle,$$

where we have used the cyclicity condition (3.1.4) and fact that the cyclic product is graded commutative. Therefore, we obtained the Bianchi identities and see that it is equivalent to the Noether theorem in the case where we have a cyclic product (because then the integral of the divergence vanishes).

3.2.2 | Batalin-Vilkovisky theory

To introduce the Batalin-Vilkovisky formalism into a homotopy theory we consider a cyclic L_∞-algebra $(\mathsf{L}, \mu_i, \langle \cdot\,, \cdot \rangle_{\mathsf{L}})$ with $|\langle \cdot\,, \cdot \rangle_{\mathsf{L}}| = -3$. The truncation (in that the complex stops at 1) of the L_∞-algebra to include fields and ghosts:

$$\mathsf{L}_{\text{BRST}} = \bigoplus_{i\leqslant 1} \mathsf{L}_i, \tag{3.2.8}$$

is the BRST complex. In terms of our classical picture of the previous section this is the field content of complex (4.1.3) truncated after $\mathsf{L'}_1$ as higher spaces do not correspond to fields if one reinterprets (higher) gauge parameters as ghost fields. Generally, given a graded vector space V, coordinate functions are maps:

$$\xi^\alpha : \mathsf{V} \to \mathbb{R} \quad \text{such that} \quad \xi^\alpha(v) = \xi^\alpha(v^\beta \tau_\beta) = v^\alpha,$$

with degrees: $|\xi^\alpha| = -|v| \equiv -|\tau_\alpha|$. We use the convention that the degree-shifted graded vector space $\mathsf{V}[k]$ has vectors of degree $|v| - k$, which implies the coordinate functions will then be of degree $|\xi^\alpha| = -|\tau_\alpha| + k$. As is customary we will write the fields as contracted coordinate functions,

$$\xi = \xi^\alpha \tau_\alpha, \qquad \xi^\alpha \in \mathcal{C}^\infty(\mathsf{L}[1]),\ \tau_\alpha \in \mathsf{L}.$$

They are useful as one can write L_∞-algebra objects in a basis independent form. Therefore, the L_∞-algebra of interest is:

$$\hat{\mathsf{L}}_{\text{BRST}} \equiv \mathcal{C}^\infty(\mathsf{L}_{\text{BRST}}[1]) \otimes \mathsf{L}_{\text{BRST}}.$$

A few slight modifications must be made for this tensor product algebra; $\mathcal{C}^\infty(\mathsf{L}[1])$ can be understood as a differential graded commutative algebra with trivial differential therefore $\hat{\mu}_1$ only has the second term in (3.1.2), and since the pairing on L itself can have a non-zero degree $k = |\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}}|$, $\zeta_1, \zeta_2 \in \mathcal{C}^\infty(\mathsf{L}[1])$ will additionally graded commute with the pairing producing a second sign. The cyclic inner product thus decomposes as:

$$\langle \zeta_1 \otimes l_1, \zeta_2 \otimes l_2 \rangle_{\hat{\mathsf{L}}} = (-1)^{k(|\zeta_1|+|\zeta_2|)+|\zeta_2||l_1|} (\zeta_1\zeta_2) \langle l_1, l_2 \rangle_{\mathsf{L}}. \tag{3.2.9}$$

$\hat{\mathsf{L}}$ as a tensor product space has elements with a bi-degree, the *ghost number* or degree in $\mathcal{C}^\infty(\mathsf{L}_{\text{BRST}}[1])$ and L_∞-degree or degree in L, and as an L_∞-algebra they have a single $\hat{\mathsf{L}}$ degree being the sum of its bi-degrees. The $\hat{\mathsf{L}}$ degree of all contracted coordinate functions of fields is then 1. Therefore we can combine all gauge fields and ghosts into a single contracted coordinate function *superfield*:

$$\mathsf{a}_{\text{BRST}} \equiv a + \sum_{i\geqslant 0} c_{-i}.$$

The action of the BRST operator Q_{BRST} is then:

$$Q_{\text{BRST}}\mathsf{a}_{\text{BRST}} = -\sum_{i\geqslant 1}\frac{1}{i!}\hat{\mu}_i(\mathsf{a}_{\text{BRST}},\ldots,\mathsf{a}_{\text{BRST}}). \tag{3.2.10}$$

In this truncated L_∞-algebra however, Q_{BRST} is in general only nilpotent up to the Maurer-Cartan equation. Since this violation of nilpotency stems precisely from the truncation, by extending this complex to all homogeneous subspaces (essentially letting $i \in \mathbb{Z}$ in (3.2.8)) one regains Q as a homological operator again, this is the BV complex.

In the language of field theory this is simply a reformulation of the requirement for BV in open algebra gauge cases, in other words this is nothing more than the addition of antifields for every physical and ghost field. We associate these antifields $a^\dagger$ and $c^\dagger_{-k}$ ($k \geqslant 0$) to each gauge or ghost field. The BV *superfield* is then also by extension a combination of all the gauge and ghost fields, and antifields:

$$\mathsf{a} \equiv a + a^\dagger + \sum_{i\geqslant 0}(c_{-i} + c^\dagger_{-i}).$$

The curvature of a is as in any L_∞-algebra given by:

$$\mathsf{f} = \sum_{i\geqslant 1}\frac{1}{i!}\hat{\mu}_i(\mathsf{a},\ldots,\mathsf{a}). \tag{3.2.11}$$

Realising the operator Q_{BV} simply reduces to Q_{BRST} when a is truncated, implies Q_{BV} is defined as in (3.2.10) with a the full BV superfield. This means we have the action of Q_{BV} given by

$$Q_{\text{BV}}\mathsf{a} = -\mathsf{f}, \qquad\qquad Q_{\text{BV}}\mathsf{f} = 0. \tag{3.2.12}$$

From this it is obvious that $Q_{\text{BV}}\mathsf{a}$ will contain the classical gauge variations:

$$Q_{\text{BV}}a = \delta_{c_0}a + \cdots \qquad Q_{\text{BV}}c_{-k} = (-1)^{k+1}\delta_{c_{-k-1}}c_{-k} + \cdots$$

where $k \geqslant 0$ and equations of motion:

$$Q_{\text{BV}}a^\dagger = -f + \cdots .$$

The function S_{BV} on $\mathcal{F}_{\text{BV}}$ defined as:

$$S_{\text{BV}}[\mathsf{a}] \equiv \sum_{i\geqslant 0}\frac{1}{(i+1)!}\langle \mathsf{a}, \hat{\mu}_i(\mathsf{a},\ldots,\mathsf{a})\rangle_{\hat{\mathsf{L}}}, \tag{3.2.13}$$

is the BV extension of (3.2.6) called the **Maurer-Cartan-Batalin-Vilkovisky action** or BV action for short. For details see [76].

APPENDIX

3.A | Conventions

We provide a short dictionary between conventions of [74] and [52], and [76] that we use. The first difference is that degrees are inverted as shown in table 3.A.1 where by

$$\begin{array}{ccc} \mathring{\mathsf{L}}_{-i} & \Leftrightarrow & \mathsf{L}_i \\ |\mathring{\mu}_i| = i-2 & \Leftrightarrow & |\mu_i| = 2-i \end{array}$$

Table 3.A.1
Change of degrees between conventions.

$\circ$ we indicate the conventions of [74] and [52]. The second, and much more important, difference is in the homotopy relation that states:

$$\sum_{j+k=i}\sum_{\sigma}\chi(\sigma;l_1,\dots,l_i)(-1)^{kj}\mathring{\mu}_{k+1}(\mathring{\mu}_j(l_{\sigma(1)},\dots,l_{\sigma(j)}),l_{\sigma(j+1)},\dots,l_{\sigma(i)})=0,$$

notice the kj in the exponent of -1 as opposed to just k in (3.1.1). The relation to our convention (other than degree inversion) is given by an additional sign:

$$\mu_j \to (-1)^{\frac{1}{2}j(j-1)}\mathring{\mu}_j,$$

this sign compensates the difference between the homotopy relations (up to an overall sign dependent on i that goes away since the right-hand side is zero). However, as the sign of μ changes so will the expressions for the Maurer-Cartan equation (3.2.1), homotopy action (3.2.6):

$$S_{\mathrm{MC}}[a] = \sum_{i\geqslant 1}\frac{(-1)^{\frac{1}{2}i(i-1)}}{(i+1)!}\langle a, \mathring{\mu}'_i(a,\dots,a)\rangle_{\mathring{\mathsf{L}}'},$$

$$f = \sum_{i \geqslant 1} \frac{(-1)^{\frac{1}{2}i(i-1)}}{i!} \mathring{\mu}'_i(a, \dots, a),$$

and others, precisely as stated in [74].

CHAPTER 4

COURANT SIGMA MODEL AND L_∞-ALGEBRAS

A very general mathematical framework that encompasses the generalised concepts of symmetry discussed in this thesis is that of L_∞-algebras. As we have seen in the previous chapter L_∞-algebras are a generalisation of standard Lie algebras in which the failure of the Jacobi identity for 2-brackets is controlled by higher 3-brackets, the failure of higher Jacobi identities for 3-brackets is controlled by 4-brackets and so on. The exact relations between higher brackets defined on a graded vector space are given by the defining homotopy relations of the L_∞-algebra.

It has been shown that using the L_∞-algebra framework one can bootstrap consistent gauge theories [83]. Choosing initial data of the theory in the form of 1- and 2-brackets one can bootstrap higher brackets using the homotopy relations of an L_∞-algebra and thus find consistent, gauge invariant theories defined by their equations of motion. This is reminiscent of the deformation of a free gauge theory into an interacting one in the BV/BRST approach. There one starts with a free, kinetic part of the action and its gauge symmetry and adds systematically all possible interaction terms consistent with BRST invariance, see review Ref. [84] for more details and references. This is relevant as the Courant sigma model was also constructed this way in [19].

As we know from chapter 2 the Courant algebroid defines a membrane sigma model and its BV/BRST action known as the Courant sigma model. However it has been shown by Roytenberg and Weinstein that the algebraic structure of a Courant algebroid can be described as 2-term L_∞-algebra [52], inducing one to wonder what the L_∞ structure of the sigma model is and how it relates to its algebraic counterpart.

Therefore, in this chapter we would like to embed the intersection between background fluxes of closed strings as seen in previous chapters and in [42, 45, 47–49], gauge (or BRST) symmetries of the Courant sigma model and axioms of a Courant algebroid into an L_∞-algebra structure. Similar ideas in the context of topological open membranes were discussed in Ref. [20] and the general discussion of L_∞-algebra structures for AKSZ models (as QP-manifolds) was already presented in the original paper [15]. We shall explicitly construct the L_∞-algebra structure for the classical Courant sigma model and then show how to extend this construction to include the full BV/BRST action. In section 4.1 we shall construct the L_∞-algebra for the classical Courant sigma model, including fields, symmetries and the action functional. Moreover, we shall demonstrate how homotopy identities naturally generate the axioms of a Courant algebroid in the form of coordinate expressions. It is important to note that at the classical level we are discussing two different L_∞-algebras, the L_∞ gauge algebra (L, μ_i) and the tensor product algebra of L_∞-algebra-valued de Rham forms $(\Omega^\bullet(\Sigma_3, \mathsf{L}), \mu_i')$ (see discussion in sections 3.1.1 and 3.2.1). Next, we shall present the full BV/BRST action for the Courant sigma model reproducing Roytenbergs result [18] by explicitly constructing the L_∞-algebra, this one including the complete BV/BRST complex. In section 4.3 we present the mappings or L_∞-algebra morphism between the Courant algebroid L_∞-algebra and the gauge algebra (L, μ_i) we constructed for the Courant sigma model. This chapter is based on [75].

4.1 | L_∞ for the classical Courant sigma model

The Courant sigma model has been constructed in Ref. [19] starting from Chern-Simons theory coupled to BF theory in the BRST formalism, and in Ref. [18] using the general construction for AKSZ topological sigma models [15]. The BV/BRST structure was described in section 2.1.3 and defined by action (2.1.24) As a quick reminder the classical part of the membrane action is (2.1.14) on a, generally, undoubled manifold:

$$S[X, A, F] = \int_{\Sigma_3} F_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\hat{\eta}_{IJ} A^I \wedge \mathrm{d}A^J - \rho^i{}_I(X) A^I \wedge F_i + \tfrac{1}{6} T_{IJK}(X) A^I \wedge A^J \wedge A^K, \quad (4.1.1)$$

where $i = 1, \ldots, d$ is the target space index and $I = 1, \ldots, 2d$ the pullback bundle index.[1] As in the doubled case we have maps $X = (X^i) : \Sigma_3 \to M$, 1-forms $A \in \Omega^1(\Sigma_3, X^*E)$, and an auxiliary 2-form $F \in \Omega^2(\Sigma_3, X^*T^*M)$. The symmetric bilinear form $\hat{\eta}$ is, again, the $O(d,d)$ invariant metric, while functions $\rho(X)$ and $T(X)$ are related to the anchor and twist of the Courant algebroid, as discussed in 2.1.2. The full definition of the Courant algebroid can be found in section 2.1.1, while structures relevant for our analysis in the context of L_∞-algebras will be defined in Sect. 4.3. As discussed in Refs. [18, 19, 46] and section 2.1.2 the gauge transformations of the CSM mediated by two gauge parameters define a first-stage reducible gauge symmetry, and the algebra of transformations closes only on-shell. In the following we shall describe this rich gauge structure using the L_∞-algebra framework.

[1] As there is no possibility of confusing the bundle index with the target space index, in this chapter only we shall not use the convention that target space indices start from the beginning of the alphabet.

4.1.1 | Maurer-Cartan homotopy action

In order to construct the L_∞-algebra for the Courant sigma model we have to define relevant physical fields and assign an appropriate L_∞ grading to each one. Starting from the action (4.1.1), we choose $\{X^i, A^I, F_i\}$ as physical fields. In order to associate higher products to each of the terms in the action we interpret functions ρ and T as infinite perturbative expansions in field X via their Taylor series. Therefore the action has infinitely many interaction terms:[2]

$$\begin{aligned} S[X,A,F] = \int_{\Sigma_3} & F_i \mathrm{d}X^i + \tfrac{1}{2}\hat{\eta}_{IJ} A^I \mathrm{d}A^J - \rho^i{}_I A^I F_i - X^{i_1}\partial_{i_1}\rho^i{}_I A^I F_i - \\ & - \tfrac{1}{2} X^{i_1} X^{i_2} \partial_{i_1}\partial_{i_2}\rho^i{}_I A^I F_i - \cdots - \tfrac{1}{n!} X^{i_1}\cdots X^{i_n}\partial_{i_1}\cdots\partial_{i_n}\rho^i{}_I A^I F_i - \cdots + \\ & + \tfrac{1}{6} T_{IJK} A^I A^J A^K + \tfrac{1}{6} X^{i_1}\partial_{i_1} T_{IJK} A^I A^J A^K + \\ & + \tfrac{1}{12} X^{i_1} X^{i_2}\partial_{i_1}\partial_{i_2} T_{IJK} A^I A^J A^K + \\ & + \cdots + \tfrac{1}{6\cdot n!} X^{i_1}\cdots X^{i_n}\partial_{i_1}\cdots\partial_{i_n} T_{IJK} A^I A^J A^K + \cdots \,, \end{aligned} \tag{4.1.2}$$

all of which must be integrated into the L_∞ picture. Because of this expansion there will not be a finite number of higher products as every interaction term (of which there are infinitely many) will require a unique product. Recalling the general construction presented in section 3.2.1 we know all physical fields $a = X + A + F$ are elements of L'_1, their *curvatures* or equations of motion $f = f_X + f_A + f_F$ of L'_2, gauge parameters $c_0 = \epsilon + t$ of L'_0 and level 1 parameter $c_{-1} = v$ of L'_{-1}. For completeness, here we anticipate appearance of "ghost for ghost" gauge parameter v that becomes relevant only in the BRST setting. Thus the complex on which we base the construction of the L_∞-algebra is therefore:

$$\cdots \to \mathsf{L}'_{-1} \xrightarrow{\mu'_1} \mathsf{L}'_0 \xrightarrow{\mu'_1} \mathsf{L}'_1 \xrightarrow{\mu'_1} \mathsf{L}'_2 \to \cdots \,. \tag{4.1.3}$$

To obtain expressions in the form of action (4.1.1) or (4.1.2) we must decompose L' into the de Rham part $\Omega^\bullet(\Sigma_3)$ and the algebraic L_∞ part L. Since we have three types of fields we shall define L with three homogeneous subspaces to form the following complex:

$$\mathsf{L}_{-1} \xrightarrow{\mu_1} \mathsf{L}_0 \xrightarrow{\mu_1} \mathsf{L}_1 . \tag{4.1.4}$$

[2]In order to make such long expressions more manageable the shorthand $f(0) \equiv f$ and $\partial f\big|_0 \equiv \partial f$ for any function f of X evaluated at 0 is used, additionally, the exterior product of forms will be implied with $\wedge$ suppressed. If the full function is meant the argument will be explicitly written.

Therefore, the classical field content is given below:

$$\begin{aligned} a &= X + A + F \in \Omega^0(\Sigma_3, \mathsf{L}_1) \oplus \Omega^1(\Sigma_3, \mathsf{L}_0) \oplus \Omega^2(\Sigma_3, \mathsf{L}_{-1}), \\ c_0 &= \epsilon + t \in \Omega^0(\Sigma_3, \mathsf{L}_0) \oplus \Omega^1(\Sigma_3, \mathsf{L}_{-1}), \\ c_{-1} &= v \in \Omega^0(\Sigma_3, \mathsf{L}_{-1}), \end{aligned} \tag{4.1.5}$$

and shown in the following table:

$$\begin{array}{cccccccccccc} \cdots & \xrightarrow{\mu_1'} & \mathsf{L'}_{-1} & \xrightarrow{\mu_1'} & \mathsf{L'}_0 & \xrightarrow{\mu_1'} & \mathsf{L'}_1 & \xrightarrow{\mu_1'} & \mathsf{L'}_2 & \xrightarrow{\mu_1'} & \mathsf{L'}_3 & \xrightarrow{\mu_1'} \cdots \\ & & \substack{\text{h. gauge}\\ \text{parameters}} & & \substack{\text{gauge}\\ \text{parameters}} & & \substack{\text{physical}\\ \text{fields}} & & \substack{\text{equations}\\ \text{of motion}} & & \substack{\text{Noether}\\ \text{identities}} & \\ & \mathsf{L}_{-1} & v_i & & t_i & & F_i & & \mathcal{D}F_i & & & \\ & {\scriptstyle\mu_1}\downarrow & & & & & & & & & & \\ & \mathsf{L}_0 & & & \epsilon^I & & A^I & & \mathcal{D}A^I & & & \\ & {\scriptstyle\mu_1}\downarrow & & & & & & & & & & \\ & \mathsf{L}_1 & & & & & X^i & & \mathcal{D}X^i & & & \end{array}$$

Once we placed fields in their appropriate homogeneous subspaces, we have to define all the products. Note that the physical field A lives in the pullback bundle X^*E and there is no natural structure defined on its sections; in particular the bracket of sections A is not the Courant bracket. Thus one can think of L_∞-products as defining relations for the relevant structures on sections of the pullback bundle. Our selection for the non-vanishing higher products of $\mathsf{L} = \mathsf{L}_1 \oplus \mathsf{L}_0 \oplus \mathsf{L}_{-1}$ is:

$$\begin{aligned} \mathsf{L}_1 \ni \quad & \mu_n(l_{(1)1}, \dots, l_{(1)n-1}, l_{(0)}) = l^{i_1}_{(1)1} \cdots l^{i_{n-1}}_{(1)n-1} \partial_{i_1} \cdots \partial_{i_{n-1}} \rho^i{}_I l^I_{(0)}, \\ \mathsf{L}_0 \ni \quad & \mu_n(l_{(1)1}, \dots, l_{(1)n-1}, l_{(-1)}) = -l^{i_1}_{(1)1} \cdots l^{i_{n-1}}_{(1)n-1} \partial_{i_1} \cdots \partial_{i_{n-1}} \rho^i{}_J l_{(-1)i} \hat\eta^{IJ}, \\ \mathsf{L}_{-1} \ni \quad & \mu_m(l_{(1)1}, \dots, l_{(1)m-2}, l_{(-1)}, l_{(0)}) = -l^{i_1}_{(1)1} \cdots l^{i_{m-2}}_{(1)m-2} \partial_{i_1} \cdots \partial_{i_{m-2}} \partial_i \rho^j{}_I l_{(-1)j} l^I_{(0)}, \\ \mathsf{L}_0 \ni \quad & \mu_m(l_{(1)1}, \dots, l_{(1)m-2}, l_{(0)1}, l_{(0)2}) = l^{i_1}_{(1)1} \cdots l^{i_{m-2}}_{(1)m-2} \partial_{i_1} \cdots \partial_{i_{m-2}} T_{JKL} l^K_{(0)1} l^L_{(0)2} \hat\eta^{IJ}, \\ \mathsf{L}_{-1} \ni \quad & \mu_r(l_{(1)1}, \dots, l_{(1)r-3}, l_{(0)1}, l_{(0)2}, l_{(0)3}) = l^{i_1}_{(1)1} \cdots l^{i_{r-3}}_{(1)r-3} \partial_{i_1} \cdots \partial_{i_{r-3}} \partial_i T_{IJK} l^I_{(0)1} l^J_{(0)2} l^K_{(0)3}, \end{aligned} \tag{4.1.6}$$

where $n \geqslant 1$, $m \geqslant 2$ and $r \geqslant 3$, and $l_{(i)} \in \mathsf{L}_i$. We defined the products by comparing the general expression for the MC action (3.2.6) with (4.1.2), but they can also be obtained from the (expanded) cohomological vector Q (2.1.13) defined for a Courant algebroid (up to an overall sign that is irrelevant as it does not change $Q^2 = 0$) described as a QP2-manifold in [53] and the end of section 2.1.1. Tensoring products (4.1.6) with the de Rham complex using (3.1.2) and (3.1.3) produces for the physical fields:

$$\begin{aligned} \mu_1'(X) &= \mathrm{d}X, \\ \mu_1'(A) &= \mathrm{d}A - \mu_1(A), \end{aligned} \tag{4.1.7}$$

$$\mu'_1(F) = \mathrm{d}F + \mu_1(F),$$

and in general with $n \geqslant 2$ and $m \geqslant 3$:

$$\begin{aligned}
\mu'_n(X_1, \ldots, X_{n-1}, A) &= -X_1^{i_1} \cdots X_{n-1}^{i_{n-1}} \partial_{i_1} \cdots \partial_{i_{n-1}} \rho^i{}_I A^I, \\
\mu'_n(X_1, \ldots, X_{n-1}, F) &= -X_1^{i_1} \cdots X_{n-1}^{i_{n-1}} \partial_{i_1} \cdots \partial_{i_{n-1}} \rho^i{}_J F_i \hat{\eta}^{IJ}, \\
\mu'_n(X_1, \ldots, X_{n-2}, A_1, A_2) &= X_1^{i_1} \cdots X_{n-2}^{i_{n-2}} \partial_{i_1} \cdots \partial_{i_{n-2}} T_{JKL} A_1^K A_2^L \hat{\eta}^{IJ}, \\
\mu'_n(X_1, \ldots, X_{n-2}, F, A) &= X_1^{i_1} \cdots X_{n-2}^{i_{n-2}} \partial_{i_1} \cdots \partial_{i_{n-2}} \partial_i \rho^j{}_I A^I F_j, \\
\mu'_m(X_1, \ldots, X_{m-3}, A_1, A_2, A_3) &= -X_1^{i_1} \cdots X_{m-3}^{i_{m-3}} \partial_{i_1} \cdots \partial_{i_{m-3}} \partial_i T_{IJK} A_1^I A_2^J A_3^K.
\end{aligned} \tag{4.1.8}$$

Finally, as we are given the action (4.1.2), we need to define a consistent inner product. We find that the following choice

$$\langle l_{(0)1}, l_{(0)2} \rangle \equiv \hat{\eta}_{IJ} l^I_{(0)1} l^J_{(0)2}, \quad \langle l_{(1)}, l_{(-1)} \rangle \equiv l^i_{(1)} l_{(-1)i}, \quad \langle l_{(-1)}, l_{(1)} \rangle \equiv -l^i_{(-1)} l_{(1)i}. \tag{4.1.9}$$

satisfies the cyclicity condition (3.1.4). Moreover, it defines the pairing on the pullback bundle X^*E. Having defined all ingredients we proceed to calculate the Maurer-Cartan action using (3.2.6). From the combinatorics of the decomposition of $\mu'_n(a = X + A + F, \ldots, a = X + A + F)$ and the fact that all higher products of physical fields are symmetric in μ' we obtain:

$$\begin{aligned}
\mu'_n(a, \ldots, a) = n\mu'_n(X, \ldots, X, A) + n\mu'_n(X, \ldots, X, F) + \tfrac{1}{2} n(n-1)\mu'_n(X, \ldots, X, A, A) + \\
+ n(n-1)\mu'_n(x, \ldots, X, F, A) + \tfrac{1}{3!} n(n-1)(n-2)\mu'_n(X, \ldots, X, A, A, A).
\end{aligned} \tag{4.1.10}$$

Making use of this decomposition and (3.1.3) the Maurer-Cartan homotopy action (3.2.6):

$$\begin{aligned}
S_{\mathrm{MC}}[X, A, F] = \langle \mathrm{d}X, F \rangle + \tfrac{1}{2} \langle A, \mathrm{d}A \rangle + \sum_{n \geqslant 0} \tfrac{1}{n!} \langle A, \mu_{n+1}(X, \ldots, X, F) \rangle + \\
+ \tfrac{1}{6} \sum_{n \geqslant 0} \tfrac{1}{n!} \langle A, \mu_{n+2}(X, \ldots, X, A, A) \rangle,
\end{aligned}$$

defined by products (4.1.6) and the cyclic inner product (4.1.9), indeed corresponds to the desired action (4.1.1) or (4.1.2). One can also calculate the equations of motion (3.2.1):

$$\begin{aligned}
f_1 &= \mathrm{d}X - \sum_{n \geqslant 0} \tfrac{1}{n!} \mu_{n+1}(X, \ldots, X, A), \\
f_0 &= \mathrm{d}A + \sum_{n \geqslant 0} \tfrac{1}{n!} \mu_{n+1}(X, \ldots, X, F) + \tfrac{1}{2} \sum_{n \geqslant 0} \tfrac{1}{n!} \mu_{n+2}(X, \ldots, X, A, A), \\
f_{-1} &= \mathrm{d}F - \sum_{n \geqslant 0} \tfrac{1}{n!} \mu_{n+2}(X, \ldots, X, F, A) - \tfrac{1}{3!} \sum_{n \geqslant 0} \tfrac{1}{n!} \mu_{n+3}(X, \ldots, X, A, A, A).
\end{aligned}$$

With the aid of (4.1.6) it becomes obvious that these indeed correspond to the equations of motion of action (4.1.1) or (4.1.2) (see section 2.1.2 as a comparison):

$$\mathcal{D}X^i = \mathrm{d}X^i - \rho^i{}_J(X)A^J, \tag{4.1.11}$$

$$\mathcal{D}A^I = \mathrm{d}A^I - \hat{\eta}^{IJ}\rho^j{}_J(X)F_j + \tfrac{1}{2}\hat{\eta}^{IJ}T_{JKL}(X)A^K A^L, \tag{4.1.12}$$

$$\mathcal{D}F_i = \mathrm{d}F_i + \partial_i\rho^j{}_J(X)A^J F_j - \tfrac{1}{3!}\partial_i T_{IJK}(X)A^I A^J A^K. \tag{4.1.13}$$

4.1.2 | Gauge symmetry

Moving on now to homogeneous subspaces L'_0 and L'_{-1}, they contain gauge parameters ϵ and t, and v, respectively. The gauge variations (3.2.2) are:

$$\begin{aligned}
\delta_{(\epsilon,t)}X &= \sum_{n\geqslant 0}\tfrac{1}{n!}\mu_{n+1}(X,\dots,X,\epsilon),\\
\delta_{(\epsilon,t)}A &= \mathrm{d}\epsilon - \sum_{n\geqslant 0}\tfrac{1}{n!}\mu_{n+1}(X,\dots,X,t) + \sum_{n\geqslant 0}\tfrac{1}{n!}\mu_{n+2}(X,\dots,X,A,\epsilon),\\
\delta_{(\epsilon,t)}F &= \mathrm{d}t + \sum_{n\geqslant 0}\tfrac{1}{n!}\mu_{n+2}(X,\dots,X,A,t) + \sum_{n\geqslant 0}\tfrac{1}{n!}\mu_{n+2}(X,\dots,X,F,\epsilon) +\\
&\quad + \tfrac{1}{2}\sum_{n\geqslant 0}\tfrac{1}{n!}\mu_{n+3}(X,\dots,X,A,A,\epsilon),
\end{aligned}$$

which corresponds to the standard gauge variations of the Courant sigma model as shown in section 2.1.2 or the original ref. [19]:[3]

$$\delta_{(\epsilon,t)}X^i = \rho^i{}_J(X)\epsilon^J, \tag{4.1.14}$$

$$\delta_{(\epsilon,t)}A^I = \mathrm{d}\epsilon^I + \hat{\eta}^{IJ}\rho^j{}_J(X)t_j + \hat{\eta}^{IJ}T_{JKL}(X)A^K\epsilon^L, \tag{4.1.15}$$

$$\delta_{(\epsilon,t)}F_i = \mathrm{d}t_i + \partial_i\rho^j{}_J(X)A^J t_j - \partial_i\rho^j{}_J(X)\epsilon^J F_j + \tfrac{1}{2}\partial_i T_{IJK}(X)A^I A^J\epsilon^K. \tag{4.1.16}$$

We are left with the higher level 1 gauge transformations of parameters ϵ and t:

$$\begin{aligned}
\delta_v\epsilon &= \sum_{n\geqslant 0}\tfrac{1}{n!}\mu_{n+1}(X,\dots,X,v),\\
\delta_v t &= \mathrm{d}v + \sum_{n\geqslant 0}\tfrac{1}{n!}\mu_{n+2}(X,\dots,X,A,v),
\end{aligned}$$

obtained using (3.2.4), which, using (4.1.6), give:

$$\begin{aligned}
\delta_v\epsilon^I &= -\hat{\eta}^{IJ}\rho^j{}_J(X)v_j,\\
\delta_v t_i &= \mathrm{d}v_i + \partial_i\rho^j{}_J(X)v_j A^J.
\end{aligned}$$

[3] As commented in appendix 4.B a careful comparison to expressions (2.1.15)–(2.1.17) shows some sign differences here. This is nothing more that a difference of convention and can be rectified by a parameter redefinition $t \to -t$.

It is perhaps worth noting that while these higher transformations have no meaning in classical field theory and are just algebraic, they suggest more structure which becomes relevant as we move towards quantisation and the BV/BRST formulation of the model.

4.1.3 | Homotopy identities

One crucial element of our construction of L has been so far omitted, namely the homotopy Jacobi identities (3.1.1) of products (4.1.6). To see what constraints these conditions place on our theory we shall calculate them explicitly now. As a reminder we state the first three identities of (3.1.1) again:

$$\begin{aligned}
\mu_1(\mu_1(l)) &= 0\\
\mu_1(\mu_2(l_1,l_2)) &= \mu_2(\mu_1(l_1),l_2) - (-1)^{|l_1||l_2|}\mu_2(\mu_1(l_2),l_1)\\
\mu_1(\mu_3(l_1,l_2,l_3)) &= \mu_2(\mu_2(l_1,l_2),l_3) - (-1)^{|l_2||l_3|}\mu_2(\mu_2(l_1,l_3),l_2) +\\
&\quad + (-1)^{|l_1|(|l_2|+|l_3|)}\mu_2(\mu_2(l_2,l_3),l_1) - \mu_3(\mu_1(l_1),l_2,l_3) +\\
&\quad + (-1)^{|l_1||l_2|}\mu_3(\mu_1(l_2),l_1,l_3) - (-1)^{|l_3|(|l_1|+|l_2|)}\mu_3(\mu_1(l_3),l_1,l_2).
\end{aligned}$$

The products (4.1.6) give the following conditions for some of the first three identities:[4]

$$\begin{array}{llcr}
i=1: & l=l_{(1)}: & & \text{trivial}\\
 & l=l_{(0)}: & & \text{trivial}\\
 & l=l_{(-1)}: & \Rightarrow & \hat\eta^{IJ}\rho^i{}_I\rho^j{}_J=0\\
i=2: & l_{1,2}=l_{(1)1},l_{(1)2}: & & \text{trivial}\\
 & l_{1,2}=l_{(1)},l_{(0)}: & & \text{trivial}\\
 & l_{1,2}=l_{(1)},l_{(-1)}: & \Rightarrow & \partial_i(\hat\eta^{IJ}\rho^j{}_I\rho^k{}_J)=0\\
 & l_{1,2}=l_{(0)1},l_{(0)2}: & \Rightarrow & 2\rho^j{}_{[I}\partial_{\underline{j}}\rho^i{}_{J]}-\rho^i{}_M\hat\eta^{ML}T_{LIJ}=0\\
 & l_{1,2}=l_{(0)},l_{(-1)}: & \Rightarrow & 2\rho^j{}_{[I}\partial_{\underline{j}}\rho^i{}_{J]}-\rho^i{}_M\hat\eta^{ML}T_{LIJ}=0\\
 & l_{1,2}=l_{(-1)1},l_{(-1)2}: & \Rightarrow & \partial_i(\hat\eta^{IJ}\rho^j{}_I\rho^k{}_J)=0\\
i=3: & l_{1,2,3}=l_{(0)1},l_{(0)2},l_{(0)3}: & \Rightarrow & 3\rho^i{}_{[A}\partial_{\underline{i}}T_{BC]J}-\rho^i{}_J\partial_iT_{ABC}-3T_{JK[A}\hat\eta^{KM}T_{BC]M}=0
\end{array}$$

In general, for arbitrary $i \geqslant 1$ we have at most seven nontrivial homotopy identities of which only three are unique, as we show explicitly in Appendix 4.A. These three sets of homotopy conditions for the higher products are actually all terms in the Taylor expansions of the axioms of the Courant algebroid (taking $l_{(1)i} = X_i$) of which each order must hold separately (compare with (2.1.8)–(2.1.10)). Classically, these follow from the

[4] As before underlined indices are not antisymmetrised.

gauge invariance of the action (4.1.1) (as seen in section 2.1.2 eq. (2.1.21)–(2.1.23)):

$$\begin{aligned} \hat{\eta}^{IJ}\rho^i{}_I(X)\rho^j{}_J(X) &= 0, \\ 2\rho^j{}_{[I}(X)\partial_{\underline{j}}\rho^i{}_{J]}(X) - \rho^i{}_M(X)\hat{\eta}^{ML}T_{LIJ}(X) &= 0, \\ 3\rho^i{}_{[A}(X)\partial_{\underline{i}}T_{BC]J}(X) - \rho^i{}_J(X)\partial_i T_{ABC}(X) - 3T_{JK[A}(X)\hat{\eta}^{KM}T_{BC]M}(X) &= 0. \end{aligned} \tag{4.1.17}$$

As expected, all these identities "live" in the L'_3 space, the space of Bianchi identities (3.2.7) as they are equivalent to the homotopy relations. These are in turn equivalent to Noether identities of classical field theory because of the existence of inner products (4.1.9) (see discussion at the end of sec. 3.2.1). In the worldsheet approach to non-geometric string backgrounds, these conditions were seen to originate from generalised Wess-Zumino terms giving expressions for fluxes and their Bianchi identities. The 'non-geometric' here means that the string background fields defined over overlapping open neighbourhoods are patched using diffeomorphisms, and gauge and T-duality transformations. In the L_∞-algebra formulation of the CSM, the expressions for the corresponding fluxes and their Bianchi identities result from higher gauge symmetries encoded in the homotopy relations. The benefit of this interpretation is that one knows how to extend the obtained classical expressions to the full BV/BRST action, as we discuss in the next section.

4.2 | L_∞ for BV/BRST Courant sigma model

Moving away from our classical results and going towards quantisation one encounters the need for BRST symmetry. BRST at the most trivial level is the promotion of gauge parameters to (propagating) *ghost* fields. For certain gauge theories this is not enough and one must introduce more fields (often called antifields) to be able to quantise, this is the Batalin-Vilkovisky procedure. The Courant sigma model is one such theory as its gauge algebra is open (see e.g. [46]), one could also see this from the existence of a higher gauge parameter in the classical L_∞ picture of the previous sections. In the next two sections we shall give an overview of how to discover BV/BRST within L_∞ following section 3.2.2 and then use this to calculate the generalised BRST transformations and BV action for the Courant sigma model.

4.2.1 | Maurer-Cartan BV for the CSM

In BRST quantisation gauge parameters ϵ and t become ghost fields, however, since the gauge algebra is a reducible one the higher gauge field becomes a scalar ghost-for-ghost field: v.[5] As stated in the previous section to complete the BV formalism we extend this BRST complex with the antifields corresponding to each BRST field. Following the

[5] Denoted the same as the gauge parameter they originate from.

construction of sec. 3.2.2 we assign these fields to the three L_∞-algebra (L, μ_i) spaces as shown in table 4.1.

$\Omega^\bullet$	L_{-1}	L_0	L_1
0	v	ϵ	X
1	t	A	$F^\dagger$
2	F	$A^\dagger$	$t^\dagger$
3	$X^\dagger$	$\epsilon^\dagger$	$v^\dagger$

gh $c_{-1} = 2$, $|c_{-1}|_{\mathsf{L}'} = -1$

gh $c_0 = 1$, $|c_0|_{\mathsf{L}'} = 0$

gh $a = 0$, $|a|_{\mathsf{L}'} = 1$

gh $a^\dagger = -1$, $|a^\dagger|_{\mathsf{L}'} = 2$

gh $c_0^\dagger = -2$, $|c_0^\dagger|_{\mathsf{L}'} = 3$

gh $c_{-1} = -3$, $|c_{-1}^\dagger|_{\mathsf{L}'} = 4$

Table 4.1
Degrees of fields.

Therefore the complete BV field content is (the number in the square bracket indicates the fields' ghost degree):

$$
\begin{aligned}
a &= X + A + F \in \Omega^0(\Sigma_3, \mathsf{L}_1)[0] \oplus \Omega^1(\Sigma_3, \mathsf{L}_0)[0] \oplus \Omega^2(\Sigma_3, \mathsf{L}_{-1})[0], \\
a^\dagger &= X^\dagger + A^\dagger + F^\dagger \in \Omega^3(\Sigma_3, \mathsf{L}_{-1})[-1] \oplus \Omega^2(\Sigma_3, \mathsf{L}_0)[-1] \oplus \Omega^1(\Sigma_3, \mathsf{L}_1)[-1], \\
c_0 &= \epsilon + t \in \Omega^0(\Sigma_3, \mathsf{L}_0)[1] \oplus \Omega^1(\Sigma_3, \mathsf{L}_{-1})[1], \\
c_{-1} &= v \in \Omega^0(\Sigma_3, \mathsf{L}_{-1})[2], \\
c_0^\dagger &= \epsilon^\dagger + t^\dagger \in \Omega^3(\Sigma_3, \mathsf{L}_0)[-2] \oplus \Omega^2(\Sigma_3, \mathsf{L}_1)[-2], \\
c_{-1}^\dagger &= v^\dagger \in \Omega^3(\Sigma_3, \mathsf{L}_1)[-3],
\end{aligned}
\tag{4.2.1}
$$

where fields with the same ghost number are collected. Additionally, these L' fields can also be combined into the BV *superfield* a:

$$
\mathsf{a} = a + a^\dagger + c_0 + c_0^\dagger + c_{-1} + c_{-1}^\dagger,
$$

of $\hat{\mathsf{L}}$-degree 1. There is a slight abuse of notation here as a, $a^\dagger$, c_0, $c_0^\dagger$, c_{-1} and $c_{-1}^\dagger$ have two meanings: they are elements of L' as was the case in (4.2.1), however, in a they imply being elements of $\mathcal{C}^\infty(\mathsf{L}'[1]) \otimes \mathsf{L}'$ as well, this is for reasons of brevity and the desired meaning should be clear from context. It is important to notice this is *not* the superfield in the sense of sec. 2.1.3 rel. (2.1.25)–(2.1.27), it is simply the collection of all elements of $\hat{\mathsf{L}}$ degree 1. Using the properties of cyclic inner product for L_∞-algebras (3.1.4) and tensored L_∞-algebras (3.2.9) and the combinatorics of decomposing a just as in (4.1.10), the BV action (3.2.13) becomes:

$$\begin{aligned}
S_{BV} = \int_{\Sigma_3} & \langle \mathrm{d}X, F\rangle + \tfrac{1}{2}\langle A, \mathrm{d}A\rangle + \\
& + \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\langle F, \mu_{n+1}(X,\dots,X,A)\rangle + \tfrac{1}{6}\langle A, \mu_{n+2}(X,\dots,X,A,A)\rangle\Big) + \\
& + \int_{\Sigma_3} -\langle F^\dagger, \mathrm{d}t\rangle - \langle A^\dagger, \mathrm{d}\epsilon\rangle - \langle t^\dagger, \mathrm{d}v\rangle + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(- \langle v^\dagger, \mu_{n+2}(X,\dots,X,v,\epsilon)\rangle - \\
& - \langle t^\dagger, \mu_{n+3}(X,\dots,X,F^\dagger,v,\epsilon)\rangle - \tfrac{1}{2}\langle A, \mu_{n+3}(X,\dots,X,F^\dagger,F^\dagger,v)\rangle + \\
& + \tfrac{1}{6}\langle F^\dagger, \mu_{n+4}(X,\dots,X,F^\dagger,F^\dagger,v,\epsilon)\rangle + \langle t^\dagger, \mu_{n+2}(X,\dots,X,v,A)\rangle + \\
& + \langle A^\dagger, \mu_{n+2}(X,\dots,X,F^\dagger,v)\rangle + \langle \epsilon^\dagger, \mu_{n+1}(X,\dots,X,v)\rangle + \\
& + \tfrac{1}{2}\langle t, \mu_{n+3}(X,\dots,X,F^\dagger,F^\dagger,\epsilon)\rangle - \langle t, \mu_{n+2}(X,\dots,X,F^\dagger,A)\rangle + \\
& + \langle A^\dagger, \mu_{n+1}(X,\dots,X,t)\rangle - \langle t^\dagger, \mu_{n+2}(X,\dots,X,t,\epsilon)\rangle - \\
& - \langle F, \mu_{n+2}(X,\dots,X,F^\dagger,\epsilon)\rangle - \langle X^\dagger, \mu_{n+1}(X,\dots,X,\epsilon)\rangle - \\
& - \tfrac{1}{6}\langle v^\dagger, \mu_{n+3}(X,\dots,X,\epsilon,\epsilon,\epsilon)\rangle - \tfrac{1}{6}\langle t^\dagger, \mu_{n+4}(X,\dots,X,F^\dagger,\epsilon,\epsilon,\epsilon)\rangle + \\
& + \tfrac{1}{2}\langle t^\dagger, \mu_{n+3}(X,\dots,X,A,\epsilon,\epsilon)\rangle + \tfrac{1}{2}\langle \epsilon^\dagger, \mu_{n+2}(X,\dots,X,\epsilon,\epsilon)\rangle - \\
& - \tfrac{1}{4}\langle A, \mu_{n+4}(X,\dots,X,F^\dagger,F^\dagger,\epsilon,\epsilon)\rangle + \tfrac{1}{2}\langle A^\dagger, \mu_{n+3}(X,\dots,X,F^\dagger,\epsilon,\epsilon)\rangle + \\
& + \tfrac{1}{2}\langle A, \mu_{n+3}(X,\dots,X,F^\dagger,A,\epsilon)\rangle - \langle A^\dagger, \mu_{n+2}(X,\dots,X,A,\epsilon)\rangle + \\
& + \tfrac{1}{36}\langle F^\dagger, \mu_{n+5}(X,\dots,X,F^\dagger,F^\dagger,\epsilon,\epsilon,\epsilon)\rangle\Big).
\end{aligned}$$

As explained above the explicit writing of ghost bases will be suppressed. Therefore, in all expressions in which it is not explicitly written it will be assumed $\mu_i'(l_1',\dots,l_i')$ stands for $\zeta_1\cdots\zeta_i\ \mu_i'(l_1',\dots,l_i')$. An equivalent procedure by use of (3.2.10) produces the following components of curvature f or BV/BRST transformations (3.2.12):

$$\begin{aligned}
Q_{BV}v &= \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\mu_{n+2}(X,\dots,X,v,\epsilon) + \tfrac{1}{6}\mu_{n+3}(X,\dots,X,\epsilon,\epsilon,\epsilon)\Big), \\
Q_{BV}\epsilon &= \sum_{n=0}^{\infty}\frac{1}{n!}\Big(- \mu_{n+1}(X,\dots,X,v) - \tfrac{1}{2}\mu_{n+2}(X,\dots,X,\epsilon,\epsilon)\Big), \\
Q_{BV}X &= \sum_{n=0}^{\infty}\frac{1}{n!}\mu_{n+1}(X,\dots,X,\epsilon), \\
Q_{BV}t &= -\mathrm{d}v + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\mu_{n+2}(X,\dots,X,v,A) - \mu_{n+2}(X,\dots,X,t,\epsilon) - \\
&\quad - \mu_{n+3}(X,\dots,X,F^\dagger,v,\epsilon) + \tfrac{1}{2}\mu_{n+3}(X,\dots,X,A,\epsilon,\epsilon) - \\
&\quad - \tfrac{1}{6}\mu_{n+4}(X,\dots,X,F^\dagger,\epsilon,\epsilon,\epsilon)\Big), \\
Q_{BV}A &= \mathrm{d}\epsilon + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(- \mu_{n+1}(X,\dots,X,t) + \mu_{n+2}(X,\dots,X,A,\epsilon) -
\end{aligned}$$

$$
\begin{aligned}
&- \mu_{n+2}(X,\dots,X,F^\dagger,v) - \tfrac{1}{2}\mu_{n+3}(X,\dots,X,F^\dagger,\epsilon,\epsilon)\Big),\\
Q_{BV}F^\dagger &= -\mathrm{d}X + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\mu_{n+1}(X,\dots,X,A) - \mu_{n+2}(X,\dots,X,F^\dagger,\epsilon)\Big),\\
Q_{BV}F &= \mathrm{d}t + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\mu_{n+2}(X,\dots,X,t,A) + \mu_{n+2}(X,\dots,X,F,\epsilon) +\\
&\qquad + \tfrac{1}{2}\mu_{n+3}(X,\dots,X,A,A,\epsilon) +\\
&+ \mu_{n+3}(X,\dots,X,F^\dagger,v,A) - \mu_{n+3}(X,\dots,X,F^\dagger,t,\epsilon) +\\
&+ \mu_{n+3}(X,\dots,X,t^\dagger,v,\epsilon) + \tfrac{1}{2}\mu_{n+3}(X,\dots,X,A^\dagger,\epsilon,\epsilon) +\\
&+ \tfrac{1}{2}\mu_{n+4}(X,\dots,X,F^\dagger,A,\epsilon,\epsilon) - \tfrac{1}{12}\mu_{n+5}(X,\dots,X,F^\dagger,F^\dagger,\epsilon,\epsilon,\epsilon) +\\
&+ \tfrac{1}{6}\mu_{n+4}(X,\dots,X,t^\dagger,\epsilon,\epsilon,\epsilon) - \tfrac{1}{2}\mu_{n+4}(X,\dots,X,F^\dagger,F^\dagger,v,\epsilon) +\\
&+ \mu_{n+2}(X,\dots,X,v,A^\dagger)\Big),\\
Q_{BV}A^\dagger &= -\mathrm{d}A + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(- \tfrac{1}{2}\mu_{n+2}(X,\dots,X,A,A) - \mu_{n+1}(X,\dots,X,F) -\\
&- \mu_{n+2}(X,\dots,X,t^\dagger,v) - \mu_{n+2}(X,\dots,X,A^\dagger,\epsilon) + \mu_{n+2}(X,\dots,X,F^\dagger,t) -\\
&- \mu_{n+3}(X,\dots,X,F^\dagger,A,\epsilon) + \tfrac{1}{4}\mu_{n+4}(X,\dots,X,F^\dagger,F^\dagger,\epsilon,\epsilon) +\\
&+ \tfrac{1}{2}\mu_{n+3}(X,\dots,X,F^\dagger,F^\dagger,v) - \tfrac{1}{2}\mu_{n+3}(X,\dots,X,t^\dagger,\epsilon,\epsilon)\Big),\\
Q_{BV}t^\dagger &= -\mathrm{d}F^\dagger + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\mu_{n+1}(X,\dots,X,A^\dagger) + \mu_{n+2}(X,\dots,X,F^\dagger,A) -\\
&- \tfrac{1}{2}\mu_{n+3}(X,\dots,X,F^\dagger,F^\dagger,\epsilon) + \mu_{n+2}(X,\dots,X,t^\dagger,\epsilon)\Big),\\
Q_{BV}X^\dagger &= -\mathrm{d}F + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\mu_{n+2}(X,\dots,X,F,A) + \tfrac{1}{6}\mu_{n+3}(X,\dots,X,A,A,A) -\\
&- \mu_{n+2}(X,\dots,X,X^\dagger,\epsilon) - \mu_{n+3}(X,\dots,X,F^\dagger,v,A^\dagger) -\\
&- \mu_{n+3}(X,\dots,X,F^\dagger,F,\epsilon) - \tfrac{1}{2}\mu_{n+4}(X,\dots,X,F^\dagger,F^\dagger,v,A) +\\
&+ \tfrac{1}{6}\mu_{n+5}(X,\dots,X,F^\dagger,F^\dagger,F^\dagger,v,\epsilon) + \mu_{n+3}(X,\dots,X,t^\dagger,v,A) -\\
&- \mu_{n+4}(X,\dots,X,F^\dagger,t^\dagger,v,\epsilon) - \mu_{n+3}(X,\dots,X,v^\dagger,v,\epsilon) -\\
&- \mu_{n+3}(X,\dots,X,A^\dagger,A,\epsilon) - \mu_{n+3}(X,\dots,X,F^\dagger,t,A) -\\
&- \tfrac{1}{2}\mu_{n+4}(X,\dots,X,F^\dagger,A^\dagger,\epsilon,\epsilon) - \tfrac{1}{4}\mu_{n+5}(X,\dots,X,F^\dagger,F^\dagger,A,\epsilon,\epsilon) +\\
&+ \tfrac{1}{36}\mu_{n+6}(X,\dots,X,F^\dagger,F^\dagger,F^\dagger,\epsilon,\epsilon,\epsilon) + \tfrac{1}{2}\mu_{n+4}(X,\dots,X,t^\dagger,A,\epsilon,\epsilon) -\\
&- \tfrac{1}{6}\mu_{n+5}(X,\dots,X,F^\dagger,t^\dagger,\epsilon,\epsilon,\epsilon) - \tfrac{1}{6}\mu_{n+4}(X,\dots,X,v^\dagger,\epsilon,\epsilon,\epsilon) -\\
&- \mu_{n+2}(X,\dots,X,t,A^\dagger) + \tfrac{1}{2}\mu_{n+4}(X,\dots,X,F^\dagger,F^\dagger,t,\epsilon) -\\
&- \tfrac{1}{2}\mu_{n+4}(X,\dots,X,F^\dagger,A,A,\epsilon) - \mu_{n+3}(X,\dots,X,t^\dagger,t,\epsilon) +\\
&+ \tfrac{1}{2}\mu_{n+3}(X,\dots,X,\epsilon^\dagger,\epsilon,\epsilon) + \mu_{n+2}(X,\dots,X,v,\epsilon^\dagger)\Big),
\end{aligned}
$$

$$
\begin{aligned}
Q_{BV}\epsilon^\dagger = \mathrm{d}A^\dagger + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(& -\mu_{n+1}(X,\dots,X,X^\dagger) - \mu_{n+2}(X,\dots,X,F^\dagger,F) + \\
&+ \tfrac{1}{6}\mu_{n+4}(X,\dots,X,F^\dagger,F^\dagger,F^\dagger,v) - \mu_{n+2}(X,\dots,X,t^\dagger,t) - \\
&- \mu_{n+2}(X,\dots,X,v^\dagger,v) + \mu_{n+2}(X,\dots,X,\epsilon^\dagger,\epsilon) - \\
&- \tfrac{1}{2}\mu_{n+3}(X,\dots,X,F^\dagger,A,A) - \mu_{n+3}(X,\dots,X,F^\dagger,A^\dagger,\epsilon) - \\
&- \mu_{n+2}(X,\dots,X,A^\dagger,A) + \mu_{n+3}(X,\dots,X,t^\dagger,A,\epsilon) - \\
&- \tfrac{1}{2}\mu_{n+3}(X,\dots,X,v^\dagger,\epsilon,\epsilon) + \tfrac{1}{2}\mu_{n+3}(X,\dots,X,F^\dagger,F^\dagger,t) - \\
&- \tfrac{1}{2}\mu_{n+4}(X,\dots,X,F^\dagger,t^\dagger,\epsilon,\epsilon) - \mu_{n+3}(X,\dots,X,F^\dagger,t^\dagger,v) + \\
&+ \tfrac{1}{12}\mu_{n+5}(X,\dots,X,F^\dagger,F^\dagger,F^\dagger,\epsilon,\epsilon) - \tfrac{1}{2}\mu_{n+4}(X,\dots,X,F^\dagger,F^\dagger,A,\epsilon)\Big), \\
Q_{BV}v^\dagger = -\mathrm{d}t^\dagger + \sum_{n=0}^{\infty}\frac{1}{n!}\Big(& \mu_{n+1}(X,\dots,X,\epsilon^\dagger) - \mu_{n+2}(X,\dots,X,F^\dagger,A^\dagger) + \\
&+ \tfrac{1}{6}\mu_{n+4}(X,\dots,X,F^\dagger,F^\dagger,F^\dagger,\epsilon) + \mu_{n+2}(X,\dots,X,t^\dagger,A) - \\
&- \mu_{n+2}(X,\dots,X,v^\dagger,\epsilon) - \tfrac{1}{2}\mu_{n+3}(X,\dots,X,F^\dagger,F^\dagger,A) - \\
&- \mu_{n+3}(X,\dots,X,F^\dagger,t^\dagger,\epsilon)\Big).
\end{aligned}
$$

Introducing our selection for the higher products (4.1.6) and inner product (4.1.9), and then resumming the Taylor expansions produces the full BV action (relation (2.1.24) expanded by (2.1.25)–(2.1.27) as obtained by the AKSZ procedure in [18]). The explicit results are given in Appendix 4.B.

In the next section we would like to relate this L_∞-algebra corresponding to the Courant sigma model with the 2-term L_∞-algebra for a Courant algebroid in Ref. [52].

4.3 | L_∞-morphisms

Roytenberg has shown [18] that given the data of a Courant algebroid one can uniquely construct the corresponding Courant sigma model. Moreover, Roytenberg and Weinstein showed [52] that a Courant algebroid can be described as a 2-term L_∞-algebra. This naturally raises the question of the relation between the L_∞-algebra we constructed for the CSM and the one defined in Ref. [52]. In order to construct the morphism between these two algebras we need to extend the L_∞-algebra they constructed which essentially used a graded vector space concentrated in just two degrees, whereas the physical fields in (L,μ_i) live in a graded vector space of three homogeneous subspaces. The construction of this morphism can be thought of as reproducing Roytenbergs result [18] that given the data of a Courant algebroid one can uniquely construct the Courant sigma model (up to the additional structure of a measure on the source space) now fully in the L_∞-algebra formalism.

4.3.1 | L_∞-algebra for a Courant algebroid

In [52] it was shown an L_∞-algebra can be constructed from a Courant algebroid by starting with the exact sequence (adjusted to match our conventions of degrees):

$$\widetilde{\mathsf{L}}_{-2} \xrightarrow{\imath} \widetilde{\mathsf{L}}_{-1} \xrightarrow{\mathcal{D}} \widetilde{\mathsf{L}}_0 \to \mathrm{coker}\mathcal{D}$$

with $\widetilde{\mathsf{L}}_{-2} = \ker \mathcal{D}$, $\widetilde{\mathsf{L}}_{-1} = C^\infty(M)$ and $\widetilde{\mathsf{L}}_0 = \Gamma(E)$, and $\imath : \ker \mathcal{D} \hookrightarrow C^\infty(M)$ the inclusion of constants as functions. and non-vanishing products:

$$\begin{aligned}
\tilde{\mu}_1(c) &= \imath c, \\
\tilde{\mu}_1(f) &= \mathcal{D} f, \\
\tilde{\mu}_2(e_1, e_2) &= [e_1, e_2]_C, \\
\tilde{\mu}_2(e, f) &= \langle e, \mathcal{D} f\rangle, \\
\tilde{\mu}_3(e_1, e_2, e_3) &= \mathcal{N}(e_1, e_2, e_3),
\end{aligned} \tag{4.3.1}$$

with $c \in \ker \mathcal{D}$, $f \in C^\infty(M)$ and $e \in \Gamma(E)$. The space coker$\mathcal{D}$ is not part of the algebra as no maps lead to it or take arguments from it. The maps are defined in terms of structures on a Courant algebroid defined in section 2.1.1, i.e.: map $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ of rel. (2.1.2), a skew-symmetric bracket on sections of bundle E over manifold M, a symmetric bilinear form $\langle\ ,\ \rangle$ and tensor $\mathcal{N}$ (recall the definition in Property 2.1.7) representing the obstruction to the Jacobi identity of the bracket (2.1.1). In the definition of a Courant algebroid (see sec. 2.1.1) these structures satisfy five compatibility conditions (2.1.1), (2.1.3)–(2.1.6) which are in the L_∞ formulation contained in the homotopy relations. However, as the space of constants only appears due to the underlying exact sequence and plays no nontrivial role in the calculation of the homotopy relations it can be omitted without any modification. For this reason in the ensuing analysis we drop this space and the corresponding $\tilde{\mu}_1$ map from consideration without loss of generality.

To make the connection with our sigma model algebra (4.1.4) and (4.1.6), we must extend[6] the chain complex of this algebra by an additional space of degree 1, $\widetilde{\mathsf{L}}_1 = T_pM$:

$$\widetilde{\mathsf{L}}_{-1} \xrightarrow{\mathcal{D}} \widetilde{\mathsf{L}}_0 \xrightarrow{\tilde{\rho}} \widetilde{\mathsf{L}}_1, \tag{4.3.2}$$

where,

$$\tilde{\mu}_1(e) = \tilde{\rho}(e)\Big|_p, \tag{4.3.3}$$

is the map $\tilde{\rho} : E \to TM$ and $p \in M$ is a point on manifold M. We will denote elements of $\widetilde{\mathsf{L}}_1$ by $h \in T_pM$. Calculation of the homotopy identities (3.1.1) (for details see Appendix

[6] In nomenclature of Ref. [74] we have to extend the pure gauge algebra to an algebra including additional field which will correspond to the field X in CSM. The reason is that the CSM algebra is field dependent.

4.C) provides the minimal extension to the higher products (4.3.1) necessary to make (4.3.2) an L_∞-algebra:

$$\tilde{\mu}_n(h_1, \ldots, h_{n-1}, e) = h_1^{i_1} \cdots h_{n-1}^{i_{n-1}} \tilde{\partial}_{i_1} \cdots \tilde{\partial}_{i_{n-1}} (\tilde{\rho}(e)^i)\Big|_p, \qquad n \in \mathbb{N}, \tag{4.3.4}$$

where the basis of T_pM is the one induced by coordinates x^i of a coordinate patch $U \subset M$ that contains point p such that $x^i(p) = 0$. It is important to note that this extended algebra also corresponds to the Courant algebroid as did the original L_2 formulation since no new properties, other than the Courant algebroid axioms, were needed. The extended structure is more natural, however, in case we want to define a Courant algebroid being given only the L_∞-algebra. On the other hand, going in the opposite direction, Roytenberg and Weinstein showed that if starting from a Courant algebroid one can omit the L_1 homogeneous subspace from the graded space and uniquely define the L_∞ structure. Furthermore, the properties of a Courant algebroid

$$\boxed{\begin{aligned} (\rho \circ \mathcal{D}) f &= 0 \\ \rho[e_1, e_2]_C - [\rho(e_1), \rho(e_2)] &= 0 \\ \mathrm{Jac}(e_1, e_2, e_3) - \mathcal{D}\mathcal{N}_c(e_1, e_2, e_3) &= 0 \end{aligned}} \tag{4.3.5}$$

come out in this form from the homotopy relations of the extended L_∞-algebra.

4.3.2 | L_∞-morphism from CA to CSM

Now, we shall construct the morphism ϕ to our CSM algebra in a pointwise fashion i.e. $\phi : \widetilde{\mathsf{L}} \times \cdots \times \widetilde{\mathsf{L}} \to \mathsf{L}\Big|_{X(\sigma)=p}$ for $\sigma \in \Sigma_3$, since the Courant sigma model is only locally defined. We begin analogously to the extension procedure of the previous paragraph given explicitly in Appendix 4.C. The construction will follow orders of i in (3.1.5).

$\underline{i = 1}$

To begin, we start by the lowest order of (3.1.5) which encompasses two non-trivial conditions:

$$\begin{aligned} \phi_1(\tilde{\mu}_1(e)) &= \mu_1(\phi_1(e)), \\ \phi_1(\tilde{\mu}_1(f)) &= \mu_1(\phi_1(f)). \end{aligned}$$

By use of the first equation in (4.1.6) and (4.3.3) the first relation gives:

$$\phi_1(\tilde{\rho}(e)^i\Big|_p) = \rho^i{}_I \phi_1(e)^I,$$

whereas the second equation in (4.1.6) and (4.3.1) produce:

$$\phi_1(\mathcal{D}f)^I = -\hat{\eta}^{IJ}\rho^i{}_J\phi_1(f)_i.$$

These two relations imply ϕ_1 to be:

$$\phi_1(h) = X^*h, \tag{4.3.6}$$

$$\phi_1(e) = X^*e\Big|_p, \tag{4.3.7}$$

$$\phi_1(f) = -X^*\tilde{\mathrm{d}}f\Big|_p, \tag{4.3.8}$$

where we used $\langle\mathcal{D}f, e\rangle = \frac{1}{2}\rho(e)f$ and $\rho^i{}_J \equiv \rho^i{}_J(\{X^j\} = 0) = \rho^i{}_J(\{X^*x^j\} = 0)$ and $x^i(p) = 0$.

<u>$i = 2$</u>

In this case we have four non-trivial morphism conditions. The first is $(l_1, l_2) = (e_1, e_2)$:

$$-\phi_2(\tilde{\mu}_1(e_1), e_2) + \phi_2(\tilde{\mu}(e_2), e_1) + \phi_1(\tilde{\mu}_2(e_1, e_2)) = \mu_1(\phi_2(e_1, e_2)) + \mu_2(\phi_1(e_1), \phi_1(e_2))$$

$$-\phi_2(\tilde{\rho}(e_1)\Big|_p, e_2)^I + \phi_2(\tilde{\rho}(e_2)\Big|_p, e_1)^I + X^*([e_1, e_2]_C)^I\Big|_p = -\hat{\eta}^{IJ}\rho^i{}_J\phi_2(e_1, e_2)_i + \\ + \hat{\eta}^{IJ}T_{JKL}(X^*e_1\Big|_p)^K(X^*e_2\Big|_p)^L.$$

By comparison with the Courant bracket:

$$([e_1, e_2]_C)^I = \tilde{\rho}^i{}_J(e_1^J\tilde{\partial}_i e_2^I - e_2^J\tilde{\partial}_i e_1^I) - \\ - \tfrac{1}{2}\tilde{\rho}^i{}_K(e_1^J\tilde{\partial}_i e_{2J} - e_2^J\tilde{\partial}_i e_{1J})\tilde{\hat{\eta}}^{IK} + \tilde{\hat{\eta}}^{IJ}\tilde{T}_{JKL}e_1^K e_2^L, \tag{4.3.9}$$

we fix two ϕ_2 maps:

$$\phi_2(h, e)^I = X^*(h^i\tilde{\partial}_i e^I)\Big|_p, \tag{4.3.10}$$

$$\phi_2(e_1, e_2) = X^*(\hat{\eta}_{IJ}e^I_{[1}\tilde{\mathrm{d}}e^J_{2]})\Big|_p. \tag{4.3.11}$$

The second condition corresponding to $(l_1, l_2) = (e, f)$ is:

$$\phi_1(\tilde{\mu}_2(e, f)) - \phi_2(\tilde{\mu}_1(e), f) + \phi_2(\tilde{\mu}_1(f), e) = \mu_2(\phi_1(e), \phi_1(f))$$

$$-X^*(\tilde{\mathrm{d}}\langle e, \mathcal{D}f\rangle)_i\Big|_p - \phi_2(\tilde{\rho}(e)\Big|_p, f)_i + \tfrac{1}{2}(X^*(\mathcal{D}_I f\tilde{\mathrm{d}}e^I)\Big|_p)_i - \tfrac{1}{2}(X^*(e^I\tilde{\mathrm{d}}\mathcal{D}_I f)\Big|_p)_i = \\ = -\partial_i\rho^j{}_I(X^*\tilde{\mathrm{d}}f)_j\Big|_p(X^*e)^I\Big|_p,$$

from which we read off:

$$\phi_2(h, f)_i = -X^*(h^j\tilde{\partial}_j\tilde{\partial}_i f)\Big|_p. \tag{4.3.12}$$

For the third combination of elements we take $(l_1, l_2) = (e, h)$:

$$\begin{aligned}\phi_1(\tilde{\mu}_2(e,h)) - \phi_2(\tilde{\mu}_1(e), h) &= \mu_1(\phi_2(e,h)) + \mu_2(\phi_1(e), \phi_1(h)) \\ X^*(h^j \tilde{\partial}_j(\tilde{\rho}(e)^i)\Big|_p) + \phi_2(\tilde{\rho}(e)\Big|_p, h)^i &= \rho^i{}_I(X^*(h^j\tilde{\partial}_j e)\Big|_p)^I + (X^*h)^j \partial_j \rho^i{}_I (X^* e\Big|_p)^I,\end{aligned}$$

that allows us to set:

$$\phi_2(h_1, h_2) = 0. \tag{4.3.13}$$

The final relation with $(l_1, l_2) = (f, h)$:

$$-\phi_2(\tilde{\mu}_1(f), h) = \mu_1(\phi_2(f,h)) + \mu_2(\phi_1(f), \phi_1(h)),$$

is just a consistency check.

$\underline{i = 3}$

Out of the five non-trivial conditions that exist for $i \geqslant 3$ we begin with the combination $(l_1, l_2, l_3) = (e_1, e_2, e_3)$:

$$\begin{aligned}&\tfrac{1}{3}\phi_1(\tilde{\mu}_3(e_1,e_2,e_3)) - \phi_2(\tilde{\mu}_2(e_1,e_2), e_3) + \phi_3(\tilde{\mu}_1(e_1), e_2, e_3) + \text{cyclic} = \\ &\qquad = \tfrac{1}{3}\mu_3(\phi_1(e_1), \phi_1(e_2), \phi_1(e_3)) - \mu_2(\phi_1(e_3), \phi_2(e_1, e_2)) + \text{cyclic}.\end{aligned}$$

This condition implies:

$$\phi_3(h, e_1, e_2)_i = X^*(h^j \tilde{\partial}_j (\hat{\eta}^{IJ} e^I_{[1} \tilde{\partial}_i e^J_{2]}))\Big|_p. \tag{4.3.14}$$

The next combination of elements $(l_1, l_2, l_3) = (e_1, e_2, h)$ gives:

$$\begin{aligned}&\phi_3(\tilde{\mu}_1(e_1), e_2, h) + \phi_2(\tilde{\mu}_2(e_1, h), e_2) + \tfrac{1}{2}\phi_2(\tilde{\mu}_2(e_2, e_1), h) - e_1 \leftrightarrow e_2 = \\ &\quad = \tfrac{1}{2}\mu_1(\phi_3(e_1, e_2, h)) + \mu_2(\phi_1(e_2), \phi_2(e_1, h)) + \tfrac{1}{2}\mu_2(\phi_1(h), \phi_2(e_1, e_2)) + \\ &\qquad + \tfrac{1}{2}\mu_3(\phi_1(e_1), \phi_1(e_2), \phi_1(h)) - e_1 \leftrightarrow e_2.\end{aligned}$$

Expanding this expression and plugging in the previously set definitions for ϕ one can consistently set:

$$\phi_3(e, h_1, h_2)^I = X^*(h_1^{i_1} h_2^{i_2} \tilde{\partial}_{i_1} \tilde{\partial}_{i_2} e^I)\Big|_p. \tag{4.3.15}$$

The third possibility is for $(l_1, l_2, l_3) = (h, f, e)$:

$$\begin{aligned}&\phi_3(\tilde{\mu}_1(f), h, e) + \phi_3(\tilde{\mu}_1(e), h, f) + \phi_2(\tilde{\mu}_2(h, e), f) + \phi_2(\tilde{\mu}_2(f, e), h) = \\ &\qquad = \mu_2(\phi_1(f), \phi_2(h, e)) - \mu_2(\phi_1(e), \phi_2(h, f)) + \mu_3(\phi_1(h), \phi_1(f), \phi_1(e)).\end{aligned}$$

Analogously to the previous cases, here we can set:

$$\phi_3(h_1,h_2,f)_i = -X^*(h_1^{i_1}h_2^{i_2}\tilde{\partial}_{i_1}\tilde{\partial}_{i_2}\tilde{\partial}_i f)\Big|_p. \tag{4.3.16}$$

There are two more combinations of elements with non-trivial conditions: $(l_1,l_2,l_3) = (h_1,h_2,e)$ and $(l_1,l_2,l_3) = (f,h_1,h_2)$, however, these are simply consistency checks that all other ϕ_3 can be set to vanish. We state them for completeness:

$$\begin{aligned}
&\tfrac{1}{2}\phi_3(\tilde{\mu}_1(e),h_1,h_2) + \phi_2(\tilde{\mu}_2(h_1,e),h_2) + \tfrac{1}{2}\phi_1(\tilde{\mu}_3(h_1,h_2,e)) + h_1 \leftrightarrow h_2 = \\
&\quad = \tfrac{1}{2}\mu_1(\phi_3(h_1,h_2,e)) + \mu_2(\phi_1(h_1),\phi_2(h_2,e)) + \tfrac{1}{2}\mu_3(\phi_1(h_1),\phi_1(h_2),\phi_1(e)) + h_1 \leftrightarrow h_2, \\
&\tfrac{1}{2}\phi_3(\tilde{\mu}_1(f),h_1,h_2) + h_1 \leftrightarrow h_2 = \\
&\quad = \tfrac{1}{2}\mu_1(\phi_3(f,h_1,h_2)) + \mu_2(\phi_1(h_1),\phi_2(f,h_2)) + \tfrac{1}{2}\mu_3(\phi_1(f),\phi_1(h_1),\phi_1(h_2)) + h_1 \leftrightarrow h_2.
\end{aligned}$$

<u>$i \geqslant 3$</u>

Since the $i = 3$ case already gives the most general morphism conditions we make the ansatz for the four possible non-vanishing ϕ_i mappings as follows:

$$\phi_i(h_1,\dots,h_{i-1},e)^I = X^*(h_1^{j_1}\cdots h_{i-1}^{j_{i-1}}\tilde{\partial}_{j_1}\cdots\tilde{\partial}_{j_{i-1}}e^I)\Big|_p, \tag{4.3.17}$$

$$\phi_i(h_1,\dots,h_{i-2},e_1,e_2)_j = X^*(h_1^{j_1}\cdots h_{i-2}^{j_{i-2}}\tilde{\partial}_{j_1}\cdots\tilde{\partial}_{j_{i-2}}(\hat{\eta}_{IJ}e^I_{[1}\tilde{\partial}_j e^J_{2]}))\Big|_p, \tag{4.3.18}$$

$$\phi_i(h_1,\dots,h_{i-1},f)_j = -X^*(h_1^{j_1}\cdots h_{i-1}^{j_{i-1}}\tilde{\partial}_{j_1}\cdots\tilde{\partial}_{j_{i-1}}\tilde{\partial}_j f)\Big|_p, \tag{4.3.19}$$

$$\phi_i(h_1,\dots,h_i) = 0. \tag{4.3.20}$$

First of the five non-trivial conditions corresponds to the choice $(l_1,\dots,l_i) = (h_1,\dots,h_{i-1},f)$:

$$(-1)^{i-1}\phi_i(\tilde{\mu}_1(f),h_1,\dots,h_{i-1}) = \sum_{n=1}^{i}\mu_n(\phi_{i-n+1}(h_1,\dots,h_{i-n},f),\phi_1(h_{i-n+1}),\dots,\phi_1(h_{i-1})) + \\ + \text{perm.}$$

which is automatically satisfied by use of the ansatz.[7] Next is the combination $(l_1,\dots,l_i) = (h_1,\dots,h_{i-1},e)$:

$$\phi_1(\tilde{\mu}_i(h_1,\dots,h_{i-1},e)) = \sum_{n=1}^{i}\mu_n(\phi_1(h_1),\dots,\phi_1(h_{n-1}),\phi_{i-n+1}(h_n,\dots,h_{i-1},e)) + \\ + \text{perm.}$$

[7] Here "perm." will indicate all possible unshuffles of $h_1,\dots,h_{i-1}$.

that is also automatically satisfied. The third case is $(l_1, \dots, l_i) = (h_1, \dots, h_{i-2}, f, e)$:

$$\begin{aligned}
&(-1)^{i-1}\phi_i(\tilde{\mu}_1(f), h_1, \dots, h_{i-2}, e) + \phi_i(\tilde{\mu}_2(f, e), h_1, \dots, h_{i-2}) + \\
&+ \sum_{n=1}^{i-1} \phi_{i-n+1}(\tilde{\mu}_n(h_1, \dots, h_{n-1}, e), h_n, \dots, h_{i-2}, f) + \text{perm.} = \\
&= \sum_{m=2}^{i} \sum_{n=1}^{i-m+1} \times \\
&\quad \times \mu_m(\phi_1(h_1), \dots, \phi_1(h_{m-2}), \phi_n(h_{m-1}, \dots, h_{m+n-3}, f), \phi_{i-m-n+2}(h_{m+n-2}, \dots, h_{i-2}, e)) + \\
&\quad + \text{perm.}
\end{aligned}$$

which is obviously satisfied after one resums. The fourth possibility is $(l_1, \dots, l_i) = (h_1, \dots, h_{i-2}, e_1, e_2)$:

$$\begin{aligned}
&(-1)^i \phi_{i-1}(\tilde{\mu}_2(e_1, e_2), h_1, \dots, h_{i-2}) - \Bigg(\sum_{n=1}^{i-1} \phi_{i-n+1}(\tilde{\mu}_n(h_1, \dots, h_{n-1}, e_1), h_n, \dots, h_{i-2}, e_2) + \\
&\qquad\qquad + \text{perm.} - e_1 \leftrightarrow e_2 \Bigg) = \\
&= \sum_{m=2}^{i} \sum_{n=1}^{i-m+1} \mu_m(\phi_1(h_1), \dots, \phi_1(h_{m-2}), \phi_n(h_{m-1}, \dots, h_{m+n-3}, e_1), \\
&\qquad\qquad \phi_{i-m-n+2}(h_{m+n-2}, \dots, h_{i-2}, e_2)) + \text{perm.} - e_1 \leftrightarrow e_2 + \\
&\quad + \sum_{m=1}^{i-1} \mu_m(\phi_1(h_1), \dots, \phi_1(h_{m-1}), \phi_{i-m+1}(h_m, \dots, h_{i-2}, e_1, e_2)) + \text{perm.}
\end{aligned}$$

This is satisfied by definition of the Courant bracket (4.3.9). Finally, the last condition for $(l_1, \dots, l_i) = (h_1, \dots, h_{i-3}, e_1, e_2, e_3)$ is:

$$\begin{aligned}
&\phi_{i-2}(\tilde{\mu}_3(e_1, e_2, e_3), h_1, \dots, h_{i-3}) + (-1)^i \phi_{i-1}(\tilde{\mu}_2(e_1, e_2), h_1, \dots, h_{i-3}, e_3) + \text{cycl.} + \\
&\quad + \sum_{n=1}^{i-2} \phi_{i-n+1}(\tilde{\mu}_n(h_1, \dots, h_{n-1}, e_1), h_n, \dots, h_{i-3}, e_2, e_3) + \text{perm.} + \text{cycl.} = \\
&= -\sum_{l=2}^{i-1} \sum_{n=1}^{i-l} \times \\
&\qquad \times \mu_l(\phi_1(h_1), \dots, \phi_1(h_{l-2}), \phi_n(h_{l-1}, \dots, h_{l+n-3}, e_1), \phi_{i-l-n+2}(h_{l+n-2}, \dots, h_{i-3}, e_2, e_3)) - \\
&\qquad - \text{perm.} - \text{cycl.} + \\
&\quad + \sum_{l=3}^{i} \sum_{n=1}^{i-l+1} \sum_{m=1}^{i-l-n+2} \mu_l(\phi_1(h_1), \dots, \phi_1(h_{l-3}), \phi_n(h_{l-2}, \dots, h_{l+n-4}, e_1), \\
&\qquad \phi_m(h_{l+n-3}, \dots, h_{m+n+l-5}, e_2), \phi_{i-m-n-l+3}(h_{m+n+l-4}, \dots, h_{i-3}, e_3)) + \text{perm.} + \text{cycl.}
\end{aligned}$$

where “cycl.” indicates all cycles of e_1, e_2, e_3. This is satisfied by virtue of cyclicity and the properties of a Courant algebroid as in the $i = 3$ case.

As is expected all five conditions are after resumming simply all the Taylor expansion terms as implied by their lowest orders i.e. $l = f$, $l = e$, $(l_1, l_2) = (e, f)$, $(l_1, l_2) = (e_1, e_2)$ and $(l_1, l_2, l_3) = (e_1, e_2, e_3)$.

4.4 | Summary

In this chapter we have constructed the cyclic L_∞-algebra underlying the CSM, and obtained, by tensoring it with the de Rham complex, the dynamics of the model – the action, equations of motion and gauge transformations. Finally, since the CSM has an open gauge algebra we proceeded to finalise the theory with its BV description obtained in the framework of L_∞-algebras by introducing a third algebra which includes the ghost degrees. Using this framework enabled us to obtain the exact BV/BRST transformations of both fields and antifields in our theory (physical and ghost). Then, we constructed the Courant sigma model starting from the structures of a Courant algebroid encoded in a 2-term L_∞-algebra. One has to extend the pure gauge structure of the 2-term L_∞-algebra of a CA defined in Ref. [52] to include field dependence. The morphism we constructed produces all brackets defining the CSM L_∞-algebra and thus the Maurer-Cartan equations. However, the MC action requires an additional input and can be constructed only if one can define a consistent bilinear pairing rendering the CSM L_∞-algebra cyclic.

APPENDIX

4.A | Homotopy identities of CSM algebra

In section 4.1.3 we calculated some of the homotopy identities for $n = 1, 2, 3$. Here we provide the calculation of all homotopy relations for arbitrary n. As was stated, there are seven possible combinations of elements that produce nontrivial identities. Each possibility is calculated below.

- $\underline{(l_1, \ldots, l_n) = (l_{(1)1}, \ldots, l_{(1)n-1}, l_{(-1)})}$

$$\begin{aligned}
&\mu_1(\mu_n(l_{(1)1}, \ldots, l_{(1)n-1}, l_{(-1)})) = \mu_2(\mu_{n-1}(l_{(1)1}, \ldots, l_{(1)n-2}, l_{(-1)}), l_{(1)n-1}) + \cdots + \\
&+ (-1)^{k+1}\mu_{k+1}(\mu_{n-k}(l_{(1)1}, \ldots, l_{(1)n-k-1}, l_{(-1)}), l_{(1)n-k}, \ldots, l_{(1)n-1}) + \\
&+ \text{perm.} + \cdots \\
&- l^{i_1}_{(1)1} \cdots l^{i_{n-1}}_{(1)n-1} \partial_{i_1} \cdots \partial_{i_{n-1}} \rho^j{}_J \eta^{IJ} \rho^i{}_I l_{(-1)j} = \\
&= l^{i_{n-1}}_{(1)n-1} \partial_{i_{n-1}} \rho^i{}_I \eta^{IJ} l^{i_1}_{(1)1} \cdots l^{i_{n-2}}_{(1)n-2} \partial_{i_1} \cdots \partial_{i_{n-2}} \rho^j{}_J l_{(-1)j} + \cdots + \\
&+ l^{i_1}_{(1)1} \cdots l^{i_{n-k-1}}_{(1)n-k-1} \partial_{i_1} \cdots \partial_{i_{n-k-1}} \rho^j{}_J l^{i_{n-k}}_{(1)n-k} \cdots l^{i_{n-1}}_{(1)n-1} \partial_{i_{n-k}} \cdots \partial_{i_{n-1}} \rho^i{}_I \eta^{IJ} l_{(-1)j} + \\
&+ \text{perm.} + \cdots
\end{aligned}$$

$$\Downarrow$$

$$\partial_{i_1} \cdots \partial_{i_{n-1}} (\rho^i{}_I \eta^{IJ} \rho^j{}_J) = 0 \tag{4.A.1}$$

Here "perm." denotes all possible permutations of $l_{(1)1}, \ldots, l_{(1)n-1}$ that are ordered as required by (3.1.1). All such permutations will have positive sign because the Koszul sign will exactly compensate the permutation sign since all objects are either of degree 1 or -1.

- $\underline{(l_1, \dots, l_n) = (l_{(1)1}, \dots, l_{(1)n-2}, l_{(-1)1}, l_{(-1)2})}$

$$
\begin{aligned}
&0 = \mu_2(\mu_{n-1}(l_{(1)1}, \dots, l_{(1)n-2}, l_{(-1)1}), l_{(-1)2}) + \dots + \\
&+ (-1)^{k+1}\mu_{k+1}(\mu_{n-k}(l_{(1)1}, \dots, l_{(1)n-k-1}, l_{(-1)1}), l_{(1)n-k}, \dots, l_{(1)n-2}, l_{(-1)2}) + \\
&+ \text{perm.} + \cdots \\
&0 = -\partial_k \rho^j{}_J \eta^{IJ} l^{i_1}_{(1)1} \cdots l^{i_{n-2}}_{(1)n-2} \partial_{i_1} \cdots \partial_{i_{n-2}} \rho^i{}_I l_{(-1)1i} l_{(-1)2j} - \cdots - \\
&- l^{i_{n-k}}_{(1)n-k} \cdots l^{i_{n-2}}_{(1)n-2} \partial_{i_{n-k}} \cdots \partial_{i_{n-2}} \rho^j{}_J \eta^{IJ} l^{i_1}_{(1)1} \cdots l^{i_{n-k-1}}_{(1)n-k-1} \partial_{i_1} \cdots \partial_{i_{n-k-1}} \rho^i{}_I l_{(-1)1j} l_{(-1)2j} - \\
&- \text{perm.} - \cdots
\end{aligned}
$$

$$\Downarrow$$

$$\partial_{i_1} \cdots \partial_{i_{n-2}} \partial_k (\rho^i{}_I \eta^{IJ} \rho^j{}_J) = 0 \tag{4.A.2}$$

In this case "perm." indicates all possible unshuffles of $l_{(1)1}, \dots, l_{(1)n-2}$ and also terms with $l_{(-1)1}$ and $l_{(-1)2}$ swapped. The sign of all terms will be the same for the same reason as above.

- $\underline{(l_1, \dots, l_n) = (l_{(1)1}, \dots, l_{(1)n-2}, l_{(0)1}, l_{(0)2})}$

$$
\begin{aligned}
&\mu_1(\mu_n(l_{(1)1}, \dots, l_{(1)n-2}, l_{(0)1}, l_{(0)2})) = \mu_2(\mu_{n-1}(l_{(1)1}, \dots, l_{(1)n-2}, l_{(0)1}), l_{(0)2}) - \\
&- l_{(0)1} \leftrightarrow l_{(0)2} + \\
&+ \mu_2(\mu_{n-1}(l_{(1)1}, \dots, l_{(1)n-3}, l_{(0)1}, l_{(0)2}), l_{(1)n-2}) + \dots + \\
&+ (-1)^{k+1}\mu_{k+1}(\mu_{n-k}(l_{(1)1}, \dots, l_{(1)n-k-2}, l_{(0)1}, l_{(0)2}), l_{(1)n-k-1}, \dots, l_{(1)n-2}) + \\
&+ \mu_{k+1}(\mu_{n-k}(l_{(1)1}, \dots, l_{(1)n-k-1}, l_{(0)1}), l_{(1)n-k}, \dots, l_{(1)n-2}, l_{(0)2}) - l_{(0)1} \leftrightarrow l_{(0)2} + \\
&+ \text{perm.} + \cdots \\
&l^{i_1}_{(1)1} \cdots l^{i_{n-2}}_{(1)n-2} \partial_{i_1} \cdots \partial_{i_{n-2}} T_{JKL} \eta^{IJ} l^K_{(0)1} l^L_{(0)2} \rho^i{}_I = \\
&= l^{i_1}_{(1)1} \cdots l^{i_{n-2}}_{(1)n-2} \partial_{i_1} \cdots \partial_{i_{n-2}} \rho^j{}_K \partial_j \rho^i{}_L l^K_{(0)1} l^L_{(0)2} - l_{(0)1} \leftrightarrow l_{(0)2} - \\
&- l^{i_1}_1 \cdots l^{i_{n-3}}_{(1)n-3} \partial_{i_1} \cdots \partial_{i_{n-3}} T_{JKL} \eta^{IJ} l^{i_{n-2}}_{(1)n-2} \partial_{i_{n-2}} \rho^i{}_I l^K_{(0)1} l^L_{(0)2} + \\
&+ \cdots + \\
&+ l^{i_1}_{(1)1} \cdots l^{i_{n-k-1}}_{(1)n-k-1} \partial_{i_1} \cdots \partial_{i_{n-k-1}} \rho^j{}_K l^{i_{n-k}}_{(1)n-k} \cdots l^{i_{n-2}}_{(1)n-2} \partial_{i_{n-k}} \cdots \partial_{i_{n-2}} \partial_j \rho^i{}_L l^K_{(0)1} l^L_{(0)2} - \\
&- l_{(0)1} \leftrightarrow l_{(0)2} - \\
&- l^{i_1}_1 \cdots l^{i_{n-k-2}}_{(1)n-k-2} \partial_{i_1} \cdots \partial_{i_{n-k-2}} T_{JKL} \eta^{IJ} l^{i_{n-k-1}}_{(1)n-k-1} \cdots l^{i_{n-2}}_{(1)n-2} \partial_{i_{n-k-1}} \cdots \partial_{i_{n-2}} \rho^i{}_I l^K_{(0)1} l^L_{(0)2} + \\
&+ \text{perm.} + \cdots
\end{aligned}
$$

$$\Downarrow$$

$$\partial_{i_1}\cdots\partial_{i_{n-2}}(\rho^i{}_I\eta^{IJ}T_{JKL}) = \partial_{i_1}\cdots\partial_{i_{n-2}}(2\rho^j{}_{[K}\partial_{\underline{j}}\rho^i{}_{L]}) \tag{4.A.3}$$

As above "perm." indicates all possible unshuffles of $l_{(1)1},\ldots,l_{(1)n-2}$. Swapping $l_{(0)1}$ and $l_{(0)2}$ produces a sign since the Koszul sign is + but the parity of the permutation will be flipped, this introduces the antisymmetrisation in the final relation.

- $\underline{(l_1,\ldots,l_n) = (l_{(1)1},\ldots,l_{(1)n-2},l_{(0)},l_{(-1)})}$

$$\begin{aligned}
&\mu_1(\mu_n(l_{(1)1},\ldots,l_{(1)n-2},l_{(0)},l_{(-1)})) = \mu_2(\mu_{n-1}(l_{(1)1},\ldots,l_{(1)n-2},l_{(0)}),l_{(-1)}) -\\
&-\mu_2(\mu_{n-1}(l_{(1)1},\ldots,l_{(1)n-2},l_{(-1)}),l_{(0)}) -\\
&-\mu_2(\mu_{n-1}(l_{(1)1},\ldots,l_{(1)n-3},l_{(0)},l_{(-1)}),l_{(1)n-2}) + \cdots +\\
&+\mu_{k+1}(\mu_{n-k}(l_{(1)1},\ldots,l_{(1)n-k-1},l_{(0)}),l_{(1)n-k},\ldots,l_{(1)n-2},l_{(-1)}) +\\
&+(-1)^k\mu_{k+1}(\mu_{n-k}(l_{(1)1},\ldots,l_{(1)n-k-1},l_{(-1)}),l_{(1)n-k},\ldots,l_{(1)n-2},l_{(0)}) -\\
&-\mu_{k+1}(\mu_{n-k}(l_{(1)1},\ldots,l_{(1)n-k-2},l_{(0)},l_{(-1)}),l_{(1)n-k-1},\ldots,l_{(1)n-2}) +\\
&+\text{perm.} + \cdots\\
&-l^{i_1}_{(1)1}\cdots l^{i_{n-2}}_{(1)n-2}\partial_{i_1}\cdots\partial_{i_{n-2}}\partial_i\rho^j{}_K\rho^i{}_J\eta^{IJ}l_{(-1)j}l^K_{(0)} =\\
&=\cdots +\\
&+l_{(-1)j}l^K_{(0)}(-l^{i_1}_{(1)1}\cdots l^{i_{n-k-1}}_{(1)n-k-1}\partial_{i_1}\cdots\partial_{i_{n-k-1}}\rho^i{}_K l^{i_{n-k}}_{(1)n-k}\cdots l^{i_{n-2}}_{(1)n-2}\partial_i\partial_{i_{n-k}}\cdots\partial_{i_{n-2}}\rho^j{}_J +\\
&+l^{i_1}_{(1)1}\cdots l^{i_{n-k-1}}_{(1)n-k-1}\partial_{i_1}\cdots\partial_{i_{n-k-1}}\rho^j{}_L\eta^{LM}l^{i_{n-k}}_{(1)n-k}\cdots l^{i_{n-2}}_{(1)n-2}\partial_{i_{n-k}}\cdots\partial_{i_{n-2}}T_{JMK}\eta^{IJ} +\\
&+l^{i_1}_{(1)1}\cdots l^{i_{n-k-2}}_{(1)n-k-2}\partial_{i_1}\cdots\partial_{i_{n-k-2}}\partial_i\rho^j{}_K l^{i_{n-k-1}}_{(1)n-k-1}\cdots l^{i_{n-2}}_{(1)n-2}\partial_i\partial_{i_{n-k-1}}\cdots\partial_{i_{n-2}}\rho^i{}_J\eta^{IJ}) +\\
&+\text{perm.} + \cdots
\end{aligned}$$

$$\Downarrow$$

$$\partial_{i_1}\cdots\partial_{i_{n-2}}(\rho^j{}_L\eta^{LM}T_{MJK}) = \partial_{i_1}\cdots\partial_{i_{n-2}}(2\rho^i{}_{[J}\partial_{\underline{i}}\rho^j{}_{K]}) \tag{4.A.4}$$

As above.

- $\underline{(l_1,\ldots,l_n) = (l_{(1)1},\ldots,l_{(1)n-3},l_{(0)1},l_{(0)2},l_{(0)3})}$

$$\begin{aligned}
&\mu_1(\mu_n(l_{(1)1},\ldots,l_{(1)n-3},l_{(0)1},l_{(0)2},l_{(0)3})) =\\
&=(-1)^{k+1}\mu_{k+1}(\mu_{n-k}(l_{(1)1},\ldots,l_{(1)n-k-2},l_{(0)1},l_{(0)2}),l_{(1)n-k-1},\ldots,l_{(1)n-3},l_{(0)3}) -\\
&-\mu_{k+1}(\mu_{n-k}(l_{(1)1},\ldots,l_{(1)n-k-1},l_{(0)1}),l_{(1)n-k},\ldots,l_{(1)n-3},l_{(0)2},l_{(0)3}) -\\
&-\mu_{k+1}(\mu_{n-k}(l_{(1)1},\ldots,l_{(1)n-k-3},l_{(0)1},l_{(0)2},l_{(0)3}),l_{(1)n-k-2},\ldots,l_{(1)n-3}) +\\
&+\text{perm.} + \cdots
\end{aligned}$$

$$
\begin{aligned}
&- l^{i_1}_{(1)1}\cdots l^{i_{n-3}}_{(1)n-3}\partial_{i_1}\cdots\partial_{i_{n-3}}\partial_i \mathrm{T}_{ABC}\rho^i{}_J\eta^{IJ}l^A_{(0)1}l^B_{(0)2}l^C_{(0)3} = \\
&= \cdots + l^A_{(0)1}l^B_{(0)2}l^C_{(0)3}\,\cdot \\
&\cdot\,(-l^{i_1}_{(1)1}\cdots l^{i_{n-k-1}}_{(1)n-k-1}\partial_{i_1}\cdots\partial_{i_{n-k-1}}\rho^i{}_A l^{i_{n-k}}_{(1)n-k}\cdots l^{i_{n-3}}_{(1)n-3}\partial_i\partial_{i_{n-k}}\cdots\partial_{i_{n-3}}T_{JBC} + \\
&+ l^{i_1}_{(1)1}\cdots l^{i_{n-k-2}}_{(1)n-k-2}\partial_{i_1}\cdots\partial_{i_{n-k-2}}T_{KAB}\eta^{KL}l^{i_{n-k-1}}_{(1)n-k-1}\cdots l^{i_{n-3}}_{(1)n-3}\partial_{i_{n-k-1}}\cdots\partial_{i_{n-3}}T_{JLC}\eta^{IJ} + \\
&+ l^{i_1}_{(1)1}\cdots l^{i_{n-k-3}}_{(1)n-k-3}\partial_{i_1}\cdots\partial_{i_{n-k-3}}\partial_i T_{ABC}l^{i_{n-k-2}}_{(1)n-k-2}\cdots l^{i_{n-3}}_{(1)n-3}\partial_{i_{n-k-2}}\cdots\partial_{i_{n-3}}\rho^i{}_J\eta^{IJ}) + \\
&+ \text{perm.} + \cdots
\end{aligned}
$$

$$\Downarrow$$

$$\partial_{i_1}\cdots\partial_{i_{n-3}}(\rho^i{}_J\partial_i T_{ABC} - 3\rho^i{}_{[A}\partial_{\underline{i}}T_{BC]J} + 3T_{JK[A}\eta^{KL}T_{BC]L}) = 0 \tag{4.A.5}$$

For reasons above, the graded Koszul sign induces the antisymmetrisation of $l_{(0)1}, l_{(0)2}, l_{(0)3}$.

- $\underline{(l_1,\ldots,l_n) = (l_{(1)1},\ldots,l_{(1)n-3},l_{(0)1},l_{(0)2},l_{(-1)})}$

$$
\begin{aligned}
0 &= -\mu_{k+1}(\mu_{n-k}(l_{(1)1},\ldots,l_{(1)n-k-1},l_{(0)1}),l_{(1)n-k},\ldots,l_{(1)n-3},l_{(0)2},l_{(-1)}) + \\
&+ (-1)^{k+1}\mu_{k+1}(\mu_{n-k}(l_{(1)1},\ldots,l_{(1)n-k-1},l_{(-1)}),l_{(1)n-k},\ldots,l_{(1)n-3},l_{(0)1},l_{(0)2}) + \\
&+ (-1)^{k+1}\mu_{k+1}(\mu_{n-k}(l_{(1)1},\ldots,l_{(1)n-k-2},l_{(0)1},l_{(0)2}),l_{(1)n-k-1},\ldots,l_{(1)n-3},l_{(-1)}) - \\
&- \mu_{k+1}(\mu_{n-k}(l_{(1)1},\ldots,l_{(1)n-k-2},l_{(0)1},l_{(-1)}),l_{(1)n-k-1},\ldots,l_{(1)n-3},l_{(0)2}) + \\
&+ \text{perm.} + \cdots \\
0 &= \cdots + l^K_{(0)1}l^L_{(0)2}l_{(-1)j}\,\cdot \\
&\cdot\,(-l^{i_1}_{(1)1}\cdots l^{i_{n-k-1}}_{(1)n-k-1}\partial_{i_1}\cdots\partial_{i_{n-k-1}}\rho^k{}_K l^{i_{n-k}}_{(1)n-k}\cdots l^{i_{n-3}}_{(1)n-3}\partial_k\partial_{i_{n-k}}\cdots\partial_{i_{n-3}}\partial_i\rho^j{}_L + \\
&+ l^{i_1}_{(1)1}\cdots l^{i_{n-k-1}}_{(1)n-k-1}\partial_{i_1}\cdots\partial_{i_{n-k-1}}\rho^j{}_I\eta^{IJ}l^{i_{n-k}}_{(1)n-k}\cdots l^{i_{n-3}}_{(1)n-3}\partial_{i_{n-k}}\cdots\partial_{i_{n-3}}\partial_i T_{JKL} + \\
&+ l^{i_1}_{(1)1}\cdots l^{i_{n-k-2}}_{(1)n-k-2}\partial_{i_1}\cdots\partial_{i_{n-k-2}}T_{JKL}\eta^{IJ}l^{i_{n-k-1}}_{(1)n-k-1}\cdots l^{i_{n-3}}_{(1)n-3}\partial_{i_{n-k-1}}\cdots\partial_{i_{n-3}}\rho^j{}_I + \\
&+ l^{i_1}_{(1)1}\cdots l^{i_{n-k-2}}_{(1)n-k-2}\partial_{i_1}\cdots\partial_{i_{n-k-2}}\partial_k\rho^j{}_K l^{i_{n-k-1}}_{(1)n-k-1}\cdots l^{i_{n-3}}_{(1)n-3}\partial_{i_{n-k-1}}\cdots\partial_{i_{n-3}}\partial_i\rho^k{}_L) + \\
&+ \text{perm.} + \cdots
\end{aligned}
$$

$$\Downarrow$$

$$\partial_{i_1}\cdots\partial_{i_{n-3}}\partial_i(\rho^j{}_I\eta^{IJ}T_{JKL} - 2\rho^k{}_{[K}\partial_{\underline{k}}\rho^j{}_{L]}) = 0 \tag{4.A.6}$$

As above.

- $\underline{(l_1, \ldots, l_n) = (l_{(1)1}, \ldots, l_{(1)n-4}, l_{(0)1}, l_{(0)2}, l_{(0)3}, l_{(0)4})}$

$$
\begin{aligned}
0 = &\cdots + \mu_{k+1}(\mu_{n-k}(l_{(1)1}, \ldots, l_{(1)n-k-1}, l_{(0)1}), l_{(1)n-k}, \ldots, l_{(1)n-4}, l_{(0)2}, l_{(0)3}, l_{(0)4}) + \\
&+ (-1)^{k+1}\mu_{k+1}(\mu_{n-k}(l_{(1)1}, \ldots, l_{(1)n-k-2}, l_{(0)1}, l_{(0)2}), l_{(1)n-k-1}, \ldots, l_{(1)n-4}, l_{(0)3}, l_{(0)4}) + \\
&+ \mu_{k+1}(\mu_{n-k}(l_{(1)1}, \ldots, l_{(1)n-k-3}, l_{(0)1}, l_{(0)2}, l_{(0)3}), l_{(1)n-k-2}, \ldots, l_{(1)n-4}, l_{(0)4}) + \\
&+ \text{perm.} + \cdots \\
0 = &\cdots + l^A_{(0)1} l^B_{(0)2} l^C_{(0)3} l^D_{(0)4} \cdot \\
&\cdot (l^{i_1}_{(1)1} \cdots l^{i_{n-k-1}}_{(1)n-k-1} \partial_{i_1} \cdots \partial_{i_{n-k-1}} \rho^i{}_A l^{i_{n-k}}_{(1)n-k} \cdots l^{i_{n-4}}_{(1)n-4} \partial_j \partial_{i_{n-k}} \cdots \partial_{i_{n-4}} \partial_i T_{BCD} - \\
&- l^{i_1}_{(1)1} \cdots l^{i_{n-k-2}}_{(1)n-k-2} \partial_{i_1} \cdots \partial_{i_{n-k-2}} T_{JAB} \eta^{IJ} l^{i_{n-k-1}}_{(1)n-k-1} \cdots l^{i_{n-4}}_{(1)n-4} \partial_{i_{n-k-1}} \cdots \partial_{i_{n-4}} \partial_i T_{ICD} - \\
&- l^{i_1}_{(1)1} \cdots l^{i_{n-k-3}}_{(1)n-k-3} \partial_{i_1} \cdots \partial_{i_{n-k-3}} \partial_j T_{ABC} l^{i_{n-k-2}}_{(1)n-k-2} \cdots l^{i_{n-4}}_{(1)n-4} \partial_{i_{n-k-2}} \cdots \partial_{i_{n-4}} \partial_i \rho^j{}_D) + \\
&+ \text{perm.} + \cdots
\end{aligned}
$$

$$\Downarrow$$

$$\partial_{i_1} \cdots \partial_{i_{n-4}} \partial_i (4\rho^j{}_{[A} \partial_{\underline{j}} T_{BCD]} - 3T_{J[AB} \eta^{IJ} T_{CD]I}) = 0 \tag{4.A.7}$$

Equivalently as above the L-degree of $l_{(0)i}$ and the antisymmetry of T ensures the total antisymmetry in indices A, B, C and D.

It is immediately obvious that relations (4.A.1) and (4.A.2) are equivalent, as are (4.A.3), (4.A.4) and (4.A.6), and that (4.A.5) implies (4.A.7). Therefore, we have three unique sets of conditions giving all terms in the Taylor expansions of the axioms of the Courant algebroid by taking $l_{(1)i} = X_i$:

$$
\begin{aligned}
\eta^{IJ} \rho^i{}_I(X) \rho^j{}_J(X) &= 0, \\
2\rho^j{}_{[I}(X) \partial_{\underline{j}} \rho^i{}_{J]}(X) - \rho^i{}_M(X) \eta^{ML} T_{LIJ}(X) &= 0, \\
3\rho^i{}_{[A}(X) \partial_{\underline{i}} T_{BC]J}(X) - \rho^i{}_J(X) \partial_i T_{ABC}(X) - 3T_{JK[A}(X) \eta^{KM} T_{BC]M}(X) &= 0.
\end{aligned}
$$

4.B | BV/BRST action for Courant sigma model

For completeness we write explicitly the action and BRST transformations for all fields in BV/BRST action for Courant sigma model.

$$
\begin{aligned}
S_{\mathrm{BV}} = \int_{\Sigma_3} & F_i \mathrm{d}X^i + \tfrac{1}{2}\eta_{IJ}A^I \mathrm{d}A^J - \rho^i{}_I(X)A^I F_i + \tfrac{1}{6}T_{IJK}(X)A^I A^J A^K - \\
& - \epsilon^I \rho^i{}_I(X) X^\dagger_i + \\
& + \left(\mathrm{d}\epsilon^I + \eta^{IJ}\rho^i{}_J t_i + \eta^{IJ}T_{KLJ}(X)A^K\epsilon^L\right) A^\dagger_I + \\
& + \left(\mathrm{d}t_i - t_j\partial_i\rho^j{}_I(X)A^I - F_j\partial_i\rho^j{}_I(X)\epsilon^I + \tfrac{1}{2}\partial_i T_{IJK}(X)A^I A^J\epsilon^K\right) F^{\dagger i} + \\
& + \left(-\mathrm{d}v_i - \partial_i\rho^j{}_I(X)v_j A^I + \partial_i\rho^j{}_I(X)t_j\epsilon^I + \tfrac{1}{2}\partial_i T_{IJK}(X)A^I\epsilon^J\epsilon^K\right) t^{\dagger i} + \\
& + \left(-\eta^{IJ}\rho^i{}_J(X)v_i + \tfrac{1}{2}\eta^{IJ}T_{JKL}(X)\epsilon^K\epsilon^L\right)\epsilon^\dagger_I + \\
& + \left(-\partial_i\rho^j{}_I(X)v_j\epsilon^I + \tfrac{1}{6}\partial_i T_{IJK}(X)\epsilon^I\epsilon^J\epsilon^K\right) v^{\dagger i} + \\
& + \left(\eta^{IJ}\partial_i\rho^j{}_J(X)v_j - \tfrac{1}{2}\eta^{IJ}\partial_i T_{JKL}(X)\epsilon^K\epsilon^L\right) F^{\dagger i}A^\dagger_I + \\
& + \tfrac{1}{2}\left(A^I\partial_i\partial_j\rho^k{}_I(X)v_k - t_k\partial_i\partial_j\rho^k{}_I(X)\epsilon^I - \tfrac{1}{2}A^I\partial_i\partial_j T_{IJK}(X)\epsilon^J\epsilon^K\right) F^{\dagger i}F^{\dagger j} + \\
& + \left(-\partial_i\partial_j\rho^k{}_I(X)v_k\epsilon^I + \tfrac{1}{6}\partial_i\partial_j T_{IJK}(X)\epsilon^I\epsilon^J\epsilon^K\right) F^{\dagger i}t^{\dagger j} + \\
& + \tfrac{1}{6}\left(\partial_i\partial_j\partial_k\rho^l{}_I(X)v_l\epsilon^I - \tfrac{1}{6}\partial_i\partial_j\partial_k T_{IJK}(X)\epsilon^I\epsilon^J\epsilon^K\right) F^{\dagger i}F^{\dagger j}F^{\dagger k},
\end{aligned}
\tag{4.B.1}
$$

and generalised BRST transformations for each field and antifield:

$$Q_{\mathrm{BV}}X^i = \rho^i{}_I(X)\epsilon^I, \tag{4.B.2}$$

$$
\begin{aligned}
Q_{\mathrm{BV}}A^I = {} & \mathrm{d}\epsilon^I + \eta^{IJ}\rho^i{}_J(X)t_i + \eta^{IJ}T_{JKL}(X)A^K\epsilon^L + \\
& + F^{\dagger i}\eta^{IJ}\partial_i\rho^j{}_J(X)v_j - \tfrac{1}{2}F^{\dagger i}\eta^{IJ}\partial_i T_{JKL}(X)\epsilon^K\epsilon^L,
\end{aligned}
\tag{4.B.3}
$$

$$
\begin{aligned}
Q_{\mathrm{BV}}F_i = {} & \mathrm{d}t_i - \partial_i\rho^j{}_J(X)t_j A^J - \partial_i\rho^j{}_J(X)F_j\epsilon^J + \tfrac{1}{2}\partial_i T_{IJK}(X)A^I A^J\epsilon^K - \\
& - \eta^{IJ}\partial_i\rho^j{}_J(X)v_j A^\dagger_I + \tfrac{1}{2}\eta^{IJ}\partial_i T_{JKL}(X)A^\dagger_I\epsilon^K\epsilon^L - t^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)v_k\epsilon^I + \\
& + \tfrac{1}{6}t^{\dagger j}\partial_i\partial_j T_{IJK}(X)\epsilon^I\epsilon^J\epsilon^K - F^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)v_k A^I + F^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)t_k\epsilon^I + \\
& + \tfrac{1}{2}F^{\dagger j}\partial_i\partial_j T_{IJK}(X)A^I\epsilon^J\epsilon^K + \tfrac{1}{2}F^{\dagger j}F^{\dagger k}\partial_i\partial_j\partial_k\rho^l{}_I(X)v_l\epsilon^I - \\
& - \tfrac{1}{12}F^{\dagger j}F^{\dagger k}\partial_i\partial_j\partial_k T_{IJK}(X)\epsilon^I\epsilon^J\epsilon^K,
\end{aligned}
\tag{4.B.4}
$$

$$Q_{\mathrm{BV}}\epsilon^I = \eta^{IJ}\rho^i{}_J(X)v_i - \tfrac{1}{2}\eta^{IJ}T_{JKL}(X)\epsilon^K\epsilon^L, \tag{4.B.5}$$

$$
\begin{aligned}
Q_{\mathrm{BV}}t_i = {} & -\mathrm{d}v_i - \partial_i\rho^j{}_J(X)v_j A^J + \partial_i\rho^j{}_J(X)t_j\epsilon^J + \tfrac{1}{2}\partial_i T_{IJK}(X)A^I\epsilon^J\epsilon^K + \\
& + F^{\dagger j}\partial_i\partial_j\rho^k{}_J(X)v_k\epsilon^J - \tfrac{1}{6}F^{\dagger j}\partial_i\partial_j T_{IJK}(X)\epsilon^I\epsilon^J\epsilon^K,
\end{aligned}
\tag{4.B.6}
$$

$$Q_{\mathrm{BV}}v_i = -\partial_i\rho^j{}_J(X)v_j\epsilon^J + \tfrac{1}{6}\partial_i T_{IJK}(X)\epsilon^I\epsilon^J\epsilon^K, \tag{4.B.7}$$

$$\begin{aligned}
Q_{\mathrm{BV}}X^\dagger_i &= -\mathrm{d}F_i - \partial_i\rho^j{}_J(X)F_jA^J + \tfrac{1}{6}\partial_iT_{IJK}(X)A^IA^JA^K + \\
&+ \partial_i\rho^j{}_J(X)X^\dagger_j\epsilon^J - \partial_i\rho^j{}_J(X)v_j\epsilon^\dagger_K\eta^{JK} + \partial_i\rho^j{}_J(X)t_j\eta^{JK}A^\dagger_K - \\
&- \partial_iT_{IJK}(X)\eta^{IL}A^\dagger_LA^J\epsilon^K + \tfrac{1}{2}\partial_iT_{IJK}(X)\epsilon^\dagger_L\eta^{IL}\epsilon^J\epsilon^K + \\
&+ F^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)v_k\eta^{IJ}A^\dagger_J + F^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)t_kA^I + F^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)F_k\epsilon^I - \\
&- \tfrac{1}{2}F^{\dagger j}\partial_i\partial_jT_{IJK}(X)A^IA^J\epsilon^K - \tfrac{1}{2}F^{\dagger j}\partial_i\partial_jT_{IJK}(X)\eta^{IL}A^\dagger_L\epsilon^J\epsilon^K - \\
&- t^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)v_kA^I + t^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)t_k\epsilon^I + \tfrac{1}{2}t^{\dagger j}\partial_i\partial_jT_{IJK}(X)A^I\epsilon^J\epsilon^K + \\
&+ v^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)v_k\epsilon^I - \tfrac{1}{6}v^{\dagger j}\partial_i\partial_jT_{IJK}(X)\epsilon^I\epsilon^J\epsilon^K + \\
&+ \tfrac{1}{2}F^{\dagger j}F^{\dagger k}\partial_i\partial_j\partial_k\rho^l{}_I(X)v_lA^I - \tfrac{1}{2}F^{\dagger j}F^{\dagger k}\partial_i\partial_j\partial_k\rho^l{}_I(X)t_l\epsilon^I - \\
&- \tfrac{1}{4}F^{\dagger j}F^{\dagger k}\partial_i\partial_j\partial_kT_{IJK}(X)A^I\epsilon^J\epsilon^K + F^{\dagger j}t^{\dagger k}\partial_i\partial_j\partial_k\rho^l{}_I(X)v_l\epsilon^I - \\
&- \tfrac{1}{6}F^{\dagger j}t^{\dagger k}\partial_i\partial_j\partial_kT_{IJK}(X)\epsilon^I\epsilon^J\epsilon^K - \tfrac{1}{6}F^{\dagger j}F^{\dagger k}F^{\dagger l}\partial_i\partial_j\partial_k\partial_l\rho^m{}_I(X)v_m\epsilon^I + \\
&+ \tfrac{1}{36}F^{\dagger j}F^{\dagger k}F^{\dagger l}\partial_i\partial_j\partial_k\partial_lT_{IJK}(X)\epsilon^I\epsilon^J\epsilon^K,
\end{aligned} \tag{4.B.8}$$

$$\begin{aligned}
Q_{\mathrm{BV}}A^\dagger_I &= \eta_{IJ}\left(-\mathrm{d}A^J + \eta^{JK}\rho^i{}_K(X)F_i - \tfrac{1}{2}\eta^{JK}T_{KLM}(X)A^LA^M\right) - \\
&- F^{\dagger i}\partial_i\rho^j{}_I(X)t_j - F^{\dagger i}\partial_iT_{IJK}(X)A^J\epsilon^K - \tfrac{1}{2}F^{\dagger i}F^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)v_k + \\
&+ \tfrac{1}{4}F^{\dagger i}F^{\dagger j}\partial_i\partial_jT_{IJK}(X)\epsilon^J\epsilon^K + t^{\dagger i}\partial_i\rho^j{}_I(X)v_j - \tfrac{1}{2}t^{\dagger i}\partial_iT_{IJK}(X)\epsilon^J\epsilon^K - \\
&- T_{IJK}(X)\eta^{JL}A^\dagger_L\epsilon^K,
\end{aligned} \tag{4.B.9}$$

$$\begin{aligned}
Q_{\mathrm{BV}}F^{\dagger i} &= -\mathrm{d}X^i + \rho^i{}_I(X)A^I - \\
&- F^{\dagger j}\partial_j\rho^i{}_I(X)\epsilon^I,
\end{aligned} \tag{4.B.10}$$

$$\begin{aligned}
Q_{\mathrm{BV}}\epsilon^\dagger_I &= \mathrm{d}A^\dagger_I + \rho^i{}_I(X)X^\dagger_i - T_{IJK}(X)\eta^{JL}A^\dagger_LA^K + T_{IJK}(X)\eta^{JL}\epsilon^\dagger_L\epsilon^K + \\
&+ F^{\dagger i}\partial_i\rho^j{}_I(X)F_j - \tfrac{1}{2}F^{\dagger i}\partial_iT_{IJK}(X)A^JA^K - F^{\dagger i}\partial_iT_{IJK}(X)\eta^{JL}A^\dagger_L\epsilon^K + \\
&+ t^{\dagger i}\partial_i\rho^j{}_I(X)t_j + t^{\dagger i}\partial_iT_{IJK}(X)A^J\epsilon^K + \tfrac{1}{2}F^{\dagger i}F^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)t_k - \\
&- \tfrac{1}{2}F^{\dagger i}F^{\dagger j}\partial_i\partial_jT_{IJK}(X)A^J\epsilon^K + F^{\dagger i}t^{\dagger j}\partial_i\partial_j\rho^k{}_I(X)v_k - \\
&- \tfrac{1}{2}F^{\dagger i}t^{\dagger j}\partial_i\partial_jT_{IJK}(X)\epsilon^J\epsilon^K - \tfrac{1}{6}F^{\dagger i}F^{\dagger j}F^{\dagger k}\partial_i\partial_j\partial_k\rho^l{}_I(X)v_l + \\
&+ \tfrac{1}{12}F^{\dagger i}F^{\dagger j}F^{\dagger k}\partial_i\partial_j\partial_kT_{IJK}(X)\epsilon^J\epsilon^K + v^{\dagger i}\partial_i\rho^j{}_I(X)v_j - \\
&- \tfrac{1}{2}v^{\dagger i}\partial_iT_{IJK}(X)\epsilon^J\epsilon^K,
\end{aligned} \tag{4.B.11}$$

$$\begin{aligned}
Q_{\mathrm{BV}}t^{\dagger i} &= -\mathrm{d}F^{\dagger i} + \eta^{IJ}\rho^i{}_I(X)A^\dagger_J + t^{\dagger j}\partial_j\rho^i{}_I(X)\epsilon^I + F^{\dagger j}\partial_j\rho^i{}_I(X)A^I - \\
&- \tfrac{1}{2}F^{\dagger j}F^{\dagger k}\partial_j\partial_k\rho^i{}_I(X)\epsilon^I,
\end{aligned} \tag{4.B.12}$$

$$\begin{aligned}
Q_{\mathrm{BV}}v^{\dagger i} &= -\mathrm{d}t^{\dagger i} + \eta^{IJ}\rho^i{}_I(X)\epsilon^\dagger_J - F^{\dagger j}\partial_j\rho^i{}_I(X)\eta^{IJ}A^\dagger_J + t^{\dagger j}\partial_j\rho^i{}_I(X)A^I - v^{\dagger i}\partial_j\rho^i{}_I(X)\epsilon^I - \\
&- \tfrac{1}{2}F^{\dagger j}F^{\dagger k}\partial_j\partial_k\rho^i{}_I(X)A^I - F^{\dagger j}t^{\dagger k}\partial_j\partial_k\rho^i{}_I(X)\epsilon^I + \tfrac{1}{6}F^{\dagger j}F^{\dagger k}F^{\dagger l}\partial_j\partial_k\partial_l\rho^i{}_I(X)\epsilon^I.
\end{aligned} \tag{4.B.13}$$

As was to be expected one may notice the classical part (first line in each expression) of the BRST transformations of physical and ghost fields (4.B.2)–(4.B.7) corresponds to their gauge variations (4.1.14)–(4.1.16) and antifields (4.B.8)–(4.B.10) to their equations of motion (4.1.11)–(4.1.13).

A careful reader might notice some signs differing here from the relations given in section 2.1.3 (2.1.28)–(2.1.33). This is an artefact of the field definition and the two can be brought to match by the field redefinitions $t \to -t$ and $F^\dagger \to -F^\dagger$.

4.C | Homotopy identities of extended CA algebra

Homotopy relations (3.1.1) imply the possible choices for the higher products i.e. restrict us in which can be set to vanish. In this section we will make the calculation for $\widetilde{\mathsf{L}}_1 = TM$ of which the restriction to T_pM, as in section 4.3.1, is a special case. Therefore we have the following for $i = 1, 2, 3, 4$ and $i \geqslant 4$.

- $\underline{i = 1}$
  There is only one non-trivial homotopy relation:
  $$\tilde{\mu}_1\tilde{\mu}_1(f) = \tilde{\rho} \circ \mathcal{D}(f) = 0,$$
  which is satisfied by the axioms of the Courant algebroid.

- $\underline{i = 2}$
  Of the four non-trivial relations, three will be modified by the existence of L_1. Choices $(l_1, l_2) = (e, f), (h, f), (e_1, e_2)$ produce the following conditions respectively:
  $$\begin{aligned}\tilde{\mu}_2(\tilde{\rho}(e), f) &= 0,\\ \tilde{\mu}_2(\mathcal{D}f, h) &= 0,\\ \tilde{\mu}_2(\tilde{\rho}(e_1), e_2) - \tilde{\mu}_2(\tilde{\rho}(e_2), e_1) &= [\tilde{\rho}(e_1), \tilde{\rho}(e_2)].\end{aligned}$$
  The first enables us to set $\tilde{\mu}_2(h, f) = 0$, whereas from the second and third relation it is obvious $\tilde{\mu}_2(h, e)$ cannot vanish and one can choose $\tilde{\mu}_2(h, e)^i = h^j \tilde{\partial}_j \tilde{\rho}(e)^i$.

- $\underline{i = 3}$
  In this case there are six combinations of elements that produce non-trivial homotopy relations, of which five are modified by the extension: $(h, f_1, f_2), (h, e, f), (e_1, e_2, e_3), (h_1, h_2, f)$ and (h, e_1, e_2). These combinations respectively produce the constraints:
  $$\begin{aligned}\tilde{\mu}_3(\mathcal{D}f_1, h, f_2) + \tilde{\mu}_3(\mathcal{D}f_2, h, f_1) &= 0,\\ \tilde{\mu}_3(\tilde{\rho}(e), h, f) + \tilde{\mu}_3(\mathcal{D}f, h, f) &= 0,\end{aligned}$$

$$\tilde{\mu}_3(\tilde{\rho}(e_1), e_2, e_3) + \text{cyclic} = 0,$$
$$\tilde{\rho}(\tilde{\mu}_3(h_1, h_2, f)) + \tilde{\mu}_3(\mathcal{D}f, h_1, h_2) = 0,$$
$$\tilde{\mu}_3(\tilde{\rho}(e_1), e_2, h)^i + h^j \tilde{\rho}(e_1)^k \tilde{\partial}_j \tilde{\partial}_k \tilde{\rho}(e_2)^i - e_1 \leftrightarrow e_2 = -\tilde{\rho}(\tilde{\mu}_3(h, e_1, e_2))^i.$$

The minimal extension implied by these constraints is to set all $\tilde{\mu}_3$ products involving h to zero except $\tilde{\mu}_3(h_1, h_2, e)^i = h_1^j h_2^k \tilde{\partial}_j \tilde{\partial}_k \tilde{\rho}(e)^i$.

- $\underline{i = 4}$
  Degree counting tells us that for $i \geqslant 4$ there are always exactly 7 non-trivial homotopy conditions, additionally, all of them are modified by the extension and must be calculated. The combinations are: $(h_1, h_2, f_1, f_2), (h, e_1, e_1, f), (e_1, e_2, e_3, e_4), (h_1, h_2, e, f), (h, e_1, e_2, e_3), (h_1, h_2, h_3, f)$ and (h_1, h_2, e_1, e_2). In order, each combination produces the following conditions:

$$\tilde{\mu}_4(\mathcal{D}f_1, h_1, h_2, f_2) + \tilde{\mu}_4(\mathcal{D}f_2, h_1, h_2, f_1) = 0,$$
$$\tilde{\mu}_4(\tilde{\rho}(e_1), h, e_2, f) - \tilde{\mu}_4(\tilde{\rho}(e_2), h, e_1, f) - \tilde{\mu}_4(\mathcal{D}f, h, e_1, e_2) = 0,$$
$$\tilde{\mu}_4(\tilde{\rho}(e_{[1}), e_{2]}, e_3, e_4) + \tilde{\mu}_4(\tilde{\rho}(e_{[3}), e_{4]}, e_1, e_2) = 0,$$
$$\mathcal{D}\tilde{\mu}_4(h_1, h_2, e, f) - \tilde{\mu}_4(\tilde{\rho}(e), h_1, h_2, f) + \tilde{\mu}_4(\mathcal{D}f, h_1, h_2, e) = 0,$$
$$\tilde{\mu}_4(\tilde{\rho}(e_1), e_2, e_3, h) + \text{cyclic} = \mathcal{D}\tilde{\mu}_4(e_1, e_2, e_3, h),$$
$$\tilde{\rho}(\tilde{\mu}_4(h_1, h_2, h_3, f)) - \tilde{\mu}_4(\mathcal{D}f, h_1, h_2, h_3) = 0,$$
$$\tilde{\mu}_4(\tilde{\rho}(e_1), e_2, h_1, h_2)^i + h_1^j h_2^k \tilde{\rho}(e_2)^l \tilde{\partial}_j \tilde{\partial}_k \tilde{\partial}_l \tilde{\rho}(e_1)^i - e_1 \leftrightarrow e_2 = \tilde{\rho}(\tilde{\mu}_4(e_1, e_2, h_1, h_2))^i.$$

  These constraints allow the minimal extension of non-vanishing products to be just one: $\tilde{\mu}_4(h_1, h_2, h_3, e)^i = h_1^j h_2^k h_3^l \tilde{\partial}_j \tilde{\partial}_k \tilde{\partial}_l \tilde{\rho}(e)^i$.

- $\underline{i \geqslant 4}$
  Since for i greater than four there are no new types of homotopy relations, since the combinations of elements from $i = 4$ that produce non-trivial homotopy identities all simply gain the appropriate number of h elements. Therefore, the structures of the corresponding conditions placed upon $\tilde{\mu}_i$ will be no different from the case of $i = 4$. For that reason we make the assumption that all higher products vanish except $\tilde{\mu}_i(h_1, \ldots, h_{i-1}, e)$ since it could not be made to vanish in lower cases. To be consistent with our choice for $\tilde{\mu}_4$ we make the ansatz:

$$\tilde{\mu}_i(h_1, \ldots, h_{i-1}, e)^j = h_1^{j_1} \cdots h_{i-1}^{j_i} \tilde{\partial}_{j_1} \cdots \tilde{\partial}_{j_i} \tilde{\rho}(e)^j.$$

  The consequence of this assumption is that only two homotopy identities will be non-trivial, those corresponding to combinations: $(h_1, \ldots, h_{i-1}, f)$ and $(h_1, \ldots, h_{i-2}, e_1, e_2)$. The first is directly satisfied by the axioms of a Courant algebroid:

$$\tilde{\mu}_i(h_1, \ldots, h_{i-1}, \mathcal{D}f) = 0.$$

The second is just the higher derivative analogue to the final condition in the $i = 4$ case:

$$\begin{aligned}
0 = &-\tilde{\mu}_2(\tilde{\mu}_{i-1}(h_1, \ldots, h_{i-2}, e_1), e_2) + \tilde{\mu}_2(\tilde{\mu}_{i-1}(h_1, \ldots, h_{i-2}, e_2), e_1) - \\
&- \cdots - \\
&- \tilde{\mu}_n(\tilde{\mu}_{i-n+1}(h_1, \ldots, h_{i-n}, e_1), h_{i-n+1}, \ldots, h_{i-2}, e_2) - \text{perm.} + \\
&+ \tilde{\mu}_n(\tilde{\mu}_{i-n+1}(h_1, \ldots, h_{i-n}, e_2), h_{i-n+1}, \ldots, h_{i-2}, e_1) + \text{perm.} - \\
&- \cdots - \\
&- \tilde{\mu}_{i-1}(\tilde{\mu}_2(h_1, e_1), h_2, \ldots, h_{i-2}, e_2) - \text{perm.} + \\
&+ \tilde{\mu}_{i-1}(\tilde{\mu}_2(h_1, e_2), h_2, \ldots, h_{i-2}, e_1) + \text{perm.} + \\
&+ (-1)^i \tilde{\mu}_{i-1}(\tilde{\mu}_2(e_1, e_2), h_1, \ldots, h_{i-2}) - \\
&- \tilde{\mu}_i(\tilde{\mu}_1(e_1), h_1, \ldots, h_{i-2}, e_2) + \tilde{\mu}_i(\tilde{\mu}_1(e_2), h_1, \ldots, h_{i-2}, e_1),
\end{aligned}$$

which is directly satisfied by use of the ansatz and Leibniz rule of the differential operator $\tilde{\partial}_{i_1} \cdots \tilde{\partial}_{i_n}$.

CHAPTER 5

DFT ALGEBROID AND CURVED L_∞-ALGEBRAS

Chapter 2 introduced the appropriate geometric structure of double field theory: a DFT algebroid. It was shown that there exists a more general structure (dubbed a pre-DFT algebroid in [45]) that corresponds to the metric or Vaisman algebroid [85], of which the DFT algebroid is a special case. On the other hand however, in Ref. [59] the authors suggested that the relevant geometric structure is a pre-NQ manifold. This structure is defined on non-negatively (N) graded manifolds with a degree 1 vector field (Q) which does not square to zero, with the obstruction controlled by the strong constraint. The relevant pre-NQ manifold was obtained as a half-dimensional submanifold from the Vinogradov algebroid defined over a doubled space.

In this chapter based on [86] we shall analyse the structure of a DFT algebroid from a different perspective, giving its definition in terms of a *curved* L_∞-algebra [72, 73]. This naturally extends the results of [59] and connects them to the results of [45] and chapter 2, and moreover, it implies that one should be able to formulate a DFT algebroid in terms of a Q structure. This becomes especially important when constructing the corresponding sigma-model. In section 5.1 we introduce curved L_∞-algebras in our convention and as a motivating example we recall the Courant algebroid L_∞-algebra [52, 75]. Thereafter, in sections 5.1.2 and 5.1.3 we construct the curved L_∞-algebra for a DFT algebroid on two different graded spaces underlying the L_∞-algebra. Section 5.2 is dedicated to the understanding of the strong constraint on the DFT algebroid as an L_∞-morphism. We begin by recalling the definition of L_∞-morphisms and then explicitly construct the map from a DFT algebroid to an (undoubled) Courant algebroid. The geometric structure studied is extended to a sigma model field theory in section 5.3 where we give the explicit definition of the corresponding Q-vector and gauge symmetries. Finally, appendices 5.A and5.B provide completeness for longer calculations of sections 5.1.3 and 5.2.2, and appendix 5.C proves the sigma model L_∞-algebra satisfies the homotopy relations.

5.1 | L_∞-algebra for the DFT algebroid

Our aim in this section is to show that a DFT algebroid can be understood as a *curved* L_∞-algebra. We shall start, however, with an introduction to curved L_∞-algebras.

5.1.1 | On curved L_∞-algebras

L_∞-algebras are generalizations of Lie algebras with infinitely-many higher brackets, related to each other by higher homotopy versions of the Jacobi identity [72, 73] (3.1.1). Thus far, however, we have ignored the extension stated in section 3.1.1, namely when i is allowed to be zero. The question is what changes? Three aspects are modified by the possibility of $i = 0$:

1. there is an additional map μ_0;
2. there is an additional homotopy relation, the one corresponding to the choice $i = 0$:
$$\mu_1\mu_0 = 0;$$
3. every order of the homotopy relations obtains an extra term in the sum since j too is now allowed to be 0 as is k:
$$\cdots + (-1)^i \mu_{i+1}(\mu_0, l_1, \ldots, l_i) = 0.$$

It is instructive to repeat the steps of section 3.1.1 and see how one must now interpret the first three ($i = 0, 1, 2$) orders of the homotopy relations. The first is $i = 0$ as stated in point 2, this relation can be understood as meaning μ_0 is a constant with respect to map μ_1. Second we have $i = 2$ and:

$$\mu_1\mu_1(l) = \mu_2(\mu_0, l),$$

implying, in general (if the map on the lhs does not coincidentally vanish), μ_1 is no longer a differential. This also explains the nomenclature, as an analogy with differential geometry can be made where covariant exterior derivatives square to the curvature of the bundle. Finally, $i = 2$ is now modified to:

$$\mu_1(\mu_2(l_1, l_2)) - \mu_2(\mu_1(l_1), l_2) + (-1)^{|l_1||l_2|}\mu_2(\mu_1(l_2), l_1) = -\mu_3(\mu_0, l_1, l_2),$$

breaking the Leibniz rule and therefore the interpretation of μ_1 as a derivation.

As an L_∞-algebra is a generalisation of Lie algebras by controlling the closure of lower brackets by higher ones and therefore modifying the bracket, in the same way the curving of an L_∞-algebra can be seen as now modifying not the bracket but the differential.

It is instructive to present an example: curved differential graded Lie algebras [87]. A curved differential graded Lie algebra (curved dgLa) is a triple $(\mathfrak{g}, d, R)$ where $\mathfrak{g}$ is a graded Lie algebra, d is a derivation of degree 1, and R is a curvature element of degree 2 such that $dR = 0$ and $d^2x = [R, x]$ for all $x \in \mathfrak{g}$. In the L_∞ framework we identify the map μ_0 with the constant curvature R, μ_1 with the derivation d and μ_2 with the graded Lie bracket, satisfying the homotopy relations

$$
\begin{aligned}
\mu_1\mu_0 &= 0, \\
\mu_1(\mu_1(l)) &= \mu_2(\mu_0, l).
\end{aligned}
$$

Notice that although μ_1 or d is no longer a differential it still is a graded derivation as it still satisfies the Leibniz identity. This is a special property of curved dgLas that happens only because the 3-bracket μ_3 vanishes making the $i = 2$ curved homotopy identity a regular graded Leibniz rule for μ_1.

5.1.2 | Curved L_∞-algebra for the DFT algebroid

The first proposal for an L_∞ structure relevant for DFT was given in Ref. [59] based on the graded geometry of a pre-NQ manifold. In that case, the homotopy relations of the proposed L_∞-algebra were satisfied only up to the strong constraint. Here we wish to extend this result by constructing proper, albeit curved, L_∞-algebra. We begin by defining the relevant graded vector space:

$$
\begin{array}{ccccc}
\mathsf{L}_{-1} & \oplus & \mathsf{L}_0 & \oplus & \mathsf{L}_2 \\
f \in C^\infty(\mathcal{M}) & & e \in \Gamma(L) & & \mu_0
\end{array}
$$

where L_2 is a 1-dimensional vector space spanned by the constant element μ_0. In general, there is no chain complex underlying the graded vector space of curved L_∞-algebras, thus we use the $\oplus$ symbol to define the space. The maps that do not involve space L_2 are taken in analogy with the Courant algebroid maps:

$$
\begin{aligned}
\mu_1(f) &= \mathcal{D}f, \\
\mu_2(e_1, e_2) &= [\![e_1, e_2]\!]\ ,\ \mu_2(e, f) = \langle e, \mathcal{D}f\rangle, \\
\mu_3(e_1, e_2, e_3) &= \mathcal{N}(e_1, e_2, e_3).
\end{aligned} \tag{5.1.1}
$$

The maps involving L_2 are going to be constructed from the homotopy relations. Before we begin with our construction, it is useful to see which homotopy relations will be non-trivial. To this end we prove the following statement.

For every $l_1, l_2 \in \mathsf{L}$ such that $\mu_i(l_1, l_2, \ldots) = 0$, the homotopy Jacobi identities of (3.1.1) can be written in the following way:

$$\sum_{j+k=i}\sum_{\sigma'}\chi(\sigma';l_1,\dots,l_i)(-1)^k\Bigg(\mu_{k+1}(\mu_j(l_1,l_{\sigma'(2)},\dots,l_{\sigma'(j)})l_2,l_{\sigma'(j+2)},\dots,l_{\sigma'(i)})+ \\ +(-1)^{1+l_1l_2+(l_1-l_2)\sum_{m=2}^{j}l_{\sigma'(m)}}\mu_{k+1}(\mu_j(l_2,l_{\sigma'(2)},\dots,l_{\sigma'(j)})l_1,l_{\sigma'(j+2)},\dots,l_{\sigma'(i)})\Bigg)=0. \tag{5.1.2}$$

The proof follows from the fact that since unshuffles are ordered and l_1 and l_2 must be in different products, all unshuffles will necessarily have l_1 and l_2 for the first and $j+1$-st element or vice versa. This implies we can split the homotopy relation into two sums, those that have unshuffles that begin with l_1 and those that begin with l_2. Then it is simply a matter of connecting the antisymmetric Koszul signs of these two unshuffles.

Additionally one can observe that in our case:

$$\mu_{i+1}(\mu_0,\mu_0,\dots)=0, \qquad \forall i\in\mathbb{N}, \tag{5.1.3}$$

holds due to the graded antisymmetry of the maps and the fact that $|\mu_0|=2$. Therefore, all homotopy relations of two μ_0 arguments must be trivial, which reduces the number of identities to be calculated significantly.

We proceed by constructing the maps involving the space L_2 from the homotopy relations. The homotopy identity for $i=0$ is trivial in our case, so we move on to $i=1$:

$$\mu_1\mu_1(l)=\mu_2(\mu_0,l).$$

This contains one non-trivial identity:

$$\mu_2(\mu_0,e)=0. \tag{5.1.4}$$

For $i=2$ the homotopy identity:

$$\mu_1(\mu_2(l_1,l_2))-\mu_2(\mu_1(l_1),l_2)-(-1)^{1+|l_1||l_2|}\mu_2(\mu_1(l_2),l_1)=-\mu_3(\mu_0,l_1,l_2),$$

contains three non-trivial cases: $(l_1,l_2)=\{(\mu_0,f),(e,f),(f_1,f_2)\}$. The first produces the condition:

$$\mu_2(\mathcal{D}f,\mu_0)=0,$$

which is automatically satisfied by (5.1.4). The second and third are simply the definitions of higher brackets:

$$\begin{aligned}\mu_3(\mu_0,e,f)&=[\![e,\mathcal{D}f]\!]-\mathcal{D}\langle e,\mathcal{D}f\rangle \\ &=-\tfrac{1}{2}\rho^{-1}\eta^{-1}(df)(\rho(e)),\end{aligned} \tag{5.1.5}$$

$$\begin{aligned}\mu_3(\mu_0, f_1, f_2) &= 2\langle \mathcal{D}f_1, \mathcal{D}f_2\rangle \\ &= \tfrac{1}{2}\eta^{-1}(df_1, df_2).\end{aligned} \tag{5.1.6}$$

In the case of $i = 3$,

$$\begin{aligned}\mu_1(\mu_3(l_1, l_2, l_3)) - \mu_2(\mu_2(l_1, l_2), l_3) + (-1)^{|l_2||l_3|}\mu_2(\mu_2(l_1, l_3), l_2)- \\ -(-1)^{|l_1|(|l_2|+|l_3|)}\mu_2(\mu_2(l_2, l_3), l_1) + \mu_3(\mu_1(l_1), l_2, l_3)- \\ -(-1)^{|l_1||l_2|}\mu_3(\mu_1(l_2), l_1, l_3) + (-1)^{|l_3|(|l_1|+|l_2|)}\mu_3(\mu_1(l_3), l_1, l_2) = \mu_4(\mu_0, l_1, l_2, l_3),\end{aligned}$$

there are four different identities to be satisfied: $(l_1, l_2, l_3) = \{(\mu_0, e_1, e_2), (\mu_0, f_1, f_2), (e_1, e_2, e_3), (e_1, e_2, f)\}$. The first case is a consistency condition satisfied due to (5.1.4) once one takes into account (5.1.3). For the next case, the identity is:

$$2\mathcal{D}\langle \mathcal{D}f_1, \mathcal{D}f_2\rangle = -\mu_3(\mu_0, \mathcal{D}f_1, f_2) - \mu_3(\mu_0, \mathcal{D}f_2, f_1),$$

which is directly satisfied by use of (5.1.5). For choice (e_1, e_2, e_3), the corresponding homotopy identity is a definition:

$$\mathcal{D}\mathcal{N}(e_1, e_2, e_3) - \mathrm{Jac}(e_1, e_2, e_3) = \mu_4(\mu_0, e_1, e_2, e_3),$$

or by use of (2.2.13):

$$\mu_4(\mu_0, e_1, e_2, e_3) = -\mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, e_3). \tag{5.1.7}$$

The last of the $i = 3$ expressions defines $\mu_4(\mu_0, e_1, e_2, f)$:

$$\begin{aligned}\mu_4(\mu_0, e_1, e_2, f) &= -\langle [\![e_1, e_2]\!], \mathcal{D}f\rangle - \langle e_2, \mathcal{D}\langle e_1, \mathcal{D}f\rangle\rangle + \langle e_1, \mathcal{D}\langle e_2, \mathcal{D}f\rangle\rangle + \mathcal{N}(\mathcal{D}f, e_1, e_2) \\ &= 0,\end{aligned} \tag{5.1.8}$$

where the second equality holds by (2.2.12) and the first identity of Property 2.2.15.

Next is $i = 4$ with three non-trivial conditions: $(l_1, l_2, l_3, l_4) = \{(\mu_0, e_1, e_2, f), (e_1, e_2, e_3, e_4), (\mu_0, e, f_1, f_2)\}$. The first case (μ_0, e_1, e_2, f) with condition:

$$\begin{aligned}\mu_4(\mu_1(f), \mu_0, e_1, e_2) + \mu_4(\mu_2(e_1, e_2), \mu_0, f) - \mu_3(\mu_2(e_1, f), \mu_0, e_2) + \\ +\mu_3(\mu_2(e_2, f), \mu_0, e_1) + \mu_2(\mu_3(\mu_0, e_1, f), e_2) - \mu_2(\mu_3(\mu_0, e_2, f), e_1) = 0,\end{aligned}$$

produces, after plugging in (5.1.1) and (5.1.5):

$$\mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, \mathcal{D}f) + \mathcal{D}\langle [\![e_1, e_2]\!], \mathcal{D}f\rangle + [\![\mathcal{D}f, [\![e_1, e_2]\!]]\!] +$$

$$+\mathcal{D}\langle e_2,\mathcal{D}\langle e_1,\mathcal{D}f\rangle\rangle-\mathcal{D}\langle e_1,\mathcal{D}\langle e_2,\mathcal{D}f\rangle\rangle-[\![[\![\mathcal{D}f,e_1]\!],e_2]\!]+[\![[\![\mathcal{D}f,e_2]\!],e_1]\!]=0,$$

which vanishes by use of (2.2.13) and (5.1.8). The second identity in $i=4$ is the definition:

$$\mathcal{N}([\![e_1,e_2]\!],e_3,e_4)+\langle\mathcal{D}\mathcal{N}(e_1,e_2,e_3),e_4\rangle+\text{antisymm.}(1,2,3,4)=-\mu_5(\mu_0,e_1,e_2,e_3,e_4).$$

Using the third identity of Property 2.2.15, this can be rewritten as:

$$\mu_5(\mu_0,e_1,e_2,e_3,e_4)=-\tfrac{1}{2}\langle\mathsf{SC}_{\text{Jac}}(e_1,e_2,e_3),e_4\rangle+\text{antisymm.}(1,2,3,4). \tag{5.1.9}$$

The last identity of $i=4$ is the compatibility:

$$\mu_3(\mu_2(e,f_1),\mu_0,f_2)+\mu_3(\mu_2(e,f_2),\mu_0,f_1)-\mu_2(\mu_3(\mu_0,e,f_1),f_2)-$$
$$-\mu_2(\mu_3(\mu_0,e,f_2),f_1)-\mu_2(\mu_3(\mu_0,f_1,f_2),e)=0.$$

This can easily be shown to hold using Property 2.2.15.

Moving on to $i=5$ with two non-trivial identities for: $(l_1,l_2,l_3,l_4,l_5)=\{(\mu_0,e_1,e_2,e_3,f),(\mu_0,e_1,e_2,e_3,e_4)\}$. The first is:

$$0=\tfrac{1}{2}\langle\mathsf{SC}_{\text{Jac}}(e_1,e_2,e_3),\mathcal{D}f\rangle+2\langle\mathcal{D}\mathcal{N}(e_1,e_2,e_3),\mathcal{D}f\rangle+$$
$$+\Big(\tfrac{1}{2}\langle\mathsf{SC}_{\text{Jac}}(e_1,e_2,\mathcal{D}f),e_3\rangle+\mathcal{N}([\![e_1,\mathcal{D}f]\!],e_2,e_3)-\mathcal{N}(\mathcal{D}\langle e_1,\mathcal{D}f\rangle,e_2,e_3)+\text{cyclic}(1,2,3)\Big),$$

where by utilising properties (2.2.12), (2.2.13) and Property 2.2.15, one obtains:

$$\tfrac{1}{6}\Big(\langle\mathsf{SC}_{\text{Jac}}(e_1,e_2,\mathcal{D}f),e_3\rangle+\mathsf{SC}_\rho(e_2,e_3)(\rho(e_1)f)+\text{cyclic}(1,2,3)\Big)+$$
$$+\tfrac{1}{2}\langle\mathsf{SC}_{\text{Jac}}(e_1,e_2,e_3),\mathcal{D}f\rangle=0.$$

This relation can be shown to be identically satisfied by direct calculation. The second and last identity of $i=5$, after plugging in all the appropriate definitions, states:

$$-\mathsf{SC}_{\text{Jac}}([\![e_1,e_2]\!],e_3,e_4)-[\![e_4,\mathcal{D}\mathcal{N}(e_1,e_2,e_3)]\!]+\mathcal{D}\langle e_4,\mathcal{D}\mathcal{N}(e_1,e_2,e_3)\rangle+$$
$$+[\![\mathsf{SC}_{\text{Jac}}(e_1,e_2,e_3),e_4]\!]-\tfrac{1}{2}\mathcal{D}\langle\mathsf{SC}_{\text{Jac}}(e_1,e_2,e_3),e_4\rangle+\text{antisymm.}(1,2,3,4)=0.$$

Properties (2.2.13) and the third identity of Property 2.2.15 produce:

$$[\![\text{Jac}(e_1,e_2,e_3),e_4]\!]-\text{Jac}([\![e_1,e_2]\!],e_3,e_4)+\text{antisymm.}(1,2,3,4)=0,$$

that is satisfied by direct calculation.

Finally, $i=6$ has only one non-trivial relation $(l_1,l_2,l_3,l_4,l_5,l_6)=(\mu_0,e_1,e_2,e_3,e_4,e_5)$

where, by the definitions given above, one obtains:

$$\begin{aligned}\tfrac{1}{2}\langle \mathsf{SC}_{\text{Jac}}([\![e_1,e_2]\!],e_3,e_4),e_5\rangle - \mathcal{N}(\mathsf{SC}_{\text{Jac}}(e_1,e_2,e_3),e_4,e_5) - \\ -\tfrac{1}{2}\langle \mathcal{D}\langle \mathsf{SC}_{\text{Jac}}(e_1,e_2,e_3),e_4\rangle,e_5\rangle + \text{antisymm.}(1,2,3,4,5) = 0.\end{aligned}$$

A straightforward but lengthy and rather tedious direct calculation shows this holds.

All higher homotopy identities vanish and we summarise our findings in the following table.

$$\boxed{\begin{aligned}
\mu_1(f) &= \mathcal{D}f\\
\mu_2(e_1,e_2) &= [\![e_1,e_2]\!]\\
\mu_2(e,f) &= \langle e,\mathcal{D}f\rangle\\
\mu_3(e_1,e_2,e_3) &= \mathcal{N}(e_1,e_2,e_3)\\
\mu_3(\mu_0,e,f) &= [\![e,\mathcal{D}f]\!] - \mathcal{D}\langle e,\mathcal{D}f\rangle\\
\mu_3(\mu_0,f_1,f_2) &= 2\langle \mathcal{D}f_1,\mathcal{D}f_2\rangle\\
\mu_4(\mu_0,e_1,e_2,e_3) &= \mathcal{D}\mathcal{N}(e_1,e_2,e_3) - \text{Jac}(e_1,e_2,e_3)\\
\mu_5(\mu_0,e_1,e_2,e_3,e_4) &= \tfrac{1}{2}\langle \mathcal{D}\mathcal{N}(e_1,e_2,e_3),e_4\rangle - \tfrac{1}{2}\langle \text{Jac}(e_1,e_2,e_3),e_4\rangle + \\
&\quad + \text{antisymm.}(1,2,3,4)
\end{aligned}} \tag{5.1.10}$$

All non-zero maps that include the constant element μ_0 of the space L_2 are controlled by the pairing on $T\mathcal{M}$ (2.2.8) and its inverse (2.2.10), as can be seen from (2.2.11) and Property 2.2.15. Here we choose to represent the space L_2 as the space spanned by the constant symmetric bivector η^{-1}.

5.1.3 | Extending the curved L_∞-algebra for the DFT algebroid

A better understanding of the DFT algebroid that arises from the L_∞ structure can be obtained if we extend the underlying vector space by adding L_1, containing sections of $T\mathcal{M}$. In that way, the anchor map is included in the L_∞ maps, the choice of representation of L_2 as the space spanned by the constant symmetric bivector η^{-1} is natural, and the homotopy relations reproduce the defining properties of a DFT algebroid. Therefore, we shall start with the following graded vector space:

$$\begin{array}{ccccccc}
\mathsf{L}_{-1} & \oplus & \mathsf{L}_0 & \oplus & \mathsf{L}_1 & \oplus & \mathsf{L}_2\\
f\in C^\infty(\mathcal{M}) & & e\in\Gamma(L) & & h\in\mathfrak{X}(\mathcal{M}) & & \mu_0
\end{array}$$

the boxed maps (5.1.10) and

$$\mu_{i+1}(h_1,\dots,h_i,e) = h_1^{A_1}\cdots h_i^{A_i}\partial_{A_1}\cdots\partial_{A_i}\rho(e)^B\partial_B, \qquad i\geqslant 0, \tag{5.1.11}$$

a choice based on the analogous relation for Courant algebroids. Additional maps are constructed from the homotopy identities as follows.

As in the previous subsection we begin our construction with the $i = 1$ homotopy identity since the $i = 0$ case is trivial. This case has two non-trivial possibilities $l = f$ and $l = e$. The first produces:[1]

$$\mu_2(\mu_0, f) = \mu_1(\mu_1(f)) = \rho \circ \mathcal{D}f = \tfrac{1}{2}\eta^{AB}\partial_B f \partial_A, \tag{5.1.12}$$

whereas the second:

$$\mu_1\rho(e) = \mu_2(\mu_0, e) \in \mathsf{L}_2,$$

must be trivial since L_2 is by construction spanned by the constant element μ_0 and cannot, therefore, non-trivially depend on an arbitrary section e of L. Thus the following must hold:

$$\mu_1(h) = 0 \quad \text{and} \quad \mu_2(\mu_0, e) = 0. \tag{5.1.13}$$

The same reasoning implies that all homotopy identities in the space L_2 must be trivially satisfied:

$$\begin{aligned} \mu_i(h_1, \ldots, h_i) &= 0, \\ \mu_{i+2}(h_1, \ldots, h_i, \mu_0, e) &= 0. \end{aligned} \tag{5.1.14}$$

Using (5.1.2) one can show that in general we can have at most 15 non-trivial identities for each i.

Moving on to $i = 2$, we find 4 non-trivial identities, however, only three of these are different from the $\mathsf{L}_1 = \varnothing$ case above: $(l_1, l_2) = \{(e, f), (e_1, e_2), (h, f)\}$. These give, respectively:

$$\begin{aligned} \mu_2(\rho(e), f) &= 0, \\ \rho[\![e_1, e_2]\!] - [\rho(e_1), \rho(e_2)] &= -\mu_3(\mu_0, e_1, e_2), \\ \rho\mu_2(h, f) + \tfrac{1}{2}\eta^{BC}h^A\partial_A\partial_C f\partial_B &= -\mu_3(\mu_0, h, f), \end{aligned}$$

that result in:

$$\begin{aligned} \mu_2(h, f) &= 0, \\ \mu_3(\mu_0, e_1, e_2) &= \mathsf{SC}_\rho(e_1, e_2), \\ \mu_3(\mu_0, h, f) &= -\tfrac{1}{2}\eta^{BC}h^A\partial_A\partial_C f\partial_B. \end{aligned} \tag{5.1.15}$$

Continuing to the $i = 3$ case, one has 8 non-trivial identities, of these only 5 are new in comparison to the previous subsection. They are: $(l_1, l_2, l_3) = \{(h, f_1, f_2), (h, e, f),$

[1]It is interesting to note that, since there is an L_1 space in this extension, one can explicitly see the curving of our "differential" μ_1 on functions.

$(h_1, h_2, f), (e_1, e_2, h), (\mu_0, e, f)\}$ with the corresponding homotopy expressions:

$$\begin{aligned}
\mu_4(\mu_0, h, f_1, f_2) &= 0, \\
\mu_4(\mu_0, h, e, f) &= 0, \\
\mu_3(\mathcal{D}f, h_1, h_2) &= \mu_4(\mu_0, h_1, h_2, f), \\
\mu_2(h, [\![e_1, e_2]\!]) + \mu_2(\mu_2(e_1, h), e_2) + \mu_3(\rho(e_1), e_2, h) - e_1 \leftrightarrow e_2 &= \mu_4(\mu_0, e_1, e_2, h), \\
\rho[\![e, \mathcal{D}f]\!] - i_{\eta^{-1}} d\langle e, \mathcal{D}f\rangle + \tfrac{1}{2}\mu_2(i_{df}\eta^{-1}, e) - & \\
-\mu_2(\langle e, \mathcal{D}f\rangle, \mu_0) - \mu_3(\rho(e), \mu_0, f) + \mu_3(\mathcal{D}f, \mu_0, e) &= 0.
\end{aligned}$$

The first four are definitions of higher maps:

$$\begin{aligned}
\mu_4(\mu_0, h_1, h_2, f) &= \tfrac{1}{2}\eta^{BC} h_1^{A_1} h_2^{A_2} \partial_{A_1}\partial_{A_2}\partial_C f \partial_B, \\
\mu_4(\mu_0, e_1, e_2, h) &= -h^A \partial_A \mathsf{SC}_\rho(e_1, e_2)^B \partial_B,
\end{aligned} \tag{5.1.16}$$

whereas the last is a condition satisfied by use of (2.2.12), and the maps defined thus far. Definitions (5.1.15) and (5.1.16) suggest, in the spirit of (5.1.11), the following Ansatz for the non-vanishing maps:

$$\begin{aligned}
\mu_{i+2}(h_1, \ldots, h_i, \mu_0, f) &= \tfrac{1}{2}\eta^{BC} h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \partial_C f \partial_B, \\
\mu_{i+3}(h_1, \ldots, h_i, \mu_0, e_1, e_2) &= h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \mathsf{SC}_\rho(e_1, e_2)^B \partial_B.
\end{aligned} \tag{5.1.17}$$

Using this Ansatz one can show that all higher identities, which are infinite in number, are satisfied, see appendix 5.A. We collect the maps for the extended L_∞-algebra corresponding to a DFT algebroid in the following list (where $i \geqslant 0$).

$$\begin{aligned}
\mu_1(f) &= \mathcal{D}f \\
\mu_2(e_1, e_2) &= [\![e_1, e_2]\!] \\
\mu_2(e, f) &= \langle e, \mathcal{D}f\rangle \\
\mu_3(e_1, e_2, e_3) &= \mathcal{N}(e_1, e_2, e_3) \\
\mu_3(\mu_0, e, f) &= [\![e, \mathcal{D}f]\!] - \mathcal{D}\langle e, \mathcal{D}f\rangle \\
\mu_3(\mu_0, f_1, f_2) &= 2\langle \mathcal{D}f_1, \mathcal{D}f_2\rangle \\
\mu_4(\mu_0, e_1, e_2, e_3) &= \mathcal{D}\mathcal{N}(e_1, e_2, e_3) - \mathrm{Jac}(e_1, e_2, e_3) \\
\mu_5(\mu_0, e_1, e_2, e_3, e_4) &= \tfrac{1}{2}\langle \mathcal{D}\mathcal{N}(e_1, e_2, e_3), e_4\rangle - \tfrac{1}{2}\langle \mathrm{Jac}(e_1, e_2, e_3), e_4\rangle + \\
&\quad + \text{antisymm.}(1,2,3,4) \\
\mu_{i+1}(h_1, \ldots, h_i, e) &= h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \rho(e)^B \partial_B \\
\mu_{i+2}(h_1, \ldots, h_i, \mu_0, f) &= \tfrac{1}{2}\eta^{BC} h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \partial_C f \partial_B \\
\mu_{i+3}(h_1, \ldots, h_i, \mu_0, e_1, e_2) &= h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \mathsf{SC}_\rho(e_1, e_2)^B \partial_B
\end{aligned} \tag{5.1.18}$$

The homotopy relations reproduce the defining properties of a DFT algebroid as discussed in Sect. 2.2.1:

$$\boxed{\begin{aligned}(\rho\circ\mathcal{D})f &= \tfrac{1}{2}\eta^{-1}(df)\\ \rho[\![e_1,e_2]\!]_C - [\rho(e_1),\rho(e_2)] &= -\mathsf{SC}_\rho(e_1,e_2)\\ \mathrm{Jac}(e_1,e_2,e_3) - \mathcal{D}\mathcal{N}(e_1,e_2,e_3) &= \mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,e_3)\end{aligned}} \tag{5.1.19}$$

and their higher derivatives.

5.2 | L_∞-morphism as the strong constraint

In order to complete the description of a DFT algebroid in terms of an L_∞-algebra, we would also like to include the strong constraint in this framework. Since we know that on the solution of the strong constraint the C-bracket of double field theory reduces to the Courant bracket, we are looking for a relation between the L_∞-algebra for a DFT algebroid and the one for a Courant algebroid. The natural relation between L_∞-algebras is an L_∞-algebra morphism or L_∞-morphism for short. In the following, we explicitly construct an L_∞-morphism from DFT to a Courant algebroid implementing the strong constraint.

5.2.1 | On curved L_∞-morphisms

Before we start with the construction of mappings, we first recall the definition of an L_∞-morphism (3.1.5) as this can be curved too. Similarly to the expression for the homotopy relations, the condition of an L_∞-morphism from (L,μ_i) to (L',μ'_i) is actually a possibly infinite series of relations, one for each $i\in\mathbb{N}_0$.[2] Here we explicitly state the first three:

- $i=0$

$$\phi_1(\mu_0) = \mu'_0 + \mu'_1(\phi_0) + \tfrac{1}{2!}\mu'_2(\phi_0,\phi_0) + \cdots$$

- $i=1$

$$\phi_1(\mu_1(l)) - \phi_2(\mu_0,l) = \mu'_1(\phi_1(l)) + \mu'_2(\phi_0,\phi_1(l)) + \tfrac{1}{2!}\mu'_3(\phi_0,\phi_0,\phi_1(l)) + \cdots$$

[2] In this instance L' does not denote the space of L_∞ valued de Rham forms but simply the target algebra. This is done to avoid using the tilde notation of the previous chapter in the hopes of not confusing the reader with regards to the splitting of coordinates with the duals being denoted by tilde.

- $i = 2$

$$\begin{aligned}
&\phi_3(\mu_0, l_1, l_2) - \phi_2(\mu_1(l_1), l_2) + (-1)^{l_1 l_2}\phi_2(\mu_1(l_2), l_1) + \phi_1(\mu_2(l_1, l_2)) = \\
&= \mu_1'(\phi_2(l_1, l_2)) + \mu_2'(\phi_0, \phi_2(l_1, l_2)) + \tfrac{1}{2!}\mu_3'(\phi_0, \phi_0, \phi_2(l_1, l_2)) + \cdots + \\
&\quad + \mu_2'(\phi_1(l_1), \phi_1(l_2)) + \mu_3'(\phi_0, \phi_1(l_1), \phi_1(l_2)) + \tfrac{1}{2!}\mu_4'(\phi_0, \phi_0, \phi_1(l_1), \phi_1(l_2)) + \cdots
\end{aligned}$$

In the case of non-vanishing ϕ_0 this is called a curved L_∞-morphism and the zeroth relation implies ϕ_0 is a Maurer-Cartan element if $\mu_0 = 0$. Therefore if one is mapping a flat algebra to a curved one this can always be undone by a differential redefinition in the sense of section 3.1.1. However as we are going the other way this will not be of significance to the following calculation.

5.2.2 | From a DFT algebroid to a Courant algebroid

To set the stage, we begin with L, a DFT algebroid over a doubled space $\mathcal{M}$, and L', a Courant algebroid over M, where M is a subspace of $\mathcal{M}$ and $\dim M = \dim \mathcal{M}/2$. Then introduce a mapping $\phi : \mathsf{L} \to \mathsf{L}'$ that projects the DFT algebroid to the Courant algebroid:[3]

$$\begin{array}{rccccc}
\text{DFT}: & \mathsf{L}_{-1} = C^\infty(\mathcal{M}) & \oplus & \mathsf{L}_0 = \Gamma(L) & \oplus & \mathsf{L}_2 \\
\phi \downarrow & \phi_1 \downarrow & & \phi_1 \downarrow & & \phi_1 \downarrow \\
\text{CA}: & \mathsf{L}'_{-1} = C^\infty(M) & \oplus & \mathsf{L}'_0 = \Gamma(E) & \oplus & \varnothing .
\end{array}$$

This basically means that if we pick a coordinate chart on $\mathcal{M}$, $x^A = (x^a, \tilde{x}_a)$, such that the coordinates x^a correspond to coordinates of the manifold M and M is then implicitly defined by $\tilde{x}_a = \text{const.}$, all functions $f(x^A)$ on $\mathcal{M}$ upon restriction only depend on half the coordinates, namely $f(x^a, \tilde{x}_a = \text{const.})$. However, the fibre structure remains unchanged. To verify that such a mapping is indeed an L_∞-morphism one must check that it satisfies the conditions (3.1.5). We begin with $i = 0$ that implies only $\phi_1(\mu_0) = 0$ as a Courant algebroid does not include spaces L'_1 nor L'_2. Therefore $\phi_0 = 0$ and we are dealing with flat L_∞-morphism. For the case of $i = 1$ we make the following choice:

$$\phi_1(f) = \tfrac{1}{2} f\Big|_M, \tag{5.2.1}$$

$$\phi_1(e) = e\Big|_M, \tag{5.2.2}$$

[3] In this section the Courant algebroid anchor and differential will be denoted a and D respectively to differentiate them from their DFT counterparts.

where $e\Big|_M$ means the component function is restricted and the section exists only over M. This case has only one non-trivial identity, the one corresponding to $l = f$:

$$\phi_1(\mu_1(f)) - \phi_2(\mu_0, f) = \mu_1'(\phi_1(f)).$$

By plugging in the products from 5.1.2 and Example 3.3, this becomes:

$$\phi_1(\mathcal{D}f) - \phi_2(\mu_0, f) = D\phi_1(f).$$

In $(x, \tilde{x})$ coordinates the DFT anchor splits into two: $\rho^A{}_I = (\rho^a{}_I, \tilde{\rho}_{aI})$, the first is the one we relate to the anchor $a^a{}_I$ of a Courant algebroid. This choice is consistent since for a DFT algebroid we have:

$$\rho^a{}_I \hat{\eta}^{IJ} \rho^b{}_J = 0,$$

according to axiom 1 of definition 2.2.7 (see also (2.2.16)), meaning $\rho^a{}_I$ satisfies the first identity in (4.3.5). Therefore, the DFT derivative splits into two: $\mathcal{D} = \frac{1}{2}D + \frac{1}{2}\tilde{D}$,[4] the first of which we associate with the Courant algebroid differential as its image is in the kernel of a. By using (5.2.1) and the fact that $(\partial_a f)\Big|_M = \partial_a(f\Big|_M)$ and $(\tilde{\partial}^a f)\Big|_M \neq \tilde{\partial}^a(f\Big|_M)$, one obtains:

$$\phi_2(\mu_0, f) = \tfrac{1}{2}(\tilde{D}f)\Big|_M. \tag{5.2.3}$$

The case of $i = 2$ has two non-trivial possibilities: $(l_1, l_2) = \{(f, e), (e_1, e_2)\}$. To keep calculations as simple as possible we shall make the Ansatz that all components ϕ_i for $i > 1$ not including μ_0 vanish. The first produces:

$$\phi_3(\mu_0, f, e) - \phi_1(\langle e, \mathcal{D}f\rangle) = -\langle \phi_1(e), D\phi_1(f)\rangle_C,$$

that reduces to the definition:

$$\phi_3(\mu_0, f, e) = \tfrac{1}{4}\langle e, \tilde{D}f\rangle\Big|_M. \tag{5.2.4}$$

The second identity is (again, after the choice of $\phi_2(e_1, e_2) = 0$):

$$\phi_3(\mu_0, e_1, e_2) + \phi_1([\![e_1, e_2]\!]) = [\phi_1(e_1), \phi_1(e_2)]_C.$$

Here we shall make the identification of $\hat{T}$ of DFT with the twist of the Courant algebroid, therefore trivially one has:

$$\phi_3(\mu_0, e_1, e_2) = [e_1, e_2]_C\Big|_M - [\![e_1, e_2]\!]\Big|_M. \tag{5.2.5}$$

[4]The factor of 1/2 stems from the different definitions of the derivative and pairing in DFT and in a Courant algebroid. Whereas in DFT the derivative carries the factor, in a Courant algebroid the pairing does instead.

In the case of $i = 3$ we have three possibilities: $(l_1, l_2, l_3) = \{(e_1, e_2, e_3), (\mu_0, f, e), (\mu_0, f_1, f_2)\}$, the first being a definition as before and the other two being consistency checks. The first produces the definition of $\phi_4(\mu_0, e_1, e_2, e_3)$:

$$\phi_4(\mu_0, e_1, e_2, e_3) = \left(\tfrac{1}{2}\mathcal{N}(e_1, e_2, e_3) - \mathcal{N}_c(e_1, e_2, e_3)\right)\Big|_M, \tag{5.2.6}$$

The next case is the following consistency condition:

$$\begin{aligned} -\phi_3(\mathcal{D}f, \mu_0, e) + \phi_2(\langle e, \mathcal{D}f\rangle, \mu_0) - \phi_1([\![e, \mathcal{D}f]\!] - \mathcal{D}\langle e, \mathcal{D}f\rangle) &= \\ = D\phi_3(\mu_0, f, e) - [\phi_1(e), \phi_2(\mu_0, f)]_C&, \end{aligned}$$

which, by use of (5.2.4) and (5.2.5), transforms into:

$$D\langle e, Df\rangle_C - [e, Df]_C = 0.$$

This is valid by the first relation of Property 2.1.7 as all present structures correspond to a Courant algebroid. The third non-vanishing condition gives:

$$\begin{aligned} \phi_3(\mu_0, \mathcal{D}f_1, f_2) + \phi_3(\mu_0, \mathcal{D}f_2, f_1) + \phi_1(2\langle \mathcal{D}f_1, \mathcal{D}f_2\rangle) &= \\ = \langle \phi_2(\mu_0, f_2), D\phi_1(f_1)\rangle_C + \langle \phi_2(\mu_0, f_1), D\phi_1(f_2)\rangle_C&, \end{aligned}$$

that vanishes by virtue of $\langle Df_1, Df_2\rangle_C = 0$.

For $i = 4$ there are two non-trivial possibilities for the selection of elements: $(l_1, l_2, l_3, l_4) = \{(\mu_0, e_1, e_2, e_3), (\mu_0, e_1, e_2, f)\}$, both producing compatibility conditions. The former combination yields condition:

$$\begin{aligned} \phi_3([\![e_1, e_2]\!], \mu_0, e_3) + \text{cyclic(1,2,3)} + \phi_2(\mathcal{N}(e_1, e_2, e_3), \mu_0) - \phi_1(\mathsf{SC}_{\text{Jac}}(e_1, e_2, e_3)) &= \\ = D\phi_4(\mu_0, e_1, e_2, e_3) - ([\phi_1(e_1), \phi_3(\mu_0, e_2, e_3)]_C + \text{cyclic(1,2,3)})&, \end{aligned}$$

that is satisfied by use of (2.2.13) and the third relation in (4.3.5). The condition corresponding to the latter selection of elements is:

$$\begin{aligned} &\phi_3(\langle e_2, \mathcal{D}f\rangle, \mu_0, e_1) - e_1 \leftrightarrow e_2 + \phi_3([\![e_1, e_2]\!], \mu_0, f) + \phi_4(\mathcal{D}f, \mu_0, e_1, e_2) = \\ &= \langle \phi_1(e_2), D\phi_3(\mu_0, e_1, f)\rangle_C - e_1 \leftrightarrow e_2 + \\ &\quad + \langle \phi_3(\mu_0, e_1, e_2), D\phi_1(f)\rangle_C + \mathcal{N}_c(\phi_1(e_1), \phi_1(e_2), \phi_2(\mu_0, f)), \end{aligned}$$

reducing to:

$$2\mathcal{N}_c(Df, e_1, e_2)\Big|_M = \langle [e_1, e_2]_C, Df\rangle_C\Big|_M,$$

satisfied by the second identity of Property 2.1.7.

Finally, $i = 5$ has only one condition to consider, $(l_1, l_2, l_3, l_4, l_5) = (\mu_0, e_1, e_2, e_3, e_4)$:

$$\begin{aligned}&\phi_4(\mu_0, [\![e_1, e_2]\!], e_3, e_4) + \phi_3(\mu_0, \mathcal{N}(e_1, e_2, e_3), e_4) + \phi_1(\langle \mathsf{SC}_{\text{Jac}}(e_1, e_2, e_3), e_4\rangle_+) + \\ &+ \text{antisymm.(1,2,3,4)} = \\ &= \langle \phi_1(e_1), D\phi_4(\mu_0, e_2, e_3, e_4)\rangle_C + \mathcal{N}_c(\phi_1(e_1), \phi_1(e_2), \phi_3(\mu_0, e_3, e_4)) + \text{antisymm.(1,2,3,4)},\end{aligned}$$

satisfied by the last identity of Property 2.2.15 and last identity of Property 2.1.7. To summarise we present all non-vanishing morphism components in the following table.

$$\boxed{\begin{aligned}\phi_1(f) &= \tfrac{1}{2} f\Big|_M \\ \phi_1(e) &= e\Big|_M \\ \phi_2(\mu_0, f) &= \tfrac{1}{2}(\tilde{D}f)\Big|_M \\ \phi_3(\mu_0, f, e) &= \tfrac{1}{4}\langle e, \tilde{D}f\rangle\Big|_M \\ \phi_3(\mu_0, e_1, e_2) &= [e_1, e_2]_C\Big|_M - [\![e_1, e_2]\!]\Big|_M \\ \phi_4(\mu_0, e_1, e_2, e_3) &= \Big(\tfrac{1}{2}\mathcal{N}(e_1, e_2, e_3) - \mathcal{N}_c(e_1, e_2, e_3)\Big)\Big|_M\end{aligned}}$$

We finish this section by presenting a minimal extension of the morphism above in order to encompass the algebras of subsection 5.1.3 and the extended Courant algebroid L_∞-algebra of sec. 4.3.1. To accomplish this we must make certain assumptions about this morphism. Our choice is the following:

- ϕ_1 morphism components are:

$$\phi_1 : \begin{cases} C^\infty(\mathcal{M}) \to C^\infty(M), & f \mapsto \tfrac{1}{2} f\Big|_M \\ \Gamma(L) \to \Gamma(E), & e \mapsto e\Big|_M \\ \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(M), & h^A \partial_A \mapsto h^a\Big|_M \partial_a \end{cases},$$

- all morphism components constructed above remain unchanged,
- the only non-vanishing ϕ_i without μ_0 as an argument are ϕ_1,
- the morphism is "flat" i.e. $\phi_0 = 0$.

Details of the calculation of the morphism conditions can be found in appendix 5.B. Here we simply state the maps that constitute an L_∞-morphism from a DFT algebroid to a

Courant algebroid (both viewed as L_∞-algebras) in the following boxed set of expressions.

$$
\begin{aligned}
\phi_1(f) &= \tfrac{1}{2} f\Big|_M \\
\phi_1(e) &= e\Big|_M \\
\phi_2(\mu_0, f) &= \tfrac{1}{2}(\tilde{D} f)\Big|_M \\
\phi_3(\mu_0, f, e) &= \tfrac{1}{4}\langle e, \tilde{D} f\rangle\Big|_M \\
\phi_3(\mu_0, e_1, e_2) &= [e_1, e_2]_C\Big|_M - [\![e_1, e_2]\!]\Big|_M \\
\phi_4(\mu_0, e_1, e_2, e_3) &= \Big(\tfrac{1}{2}\mathcal{N}(e_1, e_2, e_3) - \mathcal{N}_c(e_1, e_2, e_3)\Big)\Big|_M \\
\hline
\phi_1(h) &= h^a\Big|_M \partial_a \\
\phi_{i+2}(h_1, \ldots, h_i, \mu_0, e) &= \Big(h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \rho(e)^b - h_1^{a_1} \cdots h_i^{a_i} \partial_{a_1} \cdots \partial_{a_i} a(e)^b\Big)\Big|_M \partial_b
\end{aligned}
$$

5.3 | DFT sigma model as a Maurer-Cartan homotopy theory

As we have extensively studied the L_∞ properties of the DFT algebroid in the previous sections, the time has come to apply this structure to construct the corresponding sigma model in a similar way to chapter 4. This way we shall be able to better understand the structure obtained in chapter 2. Additionally through this process we will be able to obtain the exact expression for the cohomological Q-vector of DFT, the existence of which is implied by the consistency of the DFT L_∞-algebras (5.1.10) and (5.1.18). This section follows [88].

5.3.1 | Field theory symmetry algebra

Chapter 4 has taught us when regarding sigma model algebras from their corresponding algebroids one needs to start from the extended version (the one including the degree 1 homogeneous subspace) in order to be able to capture the full field content. Thus, we start from algebra (5.1.18). The second modification that must be done is to promote space L_{-1} from functions to target space 1-forms by incorporating the target de Rham differential appearing in the maps into the definition of the elements of L_{-1}. Due to this change some maps become trivial, however this also implies maps that coincidentally vanished when L_{-1} was a space of functions may not vanish now. Most maps will be nothing more than the maps (5.1.18) applied to the basis elements and pulled back by the sigma model-defining map X as in the case of the Courant sigma model. And finally, all maps must be expanded as we want to obtain the perturbative expansion of the interaction. In order to define this new algebra we must first define space L as:

$$\mathsf{L}_{-1} \oplus \mathsf{L}_0 \oplus \mathsf{L}_1 \oplus \mathsf{L}_2$$

where $\mathsf{L}_{-1} = \Gamma(X^* T^* \mathcal{M})$, $\mathsf{L}_0 = \Gamma(X^* E)$, $\mathsf{L}_1 = \{X^A \equiv X^* x^A\}$ and L_2, again, is the space of the curvature element μ_0. The non-vanishing maps are given by:[5]

$$\begin{aligned}
\mu_{i+1}(l_{(1)1}, \dots, l_{(1)i}, l_{(0)}) &= l_{(1)1}^{A_1} \cdots l_{(1)i}^{A_i} \partial_{A_1} \cdots \partial_{A_i} \rho^A{}_I l_{(0)}^I \\
\mu_{i+1}(l_{(1)1}, \dots, l_{(1)i}, l_{(-1)}) &= \hat{\eta}^{IJ} l_{(1)1}^{A_1} \cdots l_{(1)i}^{A_i} \partial_{A_1} \cdots \partial_{A_i} \rho^A{}_I l_{(-1)A} \\
\mu_{i+2}(l_{(1)1}, \dots, l_{(1)i}, l_{(0)1}, l_{(0)2}) &= \hat{\eta}^{IJ} l_{(1)1}^{A_1} \cdots l_{(1)i}^{A_i} \partial_{A_1} \cdots \partial_{A_i} T_{JKL} l_{(0)1}^K l_{(0)2}^L \\
\mu_{i+2}(l_{(1)1}, \dots, l_{(1)i}, l_{(0)}, l_{(-1)}) &= l_{(1)1}^{A_1} \cdots l_{(1)i}^{A_i} \partial_{A_1} \cdots \partial_{A_i} \partial_A \rho^B{}_J l_{(0)}^J l_{(-1)B} \\
\mu_{i+3}(l_{(1)1}, \dots, l_{(1)i}, l_{(0)1}, l_{(0)2}, l_{(0)3}) &= -l_{(1)1}^{A_1} \cdots l_{(1)i}^{A_i} \partial_{A_1} \cdots \partial_{A_i} \partial_A T_{IJK} l_{(0)1}^I l_{(0)2}^J l_{(0)3}^K
\end{aligned}$$

- -

$$\begin{aligned}
\mu_0 &= \eta^{-1} \\
\mu_2(\mu_0, l_{(-1)}) &= \eta^{AB} l_{(-1)B} \\
\mu_{i+3}(l_{(1)1}, \dots, l_{(1)i}, \mu_0, l_{(0)}, l_{(-1)}) &= -\eta^{AB} \hat{\eta}^{IJ} l_{(1)1}^{A_1} \cdots l_{(1)i}^{A_i} \partial_{A_1} \cdots \partial_{A_i} \bar{\Theta}_{BJK} l_{(0)}^K l_{(-1)A} \\
\mu_{i+4}(l_{(1)1}, \dots, l_{(1)i}, \mu_0, l_{(0)1}, l_{(0)2}, l_{(-1)}) &= \eta^{BC} l_{(1)1}^{A_1} \cdots l_{(1)i}^{A_i} \partial_{A_1} \cdots \partial_{A_i} \partial_A \bar{\Theta}_{CIJ} l_{(0)1}^I l_{(0)2}^J l_{(-1)B} \\
\mu_{i+3}(l_{(1)1}, \dots, l_{(1)i}, \mu_0, l_{(0)1}, l_{(0)2}) &= \eta^{AB} l_{(1)1}^{A_1} \cdots l_{(1)i}^{A_i} \partial_{A_1} \cdots \partial_{A_i} \bar{\Theta}_{BIJ} l_{(0)1}^I l_{(0)2}^J \\
\mu_{i+4}(l_{(1)1}, \dots, l_{(1)i}, \mu_0, l_{(0)1}, l_{(0)2}, l_{(0)3}) &= \hat{\eta}^{IJ} l_{(1)1}^{A_1} \cdots l_{(1)i}^{A_i} \partial_{A_1} \cdots \partial_{A_i} \mathcal{Z}_{JKLM} l_{(0)1}^K l_{(0)2}^L l_{(0)3}^M \\
\mu_{i+5}(l_{(1)1}, \dots, l_{(1)i}, \mu_0, l_{(0)1}, l_{(0)2}, l_{(0)3}, l_{(0)4}) &= l_{(1)1}^{A_1} \cdots l_{(1)i}^{A_i} \partial_{A_1} \cdots \partial_{A_i} \partial_A \mathcal{Z}_{IJKL} l_{(0)1}^I l_{(0)2}^J l_{(0)3}^K l_{(0)4}^L
\end{aligned} \tag{5.3.1}$$

with $\bar{\Theta}_{AIJ} \equiv \eta_{BC} \rho^C{}_{[I} \partial_{\underline{A}} \rho^B{}_{J]}$ (a rescaling of (2.2.48)) and $\mathcal{Z}_{IJKL} \equiv 3\eta_{AD}\eta_{BC}\eta^{EF} \rho^A{}_{[I} \partial_{\underline{F}} \rho^D{}_J \rho^B{}_K \partial_{\underline{E}} \rho^C{}_{L]}$ (expression (2.2.20) satisfying (2.2.19)). This is a new algebra only implied from the algebraic symmetry algebra, therefore, one must check it indeed satisfies the homotopy relations (3.1.1). We delegate this to appendix 5.C and turn our attention to the corresponding cohomological vector structure Q. In general, one constructs a Q-vector from the algebra formulation of an L_∞-algebra by applying the following three steps.

1. Using the shift isomorphism (3.1.6) one calculates the codifferential components from (3.1.7):

$$D_i(sl_1, \dots, sl_i) = (-1)^{\frac{1}{2}i(i-1) + \sum_{j=1}^i |l_j|(i-j)} s\mu_i(l_1, \dots, l_i).$$

2. These codifferential components acting on the basis $\{\tau_\alpha\}$ of L define the structure

[5] The notation here is, again, as in chapter 4 such that we do not write the evaluation of expanded functions at $X = 0$ explicitly and denote a generic element $l_{(i)} \in \mathsf{L}_i$.

constants $C^\alpha_{\beta_1\cdots\beta_i}$ of L_∞-algebra (L,μ):

$$C^\alpha_{\beta_1\cdots\beta_i}\tau_\alpha \equiv D_i(s\tau_{\beta_1},\dots,s\tau_{\beta_i}).$$

3. The Q-vector is the Chevalley-Eilenberg (CE) differential on $C^\infty(\mathsf{L}[1])$ that can always be locally identified with a cohomological vector field Q on a graded manifold $\mathcal{N}$ given by:
$$Q=\sum_{i=0}^{\infty}\frac{1}{i!}C^\alpha_{\beta_1\cdots\beta_i}\xi^{\beta_1}\cdots\xi^{\beta_i}\frac{\partial}{\partial\xi^\alpha} \tag{5.3.2}$$
where ξ are the coordinate functions of $\mathsf{L}[1]$ if viewed as a CE algebra or, locally, the graded coordinates on $\mathcal{N}$.

Applying the three steps above to (5.3.1) and resumming the expansion produces for the cohomological vector field:

$$\begin{aligned} Q={}&\eta^{AB}\frac{\partial}{\partial\mathsf{\eta}^{AB}}+\Big(\rho^A{}_I(X)A^I-\mathsf{\eta}^{AB}F_B-\tfrac12\rho_{B[I}(X)\partial_{\underline{D}}\rho^B{}_{J]}(X)\mathsf{\eta}^{AD}A^IA^J\Big)\frac{\partial}{\partial X^A}+\\ &+\Big(\hat\eta^{IM}\rho^A{}_I(X)F_A-\tfrac12\hat\eta^{IM}T_{IJK}(X)A^JA^K+\tfrac12\mathsf{\eta}^{AB}\hat\eta^{IM}\rho_{C[I}(X)\partial_{\underline{B}}\rho^C{}_{J]}(X)A^JF_A+\\ &\quad+\tfrac{1}{3!}\mathsf{\eta}^{AB}\bar{\mathcal{Z}}_{ABLIJK}(X)\hat\eta^{LM}A^IA^JA^K\Big)\frac{\partial}{\partial A^M}+\\ &+\Big(-\partial_E\rho^B{}_J(X)A^JF_B+\tfrac12\mathsf{\eta}^{AD}\partial_E\Big(\rho_{C[K}(X)\partial_{\underline{D}}\rho^C{}_{L]}(X)\Big)A^KA^LF_A-\\ &\quad-\tfrac{1}{3!}\partial_ET_{IJK}(X)A^IA^JA^K+\tfrac{1}{4!}\mathsf{\eta}^{AB}\partial_E\bar{\mathcal{Z}}_{ABIJKL}(X)A^IA^JA^KA^L\Big)\frac{\partial}{\partial F_E}, \end{aligned}$$

where we have contracted the η with lower indices and factored out η^{AB} from $\mathcal{Z}$ by defining $\bar{\mathcal{Z}}_{ABIJKL}\equiv 3\rho_{D[I}\partial_{\underline{A}}\rho^D{}_J\rho^C{}_K\partial_{\underline{B}}\rho_{\underline{C}L]}$ in order to clarify the degrees of objects. Observe that all terms have the required degree 1 since the coordinates (or coordinate functions) are, after the shift isomorphism, of degrees $(\mathsf{\eta},X,A,F)=(-1,0,1,2)$ (coordinate functions are oppositely graded to vectors).[6] Additionally it is important to notice that in the first term η is a (degree zero) structure constant since this term stems from the zero product that does not take any arguments, whereas $\mathsf{\eta}$ is the dual to the (shifted) L_2 basis of degree -1. One should not be worried here that the algebra used is defined on the pullback spaces since for the purposes of this discussion it could have been defined directly on the target manifold with no modifications other than the understanding of the slightly different nature of coordinates $(\mathsf{\eta},X,A,F)$. The existence of such a vector field implies the claim that a DFT algebroid is (at least locally) a Q-manifold. The caveat of locality arises due to the correspondence of Q-manifolds *globally* with L_∞*-algebroids* that are locally L_∞-algebras. Therefore, since we have only regarded L_∞-algebras we cannot claim

[6] In a slight abuse of notation we denote the coordinates (or coordinate functions) X, A, F the same as the appropriate components of field a and the degree 2 equation of motion after tensoring with the de Rham complex.

a Q-manifold structure globally.

5.3.2 | Classical DFT sigma model

Having understood the symmetry algebra structure of the sigma model corresponding to the DFT algebroid we move now to the classical field theory level by tensoring with the de Rham complex.

A trivial extension. Before constructing the MC field theory we make a trivial extension to the algebra (5.3.1) by adding an additional homogeneous subspace to L: $\mathsf{L}_{-2} \equiv \Gamma(\bigodot^2 T^*\mathcal{M})$, such that all maps μ act trivially on it i.e.:

$$\mu_i(G, \ldots) = 0, \qquad \forall G \in \mathsf{L}_{-2}.,$$

meaning no new homotopy relations arise from this extension. Therefore, L can be decomposed into the following homogeneous subspaces:

$$\mathsf{L}_{-2} \oplus \mathsf{L}_{-1} \oplus \mathsf{L}_0 \oplus \mathsf{L}_1 \oplus \mathsf{L}_2. \tag{5.3.3}$$

This is done so as to be able to write the sigma model with an action functional, in other words to be able to make the algebra cyclic. In addition to the pairings defined in (4.1.9) (up to the transformation of $i, \ldots$ indices into $A, \ldots$) we define the pairing between the L_{-2} and L_2 spaces as:

$$\langle l_{(-2)}, l_{(2)} \rangle = l_{(-2)AB} l_{(2)}^{AB}.$$

Tensor product with the de Rham complex. The full information of a field theory built upon an L_∞-algebra is obtained by combining the kinematical part with the algebraic. In this theory it comes from the de Rham chain on the worldvolume:

$$\Omega^0(\Sigma_3) \to \Omega^1(\Sigma_3) \to \Omega^2(\Sigma_3) \to \Omega^3(\Sigma_3).$$

When taking the tensor product of algebra (L, μ) defined by spaces (5.3.3) and maps (5.3.1) with this complex one obtains the following fields (L' degree 1 elements), gauge parameters (degree 0 elements) and higher gauge parameters (degree < 0 elements):

$$\begin{aligned}
a = X + A + F + G &\in \Omega^0(\Sigma_3, \mathsf{L}_1) \oplus \Omega^1(\Sigma_3, \mathsf{L}_0) \oplus \Omega^2(\Sigma_3, \mathsf{L}_{-1}) \oplus \Omega^3(\Sigma_3, \mathsf{L}_{-2}), \\
c_0 = \epsilon + t + r &\in \Omega^0(\Sigma_3, \mathsf{L}_0) \oplus \Omega^1(\Sigma_3, \mathsf{L}_{-1}) \oplus \Omega^2(\Sigma_3, \mathsf{L}_{-2}), \\
c_{-1} = v + u &\in \Omega^0(\Sigma_3, \mathsf{L}_{-1}) \oplus \Omega^1(\Sigma_3, \mathsf{L}_{-2}) \\
c_{-2} = s &\in \Omega^0(\Sigma_3, \mathsf{L}_{-2}.)
\end{aligned}$$

Graphically this is shown in Table 5.1 and means that in addition to ghosts ϵ and t, and ghost-for-ghost v appearing also in the Courant sigma model we now have a new tower of ghosts corresponding to subspace L_{-2} consisting of: a 2-form ghost r, 1-form ghost-for-ghost u and scalar ghost-for-ghost-for-ghost s. An important aspect to consider is that η from the space L_2 does not end up as a field since its tensor product can not have degree lower than 2. Thus, η becomes an equation of motion as we shall see immediately by solving the Maurer-Cartan equation.

$\Omega^\bullet$	L_{-2}	L_{-1}	L_0	L_1
	$\mathrm{gh}\, c_{-2} = 3$, $\lvert c_{-2}\rvert_{\mathsf{L}'} = -2$	$\mathrm{gh}\, c_{-1} = 2$, $\lvert c_{-1}\rvert_{\mathsf{L}'} = -1$	$\mathrm{gh}\, c_0 = 1$, $\lvert c_0\rvert_{\mathsf{L}'} = 0$	$\mathrm{gh}\, a = 0$, $\lvert a\rvert_{\mathsf{L}'} = 1$
0	s	v	ϵ	X
1	u	t	A	
2	r	F		
3	G			

Table 5.1
Degrees of fields in the L_∞ DFT sigma model.

Equations of motion. As seen in section 3.2.1 the equations of motion (eoms) correspond to the MC equation (3.2.1), however, as we are now working with a curved algebra it too must be extended by allowing i to start from zero:

$$f \equiv \mu_0' + \mu_1'(a) + \frac{1}{2}\mu_2'(a,a) + \cdots = \sum_{i\geqslant 0} \frac{1}{i!}\mu_i'(a,\ldots,a). \tag{5.3.4}$$

By plugging in field a, separating by L degree and resumming the expansion one obtains four equations of motion:

$$\begin{aligned}
f_{-1} &= dF_A + \partial_A \rho^B{}_J(X) A^J F_B + \tfrac{1}{3!}\partial_A T_{IJK}(X) A^I A^J A^K,\\
f_0 &= dA^I + \hat{\eta}^{IJ}\rho^A{}_J(X) F_A + \tfrac{1}{2}\hat{\eta}^{IJ} T_{JKL}(X) A^K A^L,\\
f_1 &= dX^A - \rho^A{}_J(X) A^J,\\
f_2 &= \eta^{-1},
\end{aligned}$$

making the strong constraint an on-shell requirement. There is a subtlety that must be explained here. Thus far, the strong constraint has been understood to mean the restriction of the doubled base space spanned by $\{X^A\}$ through the relation (2.2.5) or by the introduction of the bivector η^{-1}:

$$\eta^{-1}(df, dg) = 0, \qquad f, g \in C^\infty(\mathcal{M}).$$

However, this requirement can be restated by transferring the restriction from the functions f and g i.e. the space $\mathcal{M}$ to the operator η^{-1} itself (as discussed in [61]). This is precisely what has happened here, the equation of motion has restricted the operator in such a way that the strong constraint is satisfied.

Gauge transformations. It was shown in sec. 3.2.1 that a homotopy Maurer-Cartan theory comes with a built-in gauge symmetry given by the relations (3.2.2)–(3.2.4). We begin with the gauge transformations for the fields in a. Using the decomposition of a and reducing the maps μ' to μ by (3.1.2) and (3.1.3) we obtain:

$$\begin{aligned}
\delta X^A &= \rho^A{}_I(X)\epsilon^I, \\
\delta A^I &= d\epsilon^I - \hat{\eta}^{IJ}\rho^A{}_J(X)t_A + \hat{\eta}^{IJ}T_{JKL}A^K\epsilon^L, \\
\delta F_A &= dt_A - \partial_A\rho^B{}_J(X)F_B\epsilon^J - \tfrac{1}{2}\partial_A T_{IJK}(X)A^I A^J\epsilon^K + \partial_A\rho^B{}_J(X)A^J t_B, \\
\delta G_{AB} &= dr_{AB},
\end{aligned}$$

where we have summed the expansions of functions and separated according to L degree as in the case of the equations of motion. It is obvious these are equivalent to (4.1.14)–(4.1.16) up to signs due to differing sign choices in (5.3.1) and the addition of field G. This is expected because on the all products of fields coincide with their Courant counterparts since the maps containing μ_0 do not appear due to μ_0' not being a field but a constraint (or eom). Seeing that this sigma model possesses an extended gauge symmetry to that of the Courant sigma model we expect higher gauge transformations of gauge parameters or, in other words, a gauge redundancy among the parameters as well as the fields. To this end one can calculate these higher transformations using (3.2.4) by the same procedure as above to arrive at:

$$\begin{aligned}
\delta\epsilon^I &= \hat{\eta}^{IJ}\rho^A{}_J(X)v_A, \\
\delta t_A &= dv_A + \partial_A\rho^B{}_J(X)A^J v_B, \\
\delta r_{AB} &= du_{AB}, \\
\delta u_{AB} &= ds_{AB}.
\end{aligned}$$

Here we see what was expected by filling out Table 5.1, namely, due to the addition of a 3-form G we have a higher stage reducible theory requiring a new tower of ghosts ultimately with the appearance of a "ghost-for-ghost-for-ghost" s. The last symmetry transformation to consider is the gauge covariance of the eoms given by expression (3.2.3):

$$\begin{aligned}
\delta_{c_0} f_2 &= 0,\\
\delta_{c_0} f_1^A &= f_2^{AB} t_B + f_1^B \partial_B \rho^A{}_J(X)\epsilon^J,\\
\delta_{c_0} f_0^I &= \hat{\eta}^{IJ} T_{JKL}(X) f_0^K \epsilon^L - \hat{\eta}^{IJ} f_1^A \partial_A T_{JKL}(X) A^K \epsilon^L + \hat{\eta}^{IJ} f_1^A \partial_A \rho^B{}_J(X) t_B -\\
&\quad - \hat{\eta}^{IJ} \eta_{CD} f_2^{AB} \rho^C{}_{[J}(X) \partial_{\underline{B}} \rho^D{}_{K]}(X) \epsilon^K F_A + \tfrac{1}{2} \hat{\eta}^{IJ} \mathcal{Z}_{JKLM}(X) A^K A^L \epsilon^M +\\
&\quad + \hat{\eta}^{IJ} \eta_{CD} f_2^{AB} \rho^C{}_{[J}(X) \partial_{\underline{B}} \rho^D{}_{K]}(X) A^K t_A,
\end{aligned}$$

the transformation of f_{-1} was omitted for brevity. These expressions can be compared to relations (2.2.46) and (2.2.49) by taking into account the gauge fixing (2.2.34) and $f_2 = \eta$ (remember $\mathcal{Z}$ contains η).

Action functional. Lastly, to be able to interpret the equations of motion through a variational principle, we define a Maurer-Cartan action by relation (3.2.6) using the cyclic pairings defined above. The calculation produces:

$$\begin{aligned}
S_{\mathrm{MC}}[X,A,F,G] = \langle G, \eta^{-1}\rangle + \langle \mathrm{d}X, F\rangle + \tfrac{1}{2}\langle A, \mathrm{d}A\rangle + \sum_{n\geqslant 0} \tfrac{1}{n!}\langle A, \mu_{n+1}(X,\dots,X,F)\rangle +\\
+ \tfrac{1}{6}\sum_{n\geqslant 0} \tfrac{1}{n!}\langle A, \mu_{n+2}(X,\dots,X,A,A)\rangle,
\end{aligned}$$

or, by inserting the definitions of the maps and pairings, in components:

$$\begin{aligned}
S[X,A,F,G] = \int_{\Sigma_3} G_{AB}\eta^{AB} + F_A \wedge \mathrm{d}X^A + \tfrac{1}{2}\hat{\eta}_{IJ} A^I \wedge \mathrm{d}A^J - \rho^A{}_I(X) A^I \wedge F_A +\\
+ \tfrac{1}{6} T_{IJK}(X) A^I \wedge A^J \wedge A^K.
\end{aligned}$$

We now see the meaning of the three-form field G, it is a Lagrange multiplier enforcing the strong constraint $\eta = 0$ as an equation of motion. Notice this is a constraint on a constant due to the nature of our construction assuming η was a constant from the beginning. However, one could have introduced interaction terms to the field G and made η a function of X making this a constraint on the fields X.

5.4 | Summary

A DFT algebroid is a geometric structure describing properties of the C-bracket relevant for the gauge symmetry of double field theory. Here we discussed its global properties and gave a formulation in terms of an L_∞-algebra as a preparation to introduce the corresponding Maurer-Cartan homotopy theory (as indicated in [89]). Our ultimate motivation is finding a gauge invariant sigma-model without constraints. This is based on recent proposals suggesting that there exist physically relevant closed strings backgrounds which cannot be obtained as a solution of the strong constraint [45, 90]. Using the L_∞-algebra framework one could bootstrap consistent gauge theories by choosing the initial data of the theory in the form of 1- and 2-brackets and construct the appropriate higher brackets using the homotopy relations [83]. This is akin to the deformation of a free gauge theory into an interacting one in the BV/BRST approach, see e.g. [84].

In this chapter we gave the definition of a DFT algebroid in terms of a *curved* L_∞-algebra, and showed that one can formulate a DFT algebroid in terms of a Q-structure (in the same manner as a Courant algebroid can be, see (2.1.13)), going beyond the results of [59]. Additionally a sigma model was constructed on the basis of this curved algebra that showed it is a second-stage reducible gauge theory with the strong constraint implemented as a Lagrange multiplier. A promising route to tackle the issue of not having a P-structure consists of allowing for a *degenerate* symplectic structure following the construction in [91]. In that case one obtains a presymplectic generalisation of the BV formalism, which reduces to the standard one after factorising out the zero modes of the presymplectic form. However, for most applications one can employ the presymplectic structure without ever performing factorization explicitly. Thus, one expects to obtain an unconstrained gauge invariant theory relevant for understanding the implications of T-duality in a field theory setting.

APPENDIX

5.A | Homotopy conditions for a DFT algebroid L_∞-algebra when $i \geqslant 4$

Taking into account all of the argumentation in section 5.1.3, for arbitrary i one can have only 4 non-trivial distinctly new types of homotopy relations, those that are in space L_1. They will be higher orders of expressions found in the $i = 4$ case: $(l_1, \dots, l_i) = \{(h_1, \dots, h_{i-1}, f), (h_1, \dots, h_{i-2}, e_1, e_2), (\mu_0, h_1, \dots, h_{i-3}, e, f), (\mu_0, h_1, \dots, h_{i-4}, e_1, e_2, e_3)\}$, the first two being the definitions of (5.1.17) and the last two consistency conditions. In order we have:

$$\mu_n(h_1, \dots, h_{i-1}, \mathcal{D}f) = (-1)^{i+1}\mu_{i+1}(\mu_0, h_1, \dots, h_{i-1}, f) \ ,$$

from which one immediately sees the first line of (5.1.17), and:

$$\begin{aligned} (-1)^i\mu_{i-1}(\mu_2(e_1, e_2), h_1, \dots, h_{i-2}) +& \\ + \cdots -& \\ -(\mu_{i-m}(\mu_{i+1}(h_1, \dots, h_m, e_1), h_{m+1}, \dots, h_{i-2}, e_2) - e_1 \leftrightarrow e_2) +& \\ + \cdots =& \\ = (-1)^{i+1}\mu_{i+1}(\mu_0, h_1, \dots, h_{i_2}, e_1, e_2)& \ , \end{aligned}$$

where the dots indicate summation over m and terms of all unshuffles σ: $h_{\sigma(1)}, \dots, h_{\sigma(m)}$ and $h_{\sigma(m+1)}, \dots, h_{\sigma(i-2)}$. This summation is nothing more than the product rule expansion of differential operator $h_1^{A_1} \cdots h_{i-2}^{A_{i-2}}\partial_{A_1} \cdots \partial_{A_{i-2}}$ acting on $[\rho(e_1), \rho(e_2)]$ implying the second line of (5.1.17). Continuing on now to the conditions, the third combination of elements produces:

$$(-1)^{i-1}\mu_i(\mathcal{D}f, \mu_0, h_1, \dots, h_{i-3}, e) - \mu_{i-1}(\langle e, \mathcal{D}f\rangle, \mu_0, h_1, \dots, h_{i-3}) +$$

$$\begin{aligned}
+\mu_{i-2}(\mu_3(\mu_0, e, f), h_1, \ldots, h_{i-3}) +& \\
+\cdots+& \\
+(-1)^{m+1}\mu_{i-m}(\mu_{i+1}(h_1, \ldots, h_m, e), \mu_0, h_{m+1}, \ldots, h_{i-3}, f) + \cdots +& \\
+(-1)^{i-m}\mu_{i-m}(\mu_{i+1}(\mu_0, h_1, \ldots, h_{m-1}, f), h_m, \ldots, h_{m-3}, e) + \cdots +& \\
+\cdots =& \; 0 \;,
\end{aligned}$$

where, just as previously, dots indicate summation over m and term of all unshuffles of h. Again, realising this summation can be resummed produces:

$$(-1)^i h_1^{A_1} \cdots h_{i-3}^{A_{i-3}} \partial_{A_1} \cdots \partial_{A_{i-3}} (-\mathsf{SC}_\rho(e, \mathcal{D}f)^D - \tfrac{1}{4}\eta^{DC}\partial_C(\rho(e)f) + \tfrac{1}{2}\eta_{CD}\rho(e)^B\partial_B\partial_C f)\partial_D = 0 \;,$$

that vanishes in the exact same way it does in the $i = 3, 4$ cases. The final condition valid for $i \geqslant 4$ is:

$$\begin{aligned}
(-1)^i\mu_{i-1}(\mu_2(e_1, e_2), \mu_0, h_1, \ldots, h_{i-4}, e_3) + \text{cyclic} +& \\
+\mu_{i-2}(\mu_3(e_1, e_2, e_3), \mu_0, h_1, \ldots, h_{i-4}) +& \\
+\mu_{i-3}(\mu_4(\mu_0, e_1, e_2, e_3), h_1, \ldots, h_{i-4}) +& \\
+\cdots+& \\
+(-1)^{i-m-1}\mu_{i-m}(\mu_{m+1}(\mu_0, h_1, \ldots, h_{m-2}, e_1, e_2), h_{m-1}, \ldots, h_{i-4}, e_3) + \text{cyclic} + \cdots +& \\
+(-1)^m\mu_{i-m}(\mu_{m+1}(h_1, \ldots, h_m, e_1), \mu_0, h_{m+1}, \ldots, h_{i-4}, e_2, e_3) + \text{cyclic} + \cdots +& \\
+\cdots =& \; 0 \;,
\end{aligned}$$

utilising the same logic as before it is easy to see this is nothing more than a higher derivative of the condition obtained for $i = 4$:

$$\begin{aligned}
(-1)^i h_1^{A_1} \cdots h_{i-4}^{A_{i-4}} \partial_{A_1} \cdots \partial_{A_{i-4}} \Big(- \mathsf{SC}_\rho([\![e_1, e_2]\!], e_3) - [\mathsf{SC}_\rho(e_1, e_2), \rho(e_3)] + \text{cycl.} -& \\
-\rho \operatorname{Jac}(e_1, e_2, e_3)\Big) =& \; 0 \;,
\end{aligned}$$

again satisfied by the fact that $\operatorname{Jac}(\rho(e_1), \rho(e_2), \rho(e_3)) = 0$.

5.B | Calculation of morphism conditions with degree 1 spaces

In this appendix we show the calculation of morphism conditions in the case of DFT and Courant algebroids with spaces L_1 (respectively L_1'). In order to calculate the remaining morphism components we turn to the condition (3.1.5) order by order, beginning with $i = 0$ that does not change with respect to section 5.2.2 by the introduction of space L_1.

Next, $i = 1$, has only one new and non-trivial condition:

$$\phi_1(\rho(e)) - \phi_2(\mu_0, e) = a(\phi_1(e)) \ ,$$

implying:

$$\phi_2(\mu_0, e) = 0 \ . \tag{5.B.1}$$

In the case of $i = 2$, by taking into account the assumptions above we have three new and non-trivial morphism conditions. The first is a compatibility condition corresponding to $(l_1, l_2) = (\mu_0, f)$:

$$\phi_1\Big(\tfrac{1}{2}\eta^{-1}(df)\Big) = a\Big(\tfrac{1}{2}\tilde{D}f\Big|_M\Big) \ ,$$

that is satisfied by (5.2.2) and (2.2.9). Next, for $(l_1, l_2) = (h, e)$, the morphism condition produces:

$$\phi_3(\mu_0, h, e) + \phi_1(h^A \partial_A \rho(e)^B \partial_B) = \phi_1(h)^a \partial_a a(\phi_1(e))^b \partial_b \ ,$$

by splitting capital indices one obtains:

$$\phi_3(\mu_0, h, e) = -(\tilde{h}_a \tilde{\partial}^a \rho(e)^b)\Big|_M \partial_b \ .$$

Lastly for $i = 2$ we have $(l_1, l_2) = (h, f)$ that results in:

$$\phi_3(\mu_0, h, f) = 0 \ ,$$

directly from our assumptions and (3.1.5). Before moving on to higher cases of i, guided by what we have obtained thus far for $i = 1, 2$ we shall make the following Ansatz for the new components to ϕ:

$$\phi_{n+2}(h_1, \ldots, h_n, \mu_0, e) = \Big(h_1^{A_1} \cdots h_n^{A_n} \partial_{A_1} \cdots \partial_{A_n} \rho(e)^b - h_1^{a_1} \cdots h_n^{a_n} \partial_{a_1} \cdots \partial_{a_n} a(e)^b\Big)\Big|_M \partial_b \ , \tag{5.B.2}$$

or explicitly:

$$\phi_{n+2}(h_1, \ldots, h_n, \mu_0, e) = \Big(\tilde{h}_{1a_1} \cdots \tilde{h}_{na_n} \tilde{\partial}^{a_1} \cdots \tilde{\partial}^{a_n} \rho(e)^b + h_1^{a_1} \tilde{h}_{2a_2} \cdots \tilde{h}_{na_n} \partial_{a_1} \tilde{\partial}^{a_2} \cdots \tilde{\partial}^{a_n} \rho(e)^b + \\ + \cdots\Big)\Big|_M \partial_b \ ,$$

where the dots indicate all possible combinations of h and $\tilde{h}$ except the one with no $\tilde{h}$. All other possible new ϕ components vanish. Continuing on to $i = 3$ where after taking into account our assumptions and previous Ansatz one finds there are, in fact, three new and non-trivial identities to be satisfied. The first, $(l_1, l_2, l_3) = (h_1, h_2, e)$, is just the definition (5.B.2), however the second $(l_1, l_2, l_3) = (\mu_0, h, f)$ and third $(l_1, l_2, l_3) = (\mu_0, e_1, e_2)$ are consistency conditions of our Ansatz and yield, respectively:

$$\begin{aligned} -\phi_3(\mathcal{D}f,\mu_0,h)-\phi_1(\tfrac{1}{2}\eta^{BC}h^A\partial_A\partial_B f\partial_C) &= -\phi_1(h)^a\partial_a(\phi_2(\mu_0,f))^b\partial_b \ , \\ -\phi_3(\rho(e_1),\mu_0,e_2)+\phi_3(\rho(e_2),\mu_0,e_1)+\phi_1(\mathsf{SC}_\rho(e_1,e_2)) &= a(\phi_3(\mu_0,e_1,e_2)) \ . \end{aligned} \tag{5.B.3}$$

The latter is satisfied automatically once one takes into account $\rho\circ\tilde{D}=\tilde{\partial}$, whereas for the former one needs the homomorphism property of the Courant bracket and relations (2.2.12). It may be useful to note that since (5.B.2) for $n=0$ vanishes as is in accord with (5.B.1) the second relation above does not have an extra term coming from (5.B.2) that will appear in higher identities.

The analysis so far enables us to move on to the case of a general i. As ϕ only has infinite components for one combination of elements, $\phi_{n+2}(h_1,\dots,h_n,\mu_0,e)$, we need only look at identities involving this component since all others are taken care of in explicit i cases either above or in section 5.2.2 for identities that are equivalent. That withstanding we have three possibilities, starting with $(l_1,\dots,l_i)=(h_1,\dots,h_{i-1},e)$ which is simply the definition (5.B.2). Next, $(l_1,\dots,l_i)=(\mu_0,h_1,\dots,h_{i-2},f)$, is simply the higher derivative case of the first line in (5.B.3) vanishing for the same reason. Finally, the generalisation of the second line or $(l_1,\dots,l_i)=(\mu_0,h_1,\dots,h_{i-3},e_1,e_2)$ and the only slightly non-trivial identity for a general i. Expression (3.1.5) implies:

$$\begin{aligned} &\cdots+ \\ &+(-1)^j\phi_{i-j+1}(\mu_j(h_1,\dots,h_{j-1},e_1),\mu_0,h_j,\dots,h_{i-3},e_2)- \\ &-(-1)^j\phi_{i-j+1}(\mu_j(h_1,\dots,h_{j-1},e_2),\mu_0,h_j,\dots,h_{i-3},e_1)+ \\ &+\cdots+ \\ &+(-1)^i\phi_{i-1}(\mu_2(e_1,e_2),\mu_0,h_1,\dots,h_{i-3})+\phi_1(\mu_i(\mu_0,h_1,\dots,h_{i-3},e_1,e_2))= \\ &=\cdots+ \\ &\quad+(-1)^{i-j-1}\mu'_{i-j+1}(\phi_j(\mu_0,h_1,\dots,h_{j-2},e_1),\phi_1(h_1),\dots,\phi_1(h_{i-3}),\phi_1(e_2))- \\ &\quad-(-1)^{i-j-1}\mu'_{i-j+1}(\phi_j(\mu_0,h_1,\dots,h_{j-2},e_2),\phi_1(h_1),\dots,\phi_1(h_{i-3}),\phi_1(e_1))+ \\ &\quad+\cdots+ \\ &\quad+\mu'_{i-2}(\phi_3(\mu_0,e_1,e_2),\phi_1(h_1),\dots,\phi_1(h_{i-3}))\ , \end{aligned}$$

where by resumming the partial derivatives and utilising the definitions of maps ϕ_i, μ_i, and μ'_i one obtains

$$\begin{aligned} &h_1^{A_1}\cdots h_{i-3}^{A_{i-3}}\partial_{A_1}\cdots\partial_{A_{i-3}}([\rho(e_1),\rho(e_2)]^b-\mathsf{SC}_\rho(e_1,e_2)^b-\rho[\![e_1,e_2]\!]^b)\partial_b= \\ &\quad=h_1^{a_1}\cdots h_{i-3}^{a_{i-3}}\partial_{a_1}\cdots\partial_{a_{i-3}}([a(e_1),a(e_2)]^b-a[e_1,e_2]_C^b)\partial_b\ , \end{aligned}$$

with $\mathsf{SC}_\rho(e_1,e_2)^b$ defined by the splitting $\partial_B=\partial_b+\tilde{\partial}^b$. Each side of this equality vanishes on its own, the lhs because of (2.2.12) and the rhs because of the homomorphism property of a Courant algebroid anchor map a.

5.C | Homotopy identities for DFT sigma model algebra

To show the maps (5.3.1) satisfy the homotopy Jacobi identities (3.1.1) we first analyse all the possible combinations of elements producing non-trivial identities. Simply by degree counting, property (5.1.2) and observing the only homogeneous subspaces these identities can live in are L_{-1}, L_0 and L_1 as discussed in section 5.1.3, we identify at most 15 types of homotopy identities. Notice these are types since there are in fact an infinite number of relations that can, however, be grouped by counting all identities differing only in the number of degree 1 elements as one type. They are, in essence, the same relation, simply a higher order expansion term as seen already in appendices 4.A, 4.C and 5.A. To make the calculation more tractable we shall again abuse notation and denote $l_{(-1)} = F$, $l_{(0)} = A$ and $l_{(1)} = X$ and only calculate the lowest order relation with the rest in the same type obtained by taking higher derivatives and then evaluating at 0. Combinations not involving μ_0 can be understood as defining products of that combination of elements with μ_0, and those involving μ_0 as consistency relations (as explained in sec. 5.1.2). They must all be satisfied using the local axioms of a DFT algebroid (2.2.16), (2.2.18) and (2.2.19) written again for convenience:

$$\hat{\eta}^{IJ}\rho^A{}_I(X)\rho^B{}_J(X) = \eta^{AB}$$

$$2\rho^B{}_{[I}(X)\partial_{\underline{B}}\rho^A{}_{J]}(X) - \rho^A{}_M(X)\hat{\eta}^{MN}\hat{T}_{NIJ}(X) = \eta_{BC}\rho^C{}_{[I}(X)\partial^A\rho^B{}_{J]}(X),$$

$$3\hat{\eta}^{MN}\hat{T}_{M[JK}(X)\hat{T}_{IL]N}(X) + 4\rho^A{}_{[L}(X)\partial_{\underline{A}}\hat{T}_{JKI]}(X) = \mathcal{Z}_{JKIL}(X).$$

We start with the relations in subspace L_1.

- $\underline{l = (X_1, \dots, X_{i-1}, F)}$ In the lowest order this is the curved differential requirement, equivalent to expression (5.1.12).

- $\underline{l = (X_1, \dots, X_{i-2}, A_1, A_2)}$ Again for $i = 2$ this is the Leibniz rule for μ_1 and is precisely axiom 2 above.

- $\underline{l = (X_1, \dots, X_{i-3}, \mu_0, A, F)}$ This is a consistency condition stating:

$$\mu_1\mu_3(\mu_0, A, F) + \mu_2(\mu_2(\mu_0, F), A) - \mu_2(\mu_2(A, F), \mu_0) + \mu_3(\mu_1(F), \mu_0, A) = 0,$$

  in which all terms cancel by use of only axiom 1 and the fact that η is constant.

- $\underline{l = (X_1, \dots, X_{i-4}, \mu_0, A_1, A_2, A_3)}$ The homotopy relations for the lowest $i = 4$ case state:

$$\mu_4(\mu_1(A_1),\mu_0,A_2,A_3)+\mu_3(\mu_2(A_1,A_2),\mu_0,A_3)-\mu_2(\mu_3(\mu_0,A_1,A_2),A_3)+\text{cycl.}+ \\ +\mu_2(\mu_{i-1}(A_1,A_2,A_3),\mu_0)+\mu_1\mu_4(\mu_0,A_1,A_2,A_3)=0,$$

plugging in the definitions of maps μ produces condition:

$$-\eta^{AD}\partial_D T_{IJK}+\rho^A{}_L\hat{\eta}^{LM}\mathcal{Z}_{MIJK}-3\eta^{BD}\eta_{EF}\rho^E{}_{[I}\partial_{\underline{D}}\rho^F{}_J\partial_{\underline{B}}\rho^A{}_{K]}- \\ -3T_{L[IJ}\hat{\eta}^{LM}\eta_{\underline{EF}}\eta^{AD}\rho^E{}_{\underline{M}}\partial_{\underline{D}}\rho^F{}_{K]}+3\rho^B{}_{[I}\partial_{\underline{B}}(\eta_{\underline{EF}}\rho^E{}_J\partial_{\underline{D}}\rho^F{}_{K]})\eta^{AD}=0,$$

satisfied by axioms 2 and 3. This can be seen by utilising axiom 2 on the two terms in the second line, after cancelling what remains is axiom 3 contracted with ρ.

This ends the L_1 relations. We move on to L_0 relations.

- $\underline{l=(X_1,\ldots,X_{i-2},A,F)}$ As in the case $l=(X_1,\ldots,X_{i-2},A_1,A_2)$ this too is trivial in that it is plainly axiom 2 contracted with $\hat{\eta}$.
- $\underline{l=(X_1,\ldots,X_{i-3},A_1,A_2,A_3)}$ This is precisely axiom 3 with the second term antisymmetrisation split in two.
- $\underline{l=(X_1,\ldots,X_{i-3},\mu_0,F_1,F_2)}$ The consistency of this relation is obvious from axiom 1 and the symmetrisation that is appears due to the -1 grading of F.
- $\underline{l=(X_1,\ldots,X_{i-4},\mu_0,A_1,A_2,F)}$ By explicit calculation one can see this identity holds directly by that of $l=(X_1,\ldots,X_{i-4},\mu_0,A_1,A_2,A_3)$.
- $\underline{l=(X_1,\ldots,X_{i-5},\mu_0,A_1,A_2,A_3,A_4)}$ As in the case of three A elements this too will be a sum of μ maps with four A elements distributed in $\mu_5\mu_1$, $\mu_4\mu_2$, $\mu_3\mu_3$, $\mu_2\mu_4$ and $\mu_1\mu_5$. This sum yields the following condition:

$$\rho^A{}_{[I}\partial_{\underline{A}}\mathcal{Z}_{JKLM}+2T_{P[IJ}\mathcal{Z}_{KLM]N}\hat{\eta}^{NP}+2\eta_{BC}\eta^{AD}\rho^C{}_{[M}\partial_{\underline{D}}\rho^B{}_I\partial_{\underline{A}}T_{JKL}=0.$$

  It can be seen to hold when one uses axiom 2 on on the last term and axiom 3 (the definition od $\mathcal{Z}$) on the first two. This is because terms that do not cancel vanish identically on their own due to the antisymmetrisation of five indices and symmetricity of $\hat{\eta}$.

Finally, we now come to the last subspace L_{-1}.

- $\underline{l=(X_1,\ldots,X_{i-2},F_1,F_2)}$ The lowest order relation corresponding to this choice of elements is:

$$\mu_2(\mu_1(F_1),F_2)+\mu_2(\mu_1(F_2),F_1)=0,$$

producing: $\partial_C \rho^{(B}{}_J \hat{\eta}^{\underline{IJ}} \rho^{A)}{}_I = 0$. This is identically satisfied by axiom 1 since it implies antisymmetry of A and B.

- $\underline{l = (X_1, \ldots, X_{i-3}, F, A_1, A_2)}$ This choice produces precisely axiom 2 (up to an added derivative that is of no consequence).

- $\underline{l = (X_1, \ldots, X_{i-4}, A_1, A_2, A_3, A_4)}$ Just as the previous case this too gives precisely axiom 3 (again up to an overall derivative).

- $\underline{l = (X_1, \ldots, X_{i-4}, \mu_0, A, F_1, F_2)}$ The consistency condition produced holds trivially by use of axiom 1.

- $\underline{l = (X_1, \ldots, X_{i-5}, \mu_0, A_1, A_2, A_3, F)}$ A straightforward but lengthy calculation produces a derivative of the identity from $l = (X_1, \ldots, X_{i-4}, \mu_0, A_1, A_2, A_3)$ that has been shown to hold.

- $\underline{l = (X_1, \ldots, X_{i-5}, \mu_0, A_1, A_2, A_3, A_4, A_5)}$ The final condition, unsurprisingly, coincides with that of $l = (X_1, \ldots, X_{i-5}, \mu_0, A_1, A_2, A_3, A_4)$, again up to an overall derivative that does not impact its validity.

CHAPTER 6

CONCLUSION

Here we shall summarise what has been shown in the previous chapters and provide a selection of possible avenues for continuing this research based on the conclusions obtained through this thesis.

6.1 | Overview

The search for a quantum description of gravity has led us to higher dimensional fundamental objects and, as particles couple to 1-form gauge fields, 1-dimensional objects, strings, will couple to 2-form gauge fields and consequently higher d-dimensional membranes will couple to $(d+1)$-form gauge fields. This necessitates the study of generalised gauge symmetries and their corresponding field theories. We focused on the worldsheet description of two such theories: the Courant sigma model and double field theory. Let us recap the problems addressed and results obtained in the context of this dissertation. We began with the knowledge of the classical relation between Courant sigma models and a membrane sigma model realisation of double field theory. The problem was this was only known on the classical level of the sigma model, whereas, the full BV theory of Courant sigma models has been known for some time. Therefore the natural question was how does this correspondence survive on the BRST level where gauge parameters become ghosts that transform. In [46] this is the problem that was addressed. The solution was to fix the 1-form ghost and ghost-for-ghost fields in a consistent way such that the projection does not break the BRST symmetry. An expected issue occurred however, when going back to the gauge symmetry level it was observed that the equations of motion do not transform covariantly, requiring the use of the strong constraint.

Next, attention was shifted to the formalism of L_∞-algebras. As they hold great potential for the generalisation of gauge theories we explored them in the case of the two theories of interest in our previous discussion, beginning with the Courant sigma model.

The idea was to fully understand the symmetries involved and their interplay with the dynamics of fields. This was done by differentiating three different L_∞-algebras involved: the gauge symmetry algebra, the classical field algebra and the BV algebra. Through the construction of these three levels via tensor product L_∞-algebras it became apparent that in the sense of gauge structure one needed to only know the basic symmetry algebra to be able to construct the classical and BV levels without any other input. The full BV Courant sigma model was reconstructed in this way simply from the gauge symmetry related to the target Courant algebroid by an L_∞-morphism.

Finally enthralled by the power of the L_∞-formalism we sought to apply it to the DFT algebroid. This, however, presented us again with the problem of the strong constraint that we did not wish to implement. This is because once the strong constraint is solved all structure drops to the Courant case that is well known but does not live on a doubled manifold thus losing the desired manifest duality symmetry. Hence, to avoid this a rarely mentioned but very natural extension of L_∞-algebras was used called curved L_∞-algebras. It was shown that such an extension proved precisely what was needed in order to encapsulate the strong constraint violating terms. Then, to fully understand the strong constraint from an L_∞ aspect, we constructed an L_∞-morphism to the Courant algebroid showing the application of the strong constraint to be just a morphism of L_∞-algebras. And lastly, a sigma model was built on the basis of this curved algebra producing an even higher stage reducible theory with the strong constraint an equation of motion.

6.2 | Outlook

Continuing the exploration of L_∞ structures it would be interesting and a logical continuation to study the possible generalisations of DFT sigma models that no longer need the strong constraint to be consistent. The implication of our work here is that the appropriate framework for this are curved L_∞-algebras or their flat equivalents in the cases when that is possible. This would involve defining new products that could make Maurer-Cartan elements possible and therefore escape the caveat that the MC equation can produce constraints as well as equations of motion. To go beyond just equations of motion into an action functional one needs a cyclic pairing corresponding to a P-structure in the language of graded geometry. This is problematic for our curved algebras as this pairing becomes degenerate, however, there is a way out. If one is to consider presymplectic structures generalising BV this need not be an issue as one can, if necessary, factor out the zero modes to obtain the regular BV theory.

A second intriguing notion is the study of gauge anomalies through the use of L_∞. Realising one can analyse anomalies by careful consideration of the BV extension of a theory would imply this could be done via L_∞. More precisely, BV/BRST deformation can be used to construct interactive BV actions, on the other hand this can be done using

L_∞ bootstrapping and taking the tensor product with the dynamics of the fields of interest. Since this deformation procedure can be used to construct anomalies one is to expect the same can be done using L_∞-algebras. This would substantially simplify the analysis of gauge anomalies as calculations involving L_∞-algebras are, however long, completely straightforward.

As was recently proposed braided L_∞-algebras, where the maps no longer posses graded commutativity, show the way L_∞-algebras can be generalised to non-commutative physics. Another interesting avenue, thus, involves the coalgebra picture of L_∞-algebras and its relation to braided L_∞-algebras. Once one realises curved L_∞-algebras are in fact Hopf algebras on which a Drinfeld twist is axiomatically defined, the obvious question becomes do twisted Hopf algebras lead to braided L_∞-algebras?

CURICULUM VITAE

Clay James Grewcoe was born on the 25th of January 1992 in Brisbane, Australia. After completing his secondary education at “Gimnazija Dubrovnik” in 2010 he enrolled into the Physics programme at the Department of Physics, Faculty of Science, University of Zagreb. In 2014 he was chosen for the CERN Summer Studentship during which he worked for the ATLAS collaboration. In 2016 he graduated on the subject “Generalized Gauge Theories” under supervision of Larisa Jonke, PhD, and obtained the title of Master of Science in Physics. The same year he enrolled in the doctoral programme at the Department of Physics, Faculty of Science, University of Zagreb working as a research assistant in the Quantum Gravity and Mathematical Physics Group led by dr. L. Jonke at the Division of Theoretical Physics, Rudjer Boskovic Institute. His work was financed by the Croatian Science Foundation under project ‘DOK-10-2015’ and by the Faculty of Science as an external teaching assistant. His teaching activities include auditory and practical exercises as well as administering written exams to undergraduate courses: Classical Electrodynamics, Symmetries in Physics and Relativistic Quantum Physics.

List of publications and conference proceedings

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- Conference on Symmetries, Geometry and Quantum Gravity, Primošten, Croatia, 2018
- Dualities and Generalized Geometries, Corfu, Greece, 2018
- 2018 QSPACE Training School, Benasque, Spain, 2018
- Bayrischzell Workshop 2019, Bayrischzell, Germany, 2019
  Talk title: “BRST symmetry of doubled membrane sigma models”
- Seminar of the Group for Gravitation, Particles and Fields, Institute of Physics Belgrade, online, 2021
  Talk title: “BV/BRST formalism in the language of L_∞-algebras”
- Seminar of the Group for Gravitation, Particles and Fields, Institute of Physics Belgrade, online, 2021
  Talk title: “Curved L_∞-algebras”

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